OC51. Déterminer toutes les paires \((a, b)\) d’entiers non négatifs tels que \(a^b + b\) divise \(a^{2b} + 2b\). Noter que, pour ce problème, \(0^0 = 1\).

OC52. Soit \(d, d’\) deux diviseurs de \(n\) avec \(d’ > d\). Montrer que
\[
d’ > d + \frac{d^2}{n}.
\]

OC53. Trouver tous les polynômes \(P(x) \in \mathbb{R}[x]\) tels que \(P(a) \in \mathbb{Z}\) implique \(a \in \mathbb{Z}\).

OC54. On donne quatre points dans le plan tels que les cercles inscrits des quatre triangles formés par trois des quatre points sont égaux. Montrer que les quatre triangles sont égaux.

OC55. Soit \(d\) un entier positif. Montrer que, pour tout entier \(S\), il existe un entier \(n > 0\) et une suite \(\epsilon_1, \epsilon_2, \ldots, \epsilon_n\), où pour tout \(k, \epsilon_k = 1\) ou \(\epsilon_k = -1\), tels que
\[
S = \epsilon_1(1 + d)^2 + \epsilon_2(1 + 2d)^2 + \epsilon_3(1 + 3d)^2 + \cdots + \epsilon_n(1 + nd)^2.
\]

OC56. On suppose que \(f : \mathbb{N} \to \mathbb{N}\) est une fonction telle que pour tout \(a, b \in \mathbb{N}\), l’expression \(af(a) + bf(b) + 2ab\) est un carré parfait. Montrer que \(f(a) = a\) pour tout \(a \in \mathbb{N}\).

OC57. Soit \(ABC\) un triangle et \(A’, B’, C’\) les points milieu respectifs de \(BC, CA, AB\). Soit \(P\) et \(P’\) deux points dans le plan tels que \(PA = P’A’\), \(PB = P’B’\), \(PC = P’C’\). Montrer que tous les \(PP’\) passent par un même point.

OC58. Trouver le plus petit \(n\) pour lequel il existe des polynômes \(f_1(x), f_2(x), \ldots, f_n(x) \in \mathbb{Q}[x]\) tels que
\[
f_1^2(x) + f_2^2(x) + \cdots + f_n^2(x) = x^2 + 7.
\]
OC59. Soit $n$ un entier positif impair tel que $\phi(n)$ et $\phi(n + 1)$ sont des puissances de deux. Montrer que $n + 1$ est une puissance de deux ou $n = 5$.

OC60. On écrit les nombres 1, 2, ..., 20 au tableau noir. Une opération consiste à choisir deux nombres $a, b$ tels que $b \geq a + 2$, effacer $a$ et $b$ et les remplacer par $a + 1$ et $b - 1$. Trouver le nombre maximal d’opérations possibles.
In this number of the *Corner* we will complete the files of solutions from the readers to problems given in the *Corner*, and also my time as editor of the *Corner*. Thanks to those who contributed solutions to problems discussed in 2011 numbers:

Arkady Alt
Miguel Amengual Covas
George Apostolopoulos
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D.J. Smeenk
Gheorge Ghita Stanciu
Edward T.H. Wang
Dexter Wei
Konstantine Zelator
Kaiming Zhao
Titu Zvonaru

I’ve enjoyed my association with *Crux Mathematicorum* and the *Corner* since January 1988. I would like to express my thanks to all those over nearly a quarter century who have supported the *Corner* and *Crux Mathematicorum* by sending in problem sets, comments, and solutions. I would also like to express my thanks to Joanne Canape who has been transcribing my scribbles into \LaTeX\ manuscripts over the last decades. With the next volume of *Crux Mathematicorum* there will be a new team and new features to draw your interest and support.

R.E. Woodrow

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First, we look at the readers’ solutions to problems from the Croatian Mathematical Competition 2007, National Competition, given at [2010: 435–436], that we started in the last issue.

**4th Grade**

2. Sequence \((a_n)_{n \geq 0}\) is defined recursively by

\[
\begin{align*}
  a_0 &= 3, \\
  a_n &= 2 + a_0 \cdot a_1 \cdot \ldots \cdot a_{n-1}, \quad n \geq 1.
\end{align*}
\]

(a) Prove that any two terms of the sequence are relatively prime positive integers.

(b) Determine \(a_{2007}\).
For part (a), assume there exist two integers \( m > n > 0 \) such that \( \gcd(a_m, a_n) = d > 1 \). Then \( a_0a_1 \cdots a_{m-1} = a_m - 2 \). Since the term \( a_n \) is one of the factors on the left side of this equation, it follows that \( d \) divides 2. But each \( a_n \) \((n \geq 0)\) is odd, therefore \( d = 2 \) is impossible. Hence, \( d = 1 \).

For part (b), we begin by showing inductively that the terms \( a_n \) of the sequence are given by the Fermat numbers \( 2^{(2^n)} + 1 \). Note that \( a_0 = 3 = 2^{(2^0)} + 1 \). Assume that \( k \geq 0 \) is a nonnegative integer and

\[
a_k = 2 + a_0a_1 \cdots a_{k-1} = 2^{(2^k)} + 1
\]

or

\[
a_k - 2 = a_0a_1 \cdots a_{k-1} = 2^{(2^k)} - 1.
\]

Then

\[
a_{k+1} = (a_0a_1 \cdots a_{k-1})a_k + 2 = (2^{(2^k)} - 1)(2^{(2^k)} + 1) + 2
\]

\[
= (2^{(2^{k+1})} - 1) + 2 = 2^{(2^{k+1})} + 1.
\]

Therefore, by induction, \( a_n = 2^{(2^n)} + 1 \) for each \( n \geq 0 \). Hence,

\[
a_{2007} = 2^{(2^{2007})} + 1.
\]

4. In acute triangle \( ABC \) let \( A_1, B_1 \) and \( C_1 \) be the midpoints of sides \( BC, CA \) and \( AB \), respectively. The radius of its circumscribed circle, with centre \( O \), is 1. Prove that

\[
\frac{1}{|OA_1|} + \frac{1}{|OB_1|} + \frac{1}{|OC_1|} \geq 6.
\]

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the solution by Zvonaru.

Since it is known that \( OA_1 = R \cos A \), we have to prove that

\[
\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \geq 6.
\]

Let \( f : (0, \frac{\pi}{2}) \to R \) be the function \( f(x) = \frac{1}{\cos x} \). We have

\[
f'(x) = \frac{\sin x}{\cos^2 x},
\]

\[
f''(x) = \frac{\cos^3 x - \sin x(-2 \cos x \sin x)}{\cos^4 x} = \frac{\cos^2 x + 2 \sin^2 x}{\cos^3 x} > 0,
\]
hence $f$ is a convex function.

Applying Jensen’s Inequality we get

$$f(A) + f(B) + f(C) \geq 3f\left(\frac{A + B + C}{3}\right),$$

that is

$$\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \geq \frac{3}{\cos \frac{\pi}{3}} = 6.$$

The equality holds if and only if $A = B = C$, that is $\triangle ABC$ is equilateral.

Next we move to readers’ solutions to problems of the 51st National Mathematical Olympiad in Slovenia, Selection Examinations for the IMO 2007, given at [2010: 436–437].

**First Selection Examination, December 2006**

1. Show that the inequality

$$(1 + a^2)(1 + b^2) \geq a(1 + b^2) + b(1 + a^2)$$

holds for any pair of real numbers $a$ and $b$.

_Solved by George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Perfetti._

We know that

$$(x - y)^2 \geq 0 \implies x^2 + y^2 \geq 2|xy|$$

thus

$$(1 + a^2)(1 + b^2) = (1 + a^2)(1 + b^2)/2 + (1 + a^2)(1 + b^2)/2 \geq |a|(1 + b^2) + |b|(1 + a^2) \geq a(1 + b^2) + b(1 + a^2)$$

since $|x| \geq x$.

2. Prove that any triangle can be decomposed into $n$ isosceles triangles for every positive integer $n \geq 4$.

_Solved by Titu Zvonaru, Comănești, Romania._
We will prove three claims.

**Claim 1.** Any triangle can be decomposed into 4 isosceles triangles.

Let $\triangle ABC$ be a triangle. Suppose that $\angle BAC$ is the greatest angle of $\triangle ABC$. Denoting by $D$ the projection of $A$ onto $BC$, it follows that $D$ lies between $B$ and $C$ (since the angles $\angle ABC$ and $\angle ACB$ are acute).

Let $M, N$ be the midpoints of the sides $AC$ and $AB$, respectively.

Since in a right-angled triangle the median to the hypotenuse is half of the hypotenuse, it follows that the triangles $BDN, NDA, DCM$ and $DMA$ are isosceles.

**Claim 2.** Any triangle can be decomposed into 5 isosceles triangles.

(a) If $\triangle ABC$ is not equilateral, then we may suppose that $AB < BC$.

We take a point $D$ on the side $BC$ such that $AB = BD$. We obtain an isosceles triangle $ABD$ and for $\triangle ADF$ we can apply claim 1.

(b) Suppose that $\triangle ABC$ is an equilateral triangle.

Let $O$ be the circumcentre of $\triangle ABC$. The perpendicular through $O$ to $OC$ meets $BC$ at $M$, and let $N$ be the midpoint of $MC$.

Since $\angle BOC = 120^\circ$, it follows that $\angle BOM = 30^\circ = \angle OBM$. We deduce that the triangles $OAB, OAC, OBM, OMN$ and $ONC$ are isosceles.

**Claim 3.** Any triangle can be decomposed into 6 isosceles triangles.

Let $I$ be the incentre of $\triangle ABC$. The incircle is tangent to the sides $BC, CA, AB$ at the points $D, E$ and $F$, respectively.

We have the following 6 isosceles triangles: $AFE, IFE, BDF, IDF, CDE$ and $IDE$.

Using Claim 1, we see that if we can decompose a triangle into $k$ isosceles triangles, we can decompose this triangle into $k + 3$ isosceles triangles, and so on.

Using Claims 1, 2 and 3 we deduce that any triangle can be decomposed into $n \geq 4$ isosceles triangles.

*Comment.* If $\triangle ABC$ is an acute-angled triangle, then it can also be decomposed into 3 isosceles triangles (namely $AOB, BOC, COA$, where $O$ is the circumcentre).

3. Let $\triangle ABC$ be a triangle with $|AC| < |BC|$ and denote its circumcircle by $K$. Let $E$ be the midpoint of the arc $AB$ that contains the point $C$ and let $D$ be a point on the segment $BC$, such that $|BD| = |AC|$. The line $DE$ meets the circle $K$ again in $F$. Prove that $A, B, C$ and $F$ are the vertices of an isosceles trapezoid.
Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Zelator.

Since \( E \) is the midpoint of the arc \( AB \) [that contains \( C \)], we have

\[ |BE| = |EA|. \tag{1} \]

Also by hypothesis,

\[ |BD| = |AC|. \tag{2} \]

And obviously,

\[ \angle EBC = \angle EAC = \theta. \tag{3} \]

From (1), (2), and (3), it follows that the triangles \( BDE \) and \( AEC \) are congruent. Therefore \( |ED| = |EC| \) and so triangle \( DEC \) is isosceles.

\[ \angle EDC = \varphi = \angle ECD. \tag{4} \]

Furthermore,

\[ \angle EDC = \varphi = \angle BDF \tag{5} \]

and

\[ \angle BFE = \varphi = \angle BCE. \tag{6} \]

From (4), (5), and (6); it follows that the triangle \( FBD \) is isosceles with

\[ |BD| = |BF|. \tag{7} \]

Hence, from (7) and (2) \( \Rightarrow |BF| = |CA| \), which further implies that \( \angle BCF = \angle CFA \), which in turn implies that \( BC \) and \( FA \) are parallel.

We have \( \{ |BF| = |CA| \) and \( |BC||FA| \) \( \Rightarrow \) \( BCAF \) is an isosceles trapezoid.

Second Selection Examination, February 2007

1. Every point in the plane with positive integer coordinates \((x, y)\) such that \( x \leq 19 \) and \( y \leq 4 \) is colored green, red or blue. Prove that there exists a rectangle with sides parallel to the coordinate axes and with vertices of the same colour.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

We rephrase the problem as follows: “Each square of a \( 4 \times 19 \) chessboard is coloured green, red or blue. Prove that there exists a rectangle whose four corner squares are all coloured the same.”
There are 19 columns each containing 4 squares. By the Pigeon-hole Principle, each column must contain some pair of squares of the same colour (we have 4 squares per column and only 3 colours).

In order for a rectangle to have its four corners coloured the same colour, there must be two different columns in which squares of the same colour are placed in the same two rows.

There are \( \binom{4}{2} \) ways to place a pair of green squares, \( \binom{4}{2} \) ways to place a pair of red squares and \( \binom{4}{2} \) ways to place a pair of blue squares. Thus, there are \( 3 \cdot \binom{4}{2} = 3 \cdot 6 = 18 \) ways altogether. Since there are 19 columns in the board, there must be at least 2 different columns in which a pair of squares of the same colour are placed in the same way.

It follows that there exists a rectangle whose four corner squares are coloured with the same colour.

*Comment.* In a similar way, we can solve the problem with a \((t + 1) \times (\frac{t^2(t+1)}{2} + 1)\) chessboard and \(t\) colours.

2. The circles \( K_1 \) and \( K_2 \) of different radii meet at \( A_1 \) and \( A_2 \). Let \( t \) be the common tangent of the two circles, such that the distance from \( t \) to \( A_1 \) is shorter than the distance from \( t \) to \( A_2 \). Let \( B_1 \) and \( B_2 \) be the points in which \( t \) touches \( K_1 \) and \( K_2 \), respectively.

Let \( K_3 \) and \( K_4 \) be the circles with radii \(|A_1B_1|\) and \(|A_1B_2|\) and the centres \( A_1 \). The circles \( K_1 \) and \( K_3 \) meet again at \( C_1 \), while the circles \( K_2 \) and \( K_4 \) meet again at \( C_2 \). Denote the intersection of the lines \( B_1C_1 \) and \( B_2C_2 \) by \( D \) and let \( E \) be the intersection of \( B_1C_1 \) and \( K_4 \) which lies on the same side of the line \( B_2C_2 \) as \( C_1 \).

Show that \( A_1D \) is perpendicular to \( EC_2 \).

*Solved by Oliver Geupel, Brühl, NRW, Germany.*

Let \( O_1 \) and \( O_2 \) be the centres of \( K_1 \) and \( K_2 \), respectively. Let \( F_1 \) and \( G_1 \) be the feet of the altitudes from \( B_1 \) and \( A_1 \) in triangle \( A_1B_1O_1 \), respectively. Let \( F_2 \) and \( G_2 \) be the feet of the altitudes from \( B_2 \) and \( A_1 \) in triangle \( A_1B_2O_2 \), respectively.

By \( A_1G_1 \perp B_1G_1 \) and \( A_2G_2 \perp B_2G_2 \), the quadrilateral \( B_1B_2G_2G_1 \) is a rectangle. Hence, we see that \( B_1G_1 = B_2G_2 \). Also, the triangles \( A_1B_1O_1 \) and \( A_1B_2O_2 \) are isosceles with \( O_1A_1 = O_1B_1 \) and \( O_2A_1 = O_2B_2 \), which implies that \( A_1F_1 = B_1G_1 \) and \( A_1F_2 = B_2G_2 \). Therefore,

\[
A_1F_1 = A_1F_2,
\]

that is, the lines \( B_1E \) and \( B_2C_2 \) have the same distance from the centre \( A_1 \) of \( K_4 \). Hence, the two lines are symmetric with respect to the axis \( A_1D \). Thus, the triangle \( Dc_2E \) is isosceles with \( DC_2 = DE \) and with the line \( A_1D \) as its axis of symmetry. Consequently, \( A_1D \) is perpendicular to \( EC_2 \).
3. Find a positive integer \( n \) such that \( n^2 - 1 \) has exactly 10 positive divisors. Show that \( n^2 - 4 \) cannot have exactly 10 positive divisors for any positive integer \( n \).

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geipel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution by Manes.

If \( n \) is even, then \( n^2 - 1 = (n + 1)(n - 1) \) and \( \gcd(n + 1, n - 1) = 1 \). Since \( \tau(n) \), the number of positive divisors of \( n \), is multiplicative, it follows that \( \tau(n^2 - 1) = \tau(n + 1)\tau(n - 1) = 10 = 5 \cdot 2 \). Unique factorization implies one such factorization is \( n = 82 \) and \( \tau(83) = 2 \). Accordingly, if \( n = 82 \), then \( n^2 - 1 = 6723 = 3^4 \cdot 83 \) and \( \tau(n^2 - 1) = 10 \). The solution \( n = 82 \) is not unique since \( n = 2400 \) also satisfies the property that \( \tau(n^2 - 1) = 10 \).

Note that if \( \tau(m) = 10 \), then either \( m = p^4q \) or \( m = p^9 \) for some distinct primes \( p \) and \( q \). Thus, if \( \tau(n^2 - 4) = 10 \), then either \( n^2 - 4 = p^4q \) or \( n^2 - 4 = p^9 \). If \( n^2 - 4 = (n + 2)(n - 2) = p^9 \), then unique factorization implies \( n + 2 = p^r \) and \( n - 2 = p^s \) where \( r > s \) and \( r + s = 9 \). Subtracting the two equations, it follows that \( p^r - p^s = 4 \), hence \( p = 2 \). Therefore, \( 2^{r-2} - 2^{s-2} = 1 \) implies \( r - 2 = 1 \) and \( s - 2 = 0 \), a contradiction.

If \( n^2 - 4 = (n + 2)(n - 2) = p^4q \), then either \( n + 2 = p^r \) and \( n - 2 = p^s \) or \( n + 2 = p^r \) and \( n - 2 = p^sq \). If \( n + 2 = p^r \) and \( n - 2 = p^s \), then \( r + s = 4 \) and \( r > s \). Subtracting the two equations, one obtains \( p^r - p^s = 4 \), again implying that \( p = 2 \). Therefore, \( 2^{r-2} - 2^{s-2} = 1 \), whence \( r = 2 \) or \( s = 2 \), each of which is a contradiction since \( r + s = 4 \) and \( r > s \). On the other hand, if \( n + 2 = p^r \) and \( n - 2 = p^s \), then \( r + s = 4 \). Solving for \( n \) in each equation, it follows that \( n = p^r - 2 = p^s + 2 \) or \( p^r - q^s = 4 \), whence \( p = 2 \). Therefore, \( 2^{r-2}q - 2^s = 4 \) so that \( 2^{r-2}q = 2^{s-2} + 1 \). But \( r + s = 4 \).
implies \((r - 2) + (s - 2) = 0\) or \(r = s = 2\). Hence, \(q = 2\), a contradiction since \(p\) and \(q\) are distinct primes. Consequently, all of these contradictions imply that \(\tau(n^2 - 4) \neq 10\) for any positive integer \(n\).

Third Selection Examination, March 2007

2. Let 
\[ x = 0.a_1a_2a_3a_4 \ldots \quad \text{and} \quad y = 0.b_1b_2b_3b_4 \ldots \]

be the decimal representations of two positive real numbers. The equality \(b_n = a_{2^n}\) holds for all positive integers \(n\). Given that \(x\) is a rational number, show that \(y\) is rational, too.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

(i) If the decimal representation for \(x\) terminates, then so does that for \(y\), and \(y\) is rational.

(ii) Otherwise, the decimal representation for \(x\) eventually repeats with some smallest period \(t \in \mathbb{N}\). Thus, there is a positive integer \(k_0\) such that for \(k \geq k_0\), \(a_{k+t} = a_k\). Hence, for \(j \in \{0, 1, \ldots, t-1\}\), there exists \(r_j \in \mathbb{N} \cup \{0\}\) such that for \(k \geq k_0\), \(a_k = r_j\) if \(k \equiv j \mod t\). Thus, \(b_n = a_{2^n} = r_i\) if \(2^n \equiv i \mod t\).

(a) If \(\gcd(2, t) = 1\), that is, if \(t\) is odd, let \(\alpha\) be the order of \(2 \mod t\). Then \(2^\alpha \equiv 1 \mod t\). Suppose \(2^n \equiv i \mod t\). Then \(2^{n+\alpha} = 2^n \cdot 2^\alpha \equiv 2^n \equiv i \mod t\), so that \(b_{n+\alpha} = a_{2^n+\alpha} = r_i = a_{2^n} = b_n\). Hence, \(\alpha\) is a period for \(\{b_n\}\), so \(y\) is rational.

(b) If \(t = 2^s \cdot u\), with \(u\) odd, let \(\alpha\) be the order of \(2 \mod u\). Then \(2^\alpha \equiv 1 \mod u\). For \(n \geq s\), \(2^n\) divides \(2^{n+\alpha} - 2^n\), so that \(2^{n+\alpha} = 2^n \cdot 2^\alpha \equiv 2^n \mod t\). As before, this implies that \(y\) is rational.

3. Let \(ABCD\) be a trapezoid with \(AB\) parallel to \(CD\) and \(|AB| > |CD|\). Let \(E\) and \(F\) be the points on segments \(AB\) and \(CD\), respectively, such that \(\frac{|AE|}{|EF|} = \frac{|DF|}{|FC|}\). Let \(K\) and \(L\) be two points on the segment \(EF\) such that

\[ \angle AKB = \angle DCA \quad \text{and} \quad \angle CLD = \angle CBA. \]

Show that \(K, L, B\) and \(C\) are concyclic.

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let the perpendicular to \(BC\) at \(B\) intersect the perpendicular bisector of \(AB\) at \(O\), and let the perpendicular to \(BC\) at \(C\) intersect the perpendicular bisector of \(CD\) at \(P\). The line \(OP\) intersects \(AB\) and \(CD\) at points \(G\) and \(H\), respectively, such that \(|BG| : |EG| = |CH| : |FH|\). Hence, the lines \(BC, EF\), and \(GH\) have a common intersection \(S\). From \(\angle AKB = 180^\circ - \angle ABC\) we see that the line \(BC\) is tangent to the circle \((ABK)\). Thus, \(O\) is the centre of \((ABK)\). Analogously, \(P\) is the centre of the circle \((CDL)\). Hence, \((ABK)\) and \((CDL)\) are homothetic with respect to the centre of homothety \(S\).
If $M$ is the second point of intersection of the line $SK$ and the circle $(ABK)$, then $(B, C)$, $(O, P)$, $(G, H)$, $(E, F)$, and $(M, L)$ are pairs of corresponding points under this homothety. Therefore, $\angle KMB = \angle SMB = \angle SLC$. Furthermore, $\angle KMB = \angle KBC$, because the line $BC$ is tangent to the circle $(ABKM)$. Thus, $\angle KBC = \angle SLC$. If the point $K$ is between $S$ and $L$ then

$$\angle KBC = \angle KLC. \quad (1)$$

Otherwise $L$ is between $S$ and $K$, and we conclude

$$\angle KBC = 180^\circ - \angle KLC. \quad (2)$$

Either of (1) and (2) implies that $K$, $L$, $B$, and $C$ are concyclic.

Next, we look at solutions submitted to problems of the Correspondence Mathematical Competition in Slovakia 2006/7 First Round, First Set, given at [2010: 438–439].

1. There are some pigeons and some sparrows sitting on a fence. Five sparrows flew away and there remained two pigeons for each sparrow. Then 25 pigeons flew away and there remained three sparrows for each pigeon. Find the initial numbers of sparrows and pigeons.

*Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania.*

We give the solution of Curtis.

Let $p$ and $s$ be the initial numbers of pigeons and sparrows, respectively. Then $p$ and $s$ satisfy the system

$$\begin{cases} p = 2(s - 5) \\ s - 5 = 3(p - 25). \end{cases}$$

Substituting the first equation into the second, solving for $s$, and then substituting back gives

$$\begin{cases} p = 30 \\ s = 20. \end{cases}$$

3. We have eight cubes with digits $1, 2, 3, 4, 5, 6, 7, 9$ (each cube has one digit written on one of its faces). In how many ways can we create four two-digit primes from the cubes?
We claim that there are four ways. We assume that the order in which the four primes occur is immaterial.

Note first that none of the four primes can end in 2, 4, 5, or 6. Hence these digits must be the first digits of the four primes and 1, 3, 7, and 9 must be the second digits of the primes. The prime ending in 1 must be 41 or 61. The prime ending in 3 must be 23, 43, or 53. The prime ending in 7 must be 47 or 67. The prime ending in 9 must be 29 or 59.

(i) Suppose that 41 is one of the primes. Then 47 is not one of the primes, so 67 must be. The other two primes can then be either 23 and 59 or 29 and 53.

(ii) Suppose that 61 is one of the primes. Then 67 is not one of the primes, so 47 must be. As before, the other primes can then be either 23 and 59 or 29 and 53.

Hence, there are four possible sets of primes:

\{41, 67, 23, 59\}, \{41, 67, 29, 53\}, \{61, 47, 23, 59\}, \{61, 47, 29, 53\}.

4. A nine-member committee was formed to select a chief of the KMS. There are three candidates for the chief. Each member of the committee orders the candidates and gives 3 points to the first one, 2 points to the second one and 1 point to the last one. After summing the points of the candidates it turned out that no two candidates have the same number of points, hence the order of the candidates is clear. Someone noticed that if every member of the committee selected only one candidate, the resulting order of candidates would be reversed. How many points did the candidates get?

We have

\[ s(C_i) = 3x_i + 2y_i + z_i, \quad i = 1, 2, 3 \]

with \( x_i + y_i + z_i = 9, \quad i = 1, 2, 3 \) and \( x_1 + x_2 + x_3 = 9, \quad y_1 + y_2 + y_3 = 9, \quad z_1 + z_2 + z_3 = 9 \).
The sum of the points is 54. Since $s(C_1) > s(C_2) > s(C_3)$, we deduce that $s(C_3) < 18$.

Since $x_1 < x_2 < x_3$, it results that $x_3 \geq 4$:

- if $x_3 \geq 6$, then $s(C_3) \geq 18$, a contradiction.
- if $x_3 = 5$, then $y_3 + z_3 = 4$ and $s(C_3) = 15 + 4 + y_3 > 18$, also a contradiction.

It follows that $x_3 = 4$, and from $x_1 + x_2 = 5$ and $x_1 < x_2$ we deduce that $x_1 = 2, x_2 = 3$.

We have

\[
\begin{align*}
    s(C_1) &= 6 + y_1 + z_1 = 13 + y_1 \\
    s(C_2) &= 9 + y_2 + z_2 + y_2 = 15 + y_2 \\
    s(C_3) &= 12 + y_3 + z_3 + y_3 = 17 + y_3
\end{align*}
\]

Since $s(C_3) < 18$ we get $y_3 = 0$ and $y_1 + y_2 = 9$.

Because $s(C_1) > s(C_2) > s(C_3)$ we obtain $y_1 > y_2 + 2$ and $y_2 > y_3 + 2 = 2$.

It is easy to see that the only possibility is $y_2 = 3, y_1 = 6$, hence $s(C_1) = 19, s(C_2) = 18, n(C_3) = 17$.

5.

(a) Find all positive integers $n$ such that both of the numbers $2^n - 1$ and $2^n + 1$ are primes.

(b) Find all primes $p$ such that both of the numbers $4p^2 + 1$ and $6p^2 + 1$ are primes.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the write-up of Wang.

(a) The only such integer is $n = 2$.

Let $f(n) = 2^n + 1$ and $g(n) = 2^n - 1$. Clearly $g(1) = 1$ which is not a prime. Next, $f(2) = 5$ and $g(2) = 3$ are both primes. Now suppose $n \geq 3$. If $n$ is odd, then $n = 2k + 1$ where $k \geq 1$ so $f(n) = 2^{2k+1} + 1 = 2(4^k) + 1 \equiv 2(1) + 1 \equiv 0 \pmod{3}$. Since $f(n) > 3$ and is divisible by 3, it is a composite. If $n$ is even, then $n = 2k$ where $k \geq 1$ so $g(n) = 2^{2k} - 1 = 4^k - 1 \equiv 1 - 1 \equiv 0 \pmod{3}$. Since $g(n) > 3$ and is divisible by 3, it is a composite. This completes the proof.

(b) This is the same as Problem #3 of the Finnish Math Olympiad, 2006, (Final Round) the solution of which has appeared in *Crux* 36(7), 2010; p. 447.

6. Find all positive integers $n$ such that $n + 200$ and $n - 269$ are cubes of integers.
We prove that the unique solution is $n = 1997$. Suppose that a positive integer $n$ has the desired property. Let $a^3 = n + 200, b^3 = n - 269,$ and $d = a - b$. We have $d > 0$ and

$$0 = a^3 - (a - d)^3 - 469 = 3da^2 - 3d^2a + d^3 - 469, \quad (1)$$

a quadratic equation in $a$ with the discriminant

$$9d^4 - 12d(d^3 - 469) = 3d(1876 - d^3) \geq 0.$$ 

Hence $1 \leq d \leq 12$. Moreover, the integer $d$ is a divisor of $a^3 - b^3 = 469 = 7 \cdot 67$; hence $d \in \{1, 7\}$. If $d = 7,$ then the equation (1) becomes $0 = 21(a^2 - 7a - 6), which has no integer solution, a contradiction. Consequently $d = 1.$ Equation (1) becomes $0 = 3(a + 12)(a - 13);$ thus $a = 13, b = 12$, and $n = 13^3 - 200 = 1997.$

[Ed.: Note that instead of (1) we note that $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = 469 = 7 \times 67$, which leads to four choices for $a - b$, from which the solution follows.]

7. There were 33 children at a camp. Every child answered two questions: “How many other children have the same first name as you?” and “How many other children at camp have the same family name as you?”. Among the answers each of the numbers from 0 to 10 occurred at least once. Show that there were at least two children with the same first name and the same family name.

(Mathematical Contests 1997–1998, 1.18 Russia, 29/95)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the solution by Geupel.

If the number $k$ occurs among the answers, then there is a name that occurs exactly $k + 1$ times. Hence, among the first and family names of the 33 children, there are names $N_1, N_2, \ldots, N_{11}$ that occur exactly 1, 2, \ldots, 11 times, respectively. Since $1 + 2 + \cdots + 11 = 2 \cdot 33$, the names $N_1, N_2, \ldots, N_{11}$ cover all the first and family names of the group. By symmetry there is no loss of generality in assuming that $N_{11}$ is a first name occurring 11 times. Then among the 11 children with first name $N_{11}$ at most 10 distinct family names occur. By the Pigeonhole Principle, two children among these 11 children have the same family name. The proof is complete.

9. Find all triples of integers $x, y, z$ satisfying

$$2^x + 3^y = z^2.$$
Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Soohyun Park, Student, University of Toronto Schools, Toronto, ON; Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Wang.

We show that the only solutions are \((x, y, z) = (0, 1, \pm 2), (3, 0, \pm 3)\) and \((4, 2, \pm 5)\).

Note first that \(z \neq 0\) since the left side is always positive. Thus, \(z^2 \geq 1\). If at least one of \(x\) and \(y\) is negative then the left side cannot be an integer, a contradiction. Hence, we may assume that \(x\) and \(y\) are nonnegative. We may also assume for the time being that \(z\) is positive. We first establish a lemma.

**Lemma.** If \(p\) is a prime, then the only solutions in nonnegative integers \(u\) and \(v\) to the equation \(p^u + 1 = v^2\) is \((u, v) = (3, 3)\) if \(p = 2\), and \((u, v) = (1, 2)\) if \(p = 3\). If \(p \neq 2, 3\), then there are no solutions.

**Proof.** Clearly \(u \neq 0\) since \(2\) is not a square. Also, \(v \neq 0\). We have \(p^u = v^2 - 1 = (v - 1)(v + 1)\) so \(v - 1 = p^a\) and \(v + 1 = p^b\) for some integers \(a\) and \(b\) such that \(0 < a < b\) with \(a + b = u\). Then \(p^a - p^b = 2\) or \(p^a(p^{b-a} - 1) = 2\). If \(p \neq 2\), then \(a = 0\) and \(p^{b-a} - 1 = 2\) so \(p^b = 3\) which implies that \(p = 3\) and \(b = 1\). It follows that \(u = a + b = 1\) and \(v = 2\). If \(p = 2\), then we have either (i) \(2^a = 1, 2^{b-a} - 1 = 2\); or (ii) \(2^a = 2, 2^{b-a} - 1 = 1\). In case (i) we get \(a = 0\) and \(2^b = 3\) which is impossible. In case (ii) we have \(a = 1\) and \(2^{b-1} - 1 = 1\) so \(b = 2\). It follows that \(u = a + b = 3\) and \(v = 1 + 2 = 3\) so \((u, v) = (3, 3)\) and the lemma is established.

Back to the given problem, note that if \(x = 0\), then the equation becomes \(3^y + 1 = z^2\). Hence by the lemma above the only solution is \((x, y, z) = (0, 1, 2)\). Similarly, if \(y = 0\), then the equation becomes \(2^x + 1 = z^2\), so by the lemma again, the only solution is \((x, y, z) = (3, 0, 3)\). Now we assume \(x \geq 1\) and \(y \geq 1\). If \(x = 1\) then we have \(2 + 3^y = z^2\) which is impossible since \(\text{mod } 3, 2 + 3^y \equiv 2\) while \(z^2 \equiv 0\) or \(1\). Thus, \(x \geq 2\). Since \(2^x + 3^y\) is odd so is \(z\). Then \(z^2 \equiv 1\) (mod 4). But \(\text{mod } 4, 2^x + 3^y \equiv 0 \text{ or } 3^y \equiv (-1)^y \equiv 1\). Thus, \(y\) is even. Let \(y = 2t\) where \(t \geq 1\). Then we have \(2^x = z^2 - 3^{2t} = (z - 3^t)(z + 3^t)\). Hence \(z - 3^t = 2^c\) and \(z + 3^t = 2^d\) where \(0 \leq c < d\) such that \(c + d = x\). Subtracting the two equations above we get \(2 \cdot 3^t = 2^d - 2^c\) so \(3^t = 2^{d-c-1}\). Since \(2 \cdot 3^t\) is even, \(2^c \neq 1\) so \(c \geq 1\). Therefore, \(2^{c-1} = 1\) and \(2^{d-c-1} = 3\). This yields \(c = 1\) and \(2^{d-1} - 1 = 3^t\). We now show that the only solution to this last equation is \(d = 3\) and \(t = 1\). Since \(t \geq 1\) we have \(2^{d-1} \geq 4\) so \(d \geq 3\). Thus, \(2^{d-1} \equiv 0\) (mod 4) which implies that \(2^{d-1} - 1 \equiv -1\) (mod 4) \(\Rightarrow 3^t \equiv (-1)^t \equiv -1\) (mod 4) so \(t\) is odd. Hence, we have \(2^{d-1} = 3^t + 1 = (3 + 1)(3^{t-1} - 3^{t-2} + \cdots - 3 + 1)\). But \(3^{t-1} - 3^{t-2} + \cdots - 3 + 1\), is the sum of an odd number of odd integers so it is odd. Since it divides \(2\) it must be \(1\). It then follows that \(2^{d-1} = 4\) so \(d = 3\) which implies \(t = 1\). Finally, \(y = 2t = 2, z = 3 + 2^c = 5\) from which \(x = 4\) follows. This yields the third and last solution \((4, 2, 5)\).

12. We are given an acute triangle \(ABC\) with circumcentre \(O\). Let \(T\) be the
circumcentre of $AOC$. Let $M$ be the midpoint of $AC$. The points $D$ and $E$ lie on the lines $AB$ and $CB$ respectively in such a way that the angles $MDB$ and $MEB$ are equal to the angle $ABC$. Prove that the lines $BT$ and $DE$ are perpendicular.

Solved by Titu Zvonaru, Comănești, Romania.

As usual we write $a = BC$, $b = CA$, $c = AB$. Let $R$ be the circumradius of $\triangle ABC$, and let $N$ be the midpoint of $AB$.

We denote $TA = R_1$, and we suppose that $a \geq b \geq c$.

Since $MN \parallel BE$ and $\angle MEB = \angle NBE$, the quadrilateral $BEMN$ is an isosceles trapezoid with $MN = \frac{a}{2}$, $ME = NB = \frac{c}{2}$.

Denoting by $P$ the projection of $N$ onto $BC$, we deduce that $BP = \frac{c}{2} \cos B$, hence $BE = \frac{a}{2} + c \cos B$. Applying the Law of Sines we have

$$EC = \frac{a}{2} - c \cos B$$

$$= R(\sin A - 2 \sin C \cos B) = R[\sin A - \sin(C + B) - \sin(C - B)]$$

$$= R[\sin A - \sin A + \sin(B - C)] = R \sin(B - C).$$

Similarly we obtain

$$BD = \frac{c}{2} + a \cos B; \quad AD = R \sin(A - B).$$

Since $\angle TCE = \angle TCO + \angle OCE = B + 90^\circ - A = 90^\circ - (A - B)$ by the Law of Cosines we have

$$TE^2 = TC^2 + EC^2 - 2TC \cdot EC \cos \angle TCE$$

$$= R_1^2 + EC^2 - 2RR_1 \sin(B - C) \sin(A - B)$$

and similarly

$$TD^2 = R_1^2 + AD^2 - 2RR_1 \sin(A - B) \sin(B - C),$$

hence

$$TE^2 - TD^2 = EC^2 - AD^2.$$  (1)

On the other hand we deduce that

$$EC^2 - AD^2 = (a - BE)^2 - (BD - c)^2$$

$$= a(a - 2BE) - c(c - 2BD) + BE^2 - BD^2$$

$$= a(a - a - 2c \cos B) - c(c - c - 2a \cos B) + BE^2 - BD^2,$$
hence
\[ BE^2 - BD^2 = EC^2 - AD^2. \] (2)

By (1) and (2) it results that the lines \( BT \) and \( DE \) are perpendicular.

13. A line passing through the centroid \( T \) of the triangle \( ABC \) meets the side \( AB \) at \( P \) and the side \( CA \) at \( Q \). Prove that
\[ 4 \cdot PB \cdot QC \leq PA \cdot QA. \]

(R.B. Manfrino: Inequalities, 111/3.29, Spain 1998)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the version of Zvonaru.

Let \( M \) be the midpoint of \( BC \). We denote \( a = BC, x = \frac{AP}{PB}, y = \frac{AQ}{QC} \).

We will prove that
\[ xy = x + y \] (1)

If \( PQ \parallel BC \), then \( x = y = \frac{AT}{TM} = 2 \)

and (1) is true.

Suppose that \( PQ \) meets \( BC \) at the point \( S \) such that \( B \) lies between \( S \) and \( C \). By Menelaus’ Theorem applied in \( \triangle ABC \), we have
\[ \frac{BS}{SC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = -1, \]

that is
\[ \frac{SB}{SC} = \frac{y}{x} \iff \frac{SB}{SB + a} = \frac{y}{x} \]

hence \( SB = \frac{ay}{x - y} \).

Now we apply Menelaus’ Theorem in \( \triangle ABM \) to get
\[ \frac{SB}{SM} \cdot \frac{TM}{TA} \cdot \frac{PA}{PB} = 1 \iff \frac{ay}{x - y} + \frac{a}{2} \cdot \frac{1}{x} = 1 \]
\[ \iff \frac{ay}{x - y} \cdot \frac{2(x - y)}{a(2y + x - y)} \cdot \frac{x}{2} = 1 \]

hence \( xy = x + y \).

Using (1) and AM-GM Inequality we obtain
\[ xy = x + y \geq 2\sqrt{xy} \Rightarrow (xy)^2 \geq 4xy \Rightarrow xy \geq 4, \]

that is
\[ \frac{PA}{PB} \cdot \frac{QA}{QC} \geq 4. \]
14. Integers $x$ and $y$ greater than 1 satisfy the relation $2x^2 - 1 = y^{15}$.

(a) Prove that $x$ is divisible by five.

(b) Are there such integers $x$ and $y$ greater than 1 satisfying $2x^2 - 1 = y^{15}$? Could you find all such numbers?

(Russia 2004/05)

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

(a) Note that if integers $x, y > 1$ satisfy $2x^2 - 1 = y^{15}$ then $2x^2 - 1 = (y^3)^5$. Let $z = y^3 > 1$ and consider the Diophantine equation $2x^2 - 1 = z^5$. Then $z$ is odd and

$$2x^2 = z^5 + 1 = (z + 1)(z^4 - z^3 + z^2 - z + 1).$$

Let $d = \gcd(z + 1, z^4 - z^3 + z^2 - z + 1)$. Then $d$ divides $z + 1$ and

$$d \mid (z^4 - z^3 + z^2 - z + 1) = (z^3 - 2z^2 + 3z - 4)(z + 1) + 5.$$

Hence, $d$ divides 5 and so $d = 1$ or 5. If $d = 5$, then $5 \mid 2x^2$ whence 5 divides $x$ and we are done. If $d = 1$, then $z + 1$ and $z^4 - z^3 + z^2 - z + 1$ are relatively prime and their product is twice a square. Since $z^4 - z^3 + z^2 - z + 1$ is odd, it must be a square. However, for $y > 1$,

$$(2z^2 - z)^2 < 4(z^4 - z^3 + z^2 - z + 1) < (2z^2 - z + 1)^2.$$  

Therefore, $z^4 - z^3 + z^2 - z + 1$ is not a square, a contradiction. Accordingly if $2x^2 - 1 = y^{15}$ and $x, y > 1$, then 5 divides $x$.

(b) Assume integers $x, y > 1$ exist so that $2x^2 - 1 = y^{15}$. Then integers $x, z = y^5 > 1$ exist such that $2x^2 - 1 = z^3$. By a result of Cohn, the only solution of this equation with $x, y > 1$ is $x = 78, z = 23$ (cf. Theorem 2, p. 27, J.H.E. Cohn, The Diophantine equations $x^3 = Ny^2 \pm 1$, Quart. J. Math. Oxford, 42(1991), 27–30). Since 23 is not a fifth power of any integer, it follows that the only solution of $2x^2 - 1 = y^{15}$ is $x = y = 1$.

Next we give readers’ solutions to the Latvian School Mathematical Olympiad, Grade 11, given at [2010: 440].

1. For a positive integer $n$:

(a) can the sums of digits of $n$ and $n + 2007$ be equal?

(b) can the sums of digits of $n$ and $n + 199$ be equal?
For a positive integer $n$, let $d(n)$ denote the sum of the digits of $n$. Let $m$ and $n$ be positive integers. Write

$$m = \sum_{k=0}^{r} a_k \cdot 10^k \quad \text{and} \quad n = \sum_{k=0}^{r} b_k \cdot 10^k,$$

where the $a_k$ and $b_k$ are nonnegative integers. If $r = 0$ and $a_0 + b_0 < 10$, then $d(m+n) = d((a_0+b_0)) = a_0 + b_0 = d(m) + d(n)$. If $r = 0$ and $a_0 + b_0 \geq 10$, that is if a carry is required, then $d(m+n) = d(1 \cdot 10 + (a_0 + b_0 - 10)) = a_0 + b_0 - 9 = d(m) + d(n) - 9$. The same occurs each time a carry is required, so $d(m+n) = d(m) + d(n)$ minus $9$ times the number of carries required. Hence,

$$9 \mid [d(m+n) - (d(m) + d(n))].$$

(a) If $n = 3$, then $d(n) = 3$ and $d(n + 2007) = d(2010) = 3 = d(n)$, so the answer to the first question is ‘yes’.

(b) If $d(n) = d(n + 199)$, then $9 \mid [d(n + 199) - d(n) - d(199)]$ implies that $9$ divides $d(199)$. But $d(199) = 19$, a contradiction. Hence, the answer to the second question is ‘no’.

2. Do there exist three quadratic trinomials such that each of them has at least one root, but the sum of any two quadratic trinomials doesn’t have any roots?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

The answer is YES.

Let $a, b, c$ be three real numbers such that $a \neq b \neq c \neq a$.

$$(x - a)^2 = 0 \quad \text{has the root} \quad a$$

$$(x - b)^2 = 0 \quad \text{has the root} \quad b$$

$$(x - c)^2 = 0 \quad \text{has the root} \quad c$$

$$(x - a)^2 + (x - b)^2 = 0 \quad \text{has no real roots}$$

$$(x - a)^2 + (x - c)^2 = 0 \quad \text{has no real roots}$$

$$(x - b)^2 + (x - c)^2 = 0 \quad \text{has no real roots}.$$

4. In triangle $ABC$ a point $K$ lies on median $AM$ and $\angle BAC + \angle BKC = 180^\circ$. Prove that $AB \cdot KC = AC \cdot KB$.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Apostolopoulos.
We assume $\angle BAC < 90^\circ$. On $AM$ take $MD = KM$. Then $KBDC$ is a parallelogram, since the diagonals bisect each other. So $\angle BKC = \angle BDC$ and $\angle BAC + \angle BDC = 180^\circ$ hence $ABDC$ is cyclic and $\angle ABD + \angle ACD = 180^\circ$. Thus

$$\frac{[\triangle ABD]}{[\triangle ACD]} = \frac{AB \cdot BD}{AC \cdot CD},$$

where $[\triangle ABD]$, as usual, represents the area of $\triangle ABC$. Also $[\triangle ABD] = [\triangle ACD]$ so $AB \cdot BD = AC \cdot CD$, but $BD = KC$ and $CD = KB$ namely $AB \cdot KC = AC \cdot KB$.

5. For a sequence of real numbers $a_1, a_2, a_3, \ldots$ we have $a_{11} = 4$, $a_{22} = 2$ and $a_{33} = 1$. In addition, the relation

$$\frac{a_{n+3} - a_{n+2}}{a_{n} - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_{n} + a_{n+1}}$$

holds for each $n$. Prove that:

(a) $a_i \neq 0$ for each $i$,

(b) the sequence is periodic,

(c) $a_1^k + a_2^k + \cdots + a_{100}^k$ is a square of an integer for any arbitrary positive integer $k$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the version of Zvonaru.

(a) The given relation is equivalent to

$$a_{n+3}a_n + a_{n+3}a_{n+1} - a_n a_{n+2} - a_{n+1}a_{n+2} = a_{n+3}a_n - a_{n+1}a_{n+3} + a_n a_{n+2} - a_{n+1}a_{n+2} \iff a_n a_{n+2} = a_{n+1}a_{n+3}. \quad (1)$$

By the same relation we deduce that $a_n \neq a_{n+1}$ for each $n$. Using (1) we have

$$a_ia_{i+2} = a_{i+1}a_{i+3} = a_{i+2}a_{i+4}.$$ 

Suppose that $a_i = 0$; it results that $a_{i+1} \neq 0$ and from $a_{i+1}a_{i+3} = 0$ we get $a_{i+3} = 0$. Since $a_{i+2} \neq a_{i+3} \neq a_{i+4}$, we obtain a contradiction $a_{i+2}a_{i+4} = 0$.

(b) By (1) we have

$$a_na_{n+2} = a_{n+1}a_{n+3} : \quad a_{n+1}a_{n+3} = a_{n+2}a_{n+4},$$
hence $a_n a_{n+2} = a_{n+2} a_{n+4}$.

Since $a_{n+2} \neq 0$, it results that $a_n = a_{n+4}$, that is the sequence is periodic.

(c) We have

\[
a_1 = a_5 = a_9 = \ldots = a_{33} = 1
\]
\[
a_2 = a_6 = a_{10} = \ldots = a_{22} = 2
\]
\[
a_3 = a_7 = a_{11} = 4
\]

and from $a_1 a_3 = a_2 a_4$ we obtain $a_4 = 2$.

It follows that

\[
a_1^k + \cdots + a_{100}^k = 25(a_1^k + a_2^k + a_3^k + a_4^k) = 25(1 + 2^k + 4^k + 2^k)
\]
\[
= 25((2^k)^2 + 2 \cdot 2^k + 1),
\]

hence $a_1^k + \cdots + a_{100}^k = [5(2^k + 1)]^2$.

Next we look at solutions to the Latvian Mathematical Olympiad Grade 12, given at [2010: 440–441].

1. What can be the values of nonnegative real numbers $a$ and $b$, if it is known that equations $x^2 + a^2 x + b^3 = 0$ and $x^2 + b^2 x + a^3 = 0$ have a common real root?

**Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We use the solution of Perfetti.**

**Answer:** $a = b = 0$, $a = b \geq 4$.

**Proof.** The solutions of the two equations are

\[
x_{1,2} = \frac{-a^2 \pm \sqrt{a^4 - 4b^3}}{2}, \quad x_{3,4} = \frac{-b^2 \pm \sqrt{b^4 - 4a^3}}{2}
\]

They are real if and only if

\[
\{a^4 \geq 4b^3 \land b^4 \geq 4a^3\} \Rightarrow (ab)^3 (ab) \geq 16(ab)^3 \Rightarrow \{ab = 0 \lor ab \geq 16\}
\]

Moreover we have

\[
\{a^4 \geq 4b^3 \land b^4 \geq 4a^3\} \Rightarrow \left\{ a \geq (4b^3)^{1/4} \land a \leq \left( \frac{b^4}{4} \right)^{1/3} \right\}
\]

The graphs of the functions $(4b^3)^{1/4}$ and $(b^4/4)^{1/3}$ show that $a = b = 0$ or $a, b \geq 4$.

If $a = b = 0$ the roots are all equal to zero. If $a = b \geq 4$ the two parabola coincide. If $a = b = 4$ they have one root with multiplicity two. If $a = b > 4$ they have two distinct roots.
Thus let us suppose $a \neq b$ and because of the symmetry with respect to the change $(a, b) \rightarrow (b, a)$ we may suppose $b > a$.

Moreover we note that

$$b^4 - 4a^3 > a^4 - 4b^3$$

so $x_3$ is smaller than any other root and therefore we can have only $x_4 = x_1$ or $x_4 = x_2$.

**First case** $x_4 = x_1$.

$$-b^2 + \sqrt{b^4 - 4a^3} = -a^2 - \sqrt{a^4 - 4b^3}, \quad b^2 - a^2 = \sqrt{b^4 - 4a^3} + \sqrt{a^4 - 4b^3}$$

or squaring

$$4a^3 + 4b^3 - 2a^2b^2 = \sqrt{b^4 - 4a^3} \sqrt{a^4 - 4b^3}$$

The left-hand side may be both negative and positive. For the values which make it positive we can square again and obtain

$$4(4a^6 - 8a^3b^3 - 4a^5b^2 + 4b^6 - 4b^5a^2 + 4b^7 + 4a^7) = 0 \quad (1)$$

By AGM we have

$$(a^7 + a^7 + a^7 + a^7 + b^7 + b^7)/7 \geq a^5b^2, \quad (a^6 + b^6) \geq a^3b^3$$

with equality in both cases if and only if $a = b$. If follows that (1) is never zero unless $a = b$.

**Second case** $x_4 = x_2$.

$$-b^2 + \sqrt{b^4 - 4a^3} = -a^2 + \sqrt{a^4 - 4b^3}, \quad b^2 - a^2 = \sqrt{b^4 - 4a^3} - \sqrt{a^4 - 4b^3}$$

or squaring

$$4a^3 + 4b^3 - 2a^2b^2 = -\sqrt{b^4 - 4a^3} \sqrt{a^4 - 4b^3}$$

For the values which make the left hand side negative we can square again and obtain as above

$$4(4a^6 - 8a^3b^3 - 4a^5b^2 + 4b^6 - 4b^5a^2 + 4b^7 + 4a^7) = 0$$

The consequences are the same.

The conclusions are that if $a = b = 0$ the four solutions all coincide. If $a = b \geq 4$ the solutions coincide pairwise. For any other values of $(a, b)$ the solutions are all distinct.

**3.** Solve the system of equations

$$\begin{cases}
\sin^2 x + \cos^2 y = y^2 \\
\sin^2 y + \cos^2 x = x^2
\end{cases}$$
Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.

From the two equations, we find that
\[ \sin^2 x - \sin^2 y = y^2 - 1 = 1 - x^2 = \frac{1}{2}(y^2 - x^2), \]
from which it follows that
\[ 2 \sin^2 x + x^2 = 2 \sin^2 y + y^2. \] (2)

The function \( 2 \sin^2 t + t^2 \) is an even function of \( t \) that is strictly increasing for \( 0 \leq t \leq \frac{\pi}{2} \). Since, from the original equations, we have that \( |x| \) and \( |y| \) do not exceed \( \sqrt{2} < \frac{\pi}{2} \), it follows that from (1) \( |x| = |y| \).

On the other hand, when this condition holds, we must have \( x^2 = y^2 = 1 \), since \( x^2 + y^2 = 2 \). So the equations are satisfied when \( (x, y) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \).

4. Two circles \( w_1 \) and \( w_2 \) intersect in points \( A \) and \( B \). Line \( t_1 \) is drawn through point \( B \) with other intersection point with \( w_1 \) being \( C \) and other intersection point with \( w_2 \) being \( E \). Line \( t_2 \) is drawn through point \( B \) with other intersection point with \( w_1 \) being \( D \) and other intersection point with \( w_2 \) being \( F \). Point \( B \) lies between \( C \) and \( E \) and between \( D \) and \( F \). Midpoints of segments \( CE \) and \( DF \) are denoted by \( M \) and \( N \). Prove that triangles \( ACD, AEF \) and \( AMN \) are similar.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Zelator’s write-up.

Looking at the figure and the cyclic quadrilaterals \( ACDB \) and \( ABEF \), the following statements are clear:

\[ \angle CAD = \angle CBD = \angle EBF = \angle EAF = \theta, \] (1)
\[ \angle ABF = \angle AEF = \angle ACD = \varphi, \] (2)
\[ \angle ACB = \angle ADB = \omega, \] (3)
\[ \angle BFA = \angle BEA = \gamma. \]  

From (1) and (2) we have \( \angle CAD = \angle EAF = \theta \) and \( \angle ACD = \angle AEF = \varphi \), which proves that the triangles \( ACD \) and \( AEF \) are similar (two-angle criterion).

From (3) and (4), it follows that triangles \( ACE \) and \( ADF \) are similar (two-angle criterion) (also note that \( \angle CAE = \theta + \angle DAE \) and \( \angle DAF = \angle DAE + \theta \)). The sides \( CE \) and \( DF \) lie across the congruent angles \( \angle CAE \) and \( \angle DAF \). Thus, since \( N \) and \( M \) are the midpoints of \( CE \) and \( DF \), and the triangles \( ACE \) and \( ADF \) are similar, it follows that

\[
\begin{align*}
\angle DAM &= \angle CAN, \\
\angle DAN + \angle NAM &= \angle CAD + \angle DAB, \\
\angle NAM &= \angle CAD = \theta \quad \text{by (1)}. 
\end{align*}
\]

And since \( \angle EBF = \theta \), it follows by (5) that

\[ \angle EBF = \theta = \angle CAD \]

It follows from (6) that \( ANBM \) is a cyclic quadrilateral. Thus \( \angle ANM = \angle ABM = \varphi \) and by (2) we see that

\[ \angle ANM = \angle ACD = \varphi. \]

Clearly (6) and (7) imply that the triangles \( ACD \) and \( ANM \) are similar (two-angle criterion). And since we have already shown that \( ACD \) and \( AEF \) are similar, we conclude that the triangles \( ACD, AEF, \) and \( ANM \) are similar.

5. The set of all positive integers has been split in several parts in such a way that each integer belongs exactly to one part and each of the parts contains infinitely many integers. Can this be done so that one part contains a multiple of any positive integer? Give the answer if

(a) there are a finite number of parts,

(b) there are an infinite number of parts.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Geupel.

We prove that the answer is “Yes” in both cases. Let \( A_1 \) and \( A_2 \) denote the set of even numbers and the set of odd numbers, respectively. Then, \( A_1 \) contains the double of each positive integer, which completes the proof for part (a). Let \( p_1 = 2, p_2 = 3, p_3 = 5, \ldots \) be the sequence of prime numbers. For \( k = 1, 2, 3, \ldots \), let \( B_k \) denote the set of positive integers which are divisible by \( p_k \) but not divisible by any of the numbers \( p_1, p_2, \ldots, p_{k-1} \). Then, \( \{B_1, B_2, B_3, \ldots\} \) is an infinite partition of the set of positive integers and \( B_1 \) is the set of even numbers, which completes the proof for part (b).

While this was obvious, the following question is more interesting: Is it true that, for each partition of the set of positive integers, one part contains a multiple of each positive integer?
We prove that the answer is “Yes” for partitions into a finite number of parts. Suppose the contrary. Then there exists a partition \( \{ A_1, A_2, \ldots, A_n \} \) such that for each \( A_k \) there is a number \( a_k \) with the property that no multiple of \( a_k \) is contained in \( A_k \). The number \( a = a_1 a_2 \cdots a_n \) is in some \( A_k \), say in \( A_n \). By \( a = (a_1 a_2 \cdots a_{n-1}) a_n \), it follows that a multiple of \( a_n \) is contained in \( A_n \). This is a contradiction, which completes the proof.

However, the answer is “No” if infinitely many parts can occur. A counterexample is the partition \( \{ B_1, B_2, B_3, \ldots \} \) where \( B_k \) is the set of positive integers which have exactly \( k \) distinct prime divisors.

Next we turn to solutions from our readers to problems of the Finnish National High School Mathematics Competition, Final Round, given at [2010: 441–442].

1. Show that when a prime number is divided by 30, the remainder is either a prime number or 1. Is a similar claim true when the divisor is 60 or 90?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give the solution by Manes.

Note that if \( n \) is a composite number less than 30, then \( \gcd(30, n) \neq 1 \). Let \( p \) be a prime number and assume that the remainder \( r \) when \( p \) is divided by 30 is neither 1 nor a prime. Then \( p = 30m + r \) for some integers \( m \) and \( r \) with \( 0 < r < 30 \). Therefore, \( \gcd(30, r) \neq 1 \) implies there is a prime \( q \) such that \( q \) divides both of the integers 30 and \( r \). Hence, \( q \) divides \( p \) and this implies \( q = p \), a contradiction that proves the result.

However, no such claim can be made when the divisor is 60 or 90 since the prime 229 divided by 60 or 90 leaves a remainder of 49.

2. Determine the number of real roots of the equation

\[
x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} = 0.
\]

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give the write-up of Curtis.

Let \( P(x) \) be the given polynomial. Then

\[
P(x) = \frac{1}{2} [2x^7(x - 1) + 4x^5(x - 1) + 6x^3(x - 1) + 8x(x - 1) + 5]
\]

\[
= \frac{1}{2} [2x(x - 1)(x^6 + 2x^4 + 3x^2 + 4) + 5] > 0 \text{ for } x > 1.
\]
Also,

\[ P(-x) = \frac{1}{2}(2x^8 + 2x^7 + 4x^6 + 4x^5 + 6x^4 + 6x^3 + 8x^2 + 8x + 5). \]

Thus, \( P(-x) > 0 \) for \( x > 0 \), so that \( P(x) > 0 \) for \( x < 0 \). Hence, any real zero of \( P(x) \) lies in the interval \([0, 1]\). But, neither \( P(0) \) nor \( P(1) \) equals 0, and for \( 0 < x < 1, \frac{1}{2} < 2x(x-1) < 0 \) and \( 0 < x^6 + 2x^4 + 3x^2 + 4 < 10 \). This implies that for \( 0 < x < 1 \),

\[ 0 < P(x) < 5. \]

Thus, \( P(x) \) has no real zeros.

3. There are five points in the plane, no three of which are collinear. Show that some four of these points are the vertices of a convex quadrilateral.

_Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Geupel._

The convex hull of the given points is a convex \( n \)-gon where \( n \in \{3, 4, 5\} \).

If \( n = 4 \) then we are done.

If \( n = 5 \) then any four of the five points are the vertices of a convex quadrilateral.

Finally suppose \( n = 3 \). The convex hull is a triangle, say \( ABC \), and the other two points, say \( D \) and \( E \), are in the interior of that triangle. Because no three of the points \( A, B, C, D, \) and \( E \) are collinear, the line \( DE \) does not pass through a vertex of the triangle. Hence, the line \( DE \) meets two sides of the triangle, say \( AB \) and \( AC \) at inner points \( S \) and \( T \), respectively. Assume \( DS < ES \). Then

\[ \angle CBD < \angle CBA < 180^\circ, \quad \angle BCE < \angle BCA < 180^\circ, \]

\[ \angle BDE < \angle SDE = 180^\circ, \quad \angle CED < \angle TED = 180^\circ. \]

Consequently, \( BCED \) is a convex quadrilateral. This completes the proof.

5. Show that there exists a polynomial \( P(x) \) with integer coefficients such that the equation \( P(x) = 0 \) has no integer solutions but for each positive integer \( n \) there is an \( x \in \mathbb{Z} \) such that \( n \mid P(x) \).

_Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA._

Let \( P(x) = 6x^2 + 5x + 1 = (3x + 1)(2x + 1) \). Then \( P(x) = 0 \) has no integer roots. Let \( n \) be an arbitrary positive integer and write \( n = 2^r \cdot s \), where \( r \) is an integer \( \geq 0 \) and \( s \) is odd. Since \( \gcd(2^r, 3) = 1 = \gcd(s, 2) \), it follows that there exist positive integers \( u \) and \( v \) such that

\[ 3u \equiv -1 \pmod{2^r} \quad \text{and} \quad 2v \equiv -1 \pmod{s}. \]
Moreover, \( \gcd(2^r, s) = 1 \) so that the Chinese Remainder Theorem implies there is an integer \( m \) such that

\[
3m \equiv -1 \pmod{2^r \cdot s} \quad \text{and} \quad 2m \equiv -1 \pmod{2^r \cdot s}.
\]

Therefore, \( P(m) = (3m + 1)(2m + 1) = 0 \pmod{n} \), whence \( n \mid P(m) \).

To complete this number of the Corner we turn to the solutions to problems of the IX Olimpiada Matemático de Centramérica y el Cariba 2007, given at [2010: 442–443].

1. The OMCC is an annual mathematical competition. The ninth olympiad takes place in the year 2007. Which positive integers \( n \) divide the year in which the \( n^{th} \) olympiad takes place?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Zelator.

Since the OMCC is an annual event and the 9th olympiad takes place in 2007, it follows that the first olympiad took place in the year \( 2007 - 8 = 1999 \). Thus the \( n^{th} \) olympiad takes place in the year \( 1999 + (n - 1) = 1998 + n \). So if \( n \mid (1998 + n) \) we have \( 1998 + n = kn \), for some positive integer \( k \), so that \( 1998 = (k - 1) \cdot n \), whence \( n \mid 1998 \). So, the positive integers \( n \) which divide the year in which the \( n^{th} \) olympiad takes place are precisely the positive divisors of 1998. Now, 1998 = 2(999) = 2 \cdot 9 \cdot (111) = 2 \cdot 9 \cdot 3 \cdot 37 = 2 \cdot 3^3 \cdot 37. The number of divisors of 1998 is \( \tau(1998) = (1 + 1) \cdot (3 + 1) \cdot (1 + 1) = 16 \). The sixteen divisors of 1998 are the positive integers:

\[
n = 1, 2, 2 \cdot 3, 2 \cdot 3^2, 2 \cdot 3^3, 2 \cdot 37, 3, 3^2, 3^3, 37, 3 \cdot 37,
\]

\[
3^2 \cdot 37, 3^3 \cdot 37, 2 \cdot 3^3 \cdot 37, 2 \cdot 3^2 \cdot 37, 2 \cdot 3 \cdot 37.
\]

We list these sixteen positive integers in increasing order.

\[
n = 1, 2, 3, 6, 9, 18, 27, 37, 54, 74, 111, 222, 333, 666, 999, 1998.
\]

2. Let \( ABC \) be a triangle, \( D \) and \( E \) points on the sides \( AC \) and \( AB \), respectively, such that the lines \( BD, CE \) and the angle bisector of angle \( A \) concur in an interior point \( P \) of the triangle. Prove that there is a circle tangent to the four sides of the quadrilateral \( ADPE \) if and only if \( AB = AC \).

Solved by Titu Zvonaru, Comănești, Romania.
Let $A'$ be the point of intersection of $BC$ with the angle bisector of angle $A$.  
(i) Suppose that $AB = AC$; then $A'$ is the midpoint of $BC$. By Ceva’s Theorem we obtain
\[
\frac{AE}{EB} \cdot \frac{BA'}{A'C} \cdot \frac{CD}{DA} = 1
\]
hence
\[
\frac{AE}{AB - AE} = \frac{DA}{AC - DA}
\]
\[\Leftrightarrow AE \cdot AC - AE \cdot DA = AB \cdot DA - AE \cdot DA,
\]
that is $AE = AD$.

It results that $\triangle AEP$ and $\triangle ADP$ are congruent (side-angle-side) hence $AE + DP = AD + EP$, which means that there is a circle tangent to the four sides of quadrilateral $ADPE$.

(ii) Suppose that there is a circle tangent to the four sides of quadrilateral $ADPE$. Then the centre of this circle lies on the bisector of angle $\angle EAD$, that is on $AP$. We deduce that $AP$ is the bisector of $\angle EAP$, and the triangles $AEP$ and $ADP$ are congruent (angle-side-angle), hence $AE = AD$. Because $A'$ lies on the bisector $AP$ of $\angle BAC$, $\frac{AB}{AC} = \frac{BA'}{A'C}$. Inserting the last two equalities into Ceva’s theorem gives us
\[
\frac{EA}{EB} \cdot \frac{A'B}{AC} \cdot \frac{DC}{DA} = 1
\]
\[\Rightarrow \frac{AB}{AC} = \frac{EA}{EB} \Rightarrow \frac{AB}{AC} = \frac{AB - AE}{AC - AE}
\]
\[\Rightarrow AB \cdot AC - AB \cdot AE = AB \cdot AC - AC \cdot AE,
\]
hence $AB = AC$.

3. Let $S$ be a finite set of integers. For any two integers $p, q \in S$, with $p \neq q$, there are integers $a, b, c$ in $S$, not necessarily distinct and with $a \neq 0$, such that the polynomial $F(x) = ax^2 + bx + c$ satisfies $F(p) = F(q) = 0$. Determine the maximum number of elements set $S$ can have.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geupel’s write-up.

We prove that $\max|S| = 3$.
A valid set with three elements is $S = \{-1, \ 0, \ 1\}$.
We show by contradiction that $|S| < 4$.
Suppose $S$ is such that $|S| \geq 4$. Then, there is a member $p \in S$ such that $|p| > 1$. Let $T = \{-1, \ 0, \ 1, \ p\}$.
If $p > 1$ then $-a(p+1) \notin T$ for each $a \in T \setminus \{0\}$. Hence, there are no $a \neq 0, \ b, \ c \in T$ such that
\[
a(x - 1)(x - p) = ax^2 - a(p+1)x + ap.
\]
If \( p < -1 \) then \(-a(p-1) \notin T\) for each \( a \in T \setminus \{0\}\). Hence, there are no \( a \neq 0, b, c \in T \) such that
\[
ax^2 + bx + c = a(x+1)(x-p) = ax^2 - a(p-1)x - ap.
\]

Therefore \( S \neq T \). Thus, there are distinct members \( p, q \in S \) such that \(|p| > 1\) and \(|q| > 1\). Let \( p \) and \( q \) be the elements of \( S \) with the greatest absolute values. Then, \( apq \notin S \) for each \( a \in S \setminus \{0\} \). Hence, there are no \( a \neq 0, b, c \in S \) such that
\[
ax^2 + bx + c = a(x-p)(x-q) = ax^2 - a(p+q)x + apq,
\]
a contradiction.

This completes the proof that \(|S| < 4\).

4. The inhabitants of a certain island speak a language in which every word can be written with the following letters: \( a, b, c, d, e, f, g \). A word is said to produce another one if the second word can be formed from the first one applying any of the following rules as many times as needed:

(i) Replace a letter by two letters according to one of the substitutions:
\[
\begin{align*}
a &\rightarrow bc, \\
b &\rightarrow cd, \\
c &\rightarrow de, \\
d &\rightarrow ef, \\
e &\rightarrow fg, \\
f &\rightarrow ga, \\
g &\rightarrow ab.
\end{align*}
\]
(ii) If only one letter is between two letters that are the same, these two letters can be eliminated. Example: \( dfd \rightarrow f \)

As another example, \( cafed \) produces \( bfed \), since \( cafed \rightarrow cbced \rightarrow bced \rightarrow bfed \).

Prove that every word on this island produces any other word.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Curtis.

We note first that each letter produces each letter as seen by the following progression:
\[
\begin{align*}
a &\rightarrow bc \rightarrow cdc \rightarrow d \rightarrow ef \rightarrow fgf \rightarrow g \rightarrow ab \rightarrow bcd \rightarrow c \rightarrow de \rightarrow efe \\
&\rightarrow f \rightarrow ga \rightarrow aba \rightarrow b \rightarrow cd \rightarrow ded \rightarrow e \rightarrow fg \rightarrow gag \rightarrow a.
\end{align*}
\]

Suppose word \( M \) has \( m \) letters and word \( N \) has \( n \) letters and we wish to produce \( N \) from \( M \). By the above, we can assume that each letter of \( M \) and \( N \) is ‘a’. If \( m = n \), we are done. From
\[
a \rightarrow bc \rightarrow aa
\]
and
\[
aa \rightarrow abc \rightarrow abc \rightarrow b \rightarrow a,
\]
we see that if \( m < n \), we can expand using \( a \rightarrow aa \) repeatedly; if \( m > n \), we can condense using \( aa \rightarrow a \) repeatedly. Hence we can produce \( N \) from \( M \).
5. Given two non-negative integers \( m \) and \( n \), with \( m > n \), we say that \( m \) ends in \( n \) if one can erase some consecutive digits from left of \( m \) to obtain \( n \). For example, 329 ends only in 9 and in 29. Determine how many three-digit numbers end with the product of their digits.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the write-up of Zvonaru.

Let \( \overline{abc} \) be the three-digit numbers, with \( a \neq 0 \).

(i) If \( c = abc \), then we have \( c = 0 \) or \( ab = 1 \). It results that the numbers \( \overline{ab0} \) (with \( a = 1, 2, \ldots, 9 \) and \( b = 0, 1, 2, \ldots, 9 \)) and \( \overline{11c} \) (with \( c = 1, 2, \ldots, 9 \)) end in the product of their digits.

In this case we have 90 + 9 = 99 numbers with the desired property.

(ii) If \( \overline{bc} = abc \), then we have successively

\[
\overline{bc} = abc \iff 10b + c = abc \iff b = \frac{c}{ac - 10}
\]

\[
\iff ab = \frac{ac}{ac - 10} \iff ab = 1 + \frac{10}{ac - 10}.
\]

Since \( ab \) is an integer, we obtain

\[
ac = 11, \quad ab = 11 \quad (1)
\]
\[
ac = 12, \quad ab = 6 \quad (2)
\]
\[
ac = 15, \quad ab = 3 \quad (3)
\]
\[
ac = 20, \quad ab = 2 \quad (4)
\]

The systems (1) and (4) have no solution.

From (2) we deduce the following possibilities:

\[(a = 2, b = 3, c = 6)(a = 3, b = 2, c = 4)(a = 6, b = 1, c = 2)\]

and from (3) we deduce the solution \((a = 3, b = 1, c = 5)\).

It results that the numbers 236, 324, 612 and 315 end in the product of their digits.

As a conclusion, there are 99 + 4 = 103 three-digit numbers which end in the product of their digits.

6. Let \( A \) and \( B \) be points on the circle \( \Gamma \) such that the lines \( PA \) and \( PB \) are tangent to \( \Gamma \) for an exterior point \( P \). Let \( M \) be the midpoint of \( AB \). The perpendicular bisector of \( AM \) intersects \( \Gamma \) at \( C \) which is interior to \( \triangle ABP \), the line \( AC \) intersects the line \( PM \) at \( G \), and the line \( PM \) intersects \( \Gamma \) at \( D \), which is exterior to the triangle \( \triangle ABP \). If \( BD \) is parallel to \( AC \), prove that \( G \) is the point in which the medians of \( \triangle ABP \) concur.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the write-up of Zvonaru.
Let $O$ be the centre of $\Gamma$, $E$ be the intersection of $PD$ with $\Gamma$ and $N$ be the midpoint of $AM$.

We denote $PE = a$, $PD = b$; it results that $OA = \frac{b-a}{2}$ and $PO = \frac{a+b}{2}$. By the power of the point $P$ with respect to the circle $\Gamma$ we obtain $PA^2 = ab$, and by similitude in $\triangle APO$ we have $AP^2 = PM \cdot PO$, hence $PM = \frac{2ab}{a+b}$ and $MD = b - \frac{2ab}{a+b} = \frac{(b-a)}{a+b}$.

By the Pythagorean theorem we deduce
\[
AM^2 = AP^2 - PM^2 = ab - \frac{4a^2b^2}{(a+b)^2} = \frac{ab(b-a)}{(a+b)^2},
\]
\[
AD^2 = AM^2 + MD^2 = \frac{ab(b-a)^2}{(a+b)^2} + \frac{b^2(b-a)^2}{(a+b)^2} = \frac{(ab+b^2)(b-a)^2}{(a+b)^2}.
\]

Since $AC \parallel BD$, the cyclic quadrilateral $ACBD$ is an isosceles trapezoid; because $AD = BD = BC = 2AC$, we have
\[
AB^2 = \left(\frac{BD + AC}{2}\right)^2 + AD^2 - \left(\frac{BD - AC}{2}\right)^2
\]
\[
\Leftrightarrow 4AM^2 = AD^2 + AD \cdot AC
\]
\[
\Leftrightarrow 4AM^2 = \frac{3AD^2}{2}
\]
\[
\Leftrightarrow \frac{4ab(b-a)^2}{(a+b)^2} = \frac{3(b-a)^2}{(a+b)^2} \cdot b(a+b)
\]
\[
\Leftrightarrow 8a = 3a + 3b,
\]

hence
\[
5a = 3b \quad (1)
\]

Since $AG \parallel BD$ and $AM = MB$ we deduce that $GM = MD$. Using (1) we have
\[
GM = \frac{b(b-a)}{a+b} = \frac{5a}{3} \left(\frac{5a}{7} - \frac{b}{3}\right) = \frac{5a}{12}, \quad (2)
\]
\[
PM = \frac{2ab}{a+b} = \frac{5a}{4}. \quad (3)
\]

By (2) and (3) it follows that $PM = 3GM$; since $PM$ is a median in $\triangle ABP$, we deduce that $G$ is the centroid of $\triangle ABP$.

THE END