That old root flipping trick of
Andrey Andreyevich Markov

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A straightforward fact

All the mathematics in this article is based on the following straightforward fact.

Fact. Let \( p \) and \( q \) be positive integers. If the equation \( x^2 - px + q = 0 \)
has a positive integer root, then also its second root is a positive integer.

Why do we call this fact straightforward? Well, it easily follows from Vieta’s
formulas \( x_1 + x_2 = p \) and \( x_1x_2 = q \) for the roots \( x_1 \) and \( x_2 \) of a quadratic
equation. As \( p \) and \( x_1 \) are integers, also \( x_2 = p - x_1 \) is integer. And as \( q \) and \( x_1 \) both are positive, also \( x_2 = q/x_1 \) is positive. So the fact can be proved in
two lines and indeed is simple. Then what’s the reason for spending several pages
on it? Well, the fact turns out to be surprisingly useful in the analysis of certain
Diophantine equations. This article will illustrate this usefulness by a number of
examples and also discuss some of its background.

Short history lesson

The Russian mathematician Andrey Andreyevich Markov (1856–1922) received his
Master’s degree in 1880 from the university of St Petersburg. His supervisor was
Pafnuty Lvovich Chebyshev (1821–1894), and the title of his thesis was “About
binary quadratic forms with positive determinant”. Among many other contribu-
tions, the thesis contained as a side result the complete analysis of the following
Diophantine equation:

\[
a^2 + b^2 + c^2 = 3abc. \tag{2}
\]

This equation is nowadays called the Markov equation. Let us take a closer look
at it. For a solution triple \((a, b, c)\) of (2) over the positive integers, we consider
the following quadratic equation in \( x \).

\[
x^2 - (3bc)x + (b^2 + c^2) = 0. \tag{3}
\]

The quadratic equation (3) has two roots: one of them of course is \( a \), and the other
one is — at this place we apply our straightforward fact — the positive integer
\( a' = 3bc - a \). Unless the two roots coincide, we have found a new solution
triple \((a', b, c)\) for equation (2). Symmetric observations for \( b \) and \( c \) yield the
following: whenever \((a, b, c)\) is a positive integer solution of (2), then also the
triples \((a', b, c)\), \((a, b', c)\), and \((a, b, c')\) are positive integer solutions where

\[
a' = 3bc - a, \quad b' = 3ac - b, \quad c' = 3ab - c.
\]
The weight of triple \((a, b, c)\) is defined as the maximum of \(a, b, c\). How does the weight of the new triples relate to the weight of the old triple? Let us assume for the moment that \(a, b, c\) are pairwise distinct and satisfy \(a > b > c\). We then derive
\[
(a - b)(b - a') = a^2 - b^2 + 3b^2c - 3abc \\
= a^2 - b^2 + 3b^2c - (a^2 + b^2 + c^2) \\
= 3b^2c - 2b^2 - c^2 = 2b^2(c - 1) + c(b^2 - c) > 0.
\]
This implies \(a' < b < a\), and analogous arguments lead to \(b' < a\) and \(c' > a\).
Hence by flipping the largest coordinate from \(a\) into \(a'\) we decrease the weight of the triple, and by flipping one of the smaller coordinates we increase the weight of the triple. Now let us repeatedly flip the largest coordinate, so that the weight of the resulting triples keeps decreasing. Since the weight cannot decrease below zero, we must eventually get stuck with a triple whose coordinates are not pairwise distinct anymore. By symmetry we may assume that the root flipping process terminates with \(b = c\). Then (2) becomes \(a^2 = (3a - 2)b^2\), which implies that \(b\) divides \(a\). Thus \(a = kb\), and substituting this into the equation and rewriting yields \(k(3b - k) = 2\). Hence \(k\) must be a divisor of \(2\): if \(k = 1\) then \(b = c = 1\) and \(a = 1\); if \(k = 2\) then \(b = c = 1\) and \(a = 2\).

Let us summarize our findings. For every positive integer solution \((a, b, c)\) of the Markov equation (2) with pairwise distinct coordinates, root flipping produces three neighbor solutions. One of these neighbors has smaller weight, and is called the predecessor of \((a, b, c)\). The other two neighbors have larger weight, and are called the successors of \((a, b, c)\). If we start in an arbitrary solution and follow the chain of predecessors, we eventually arrive in one of the special solutions \((1, 1, 1)\) or \((2, 1, 1)\). Vice versa, by starting in \((1, 1, 1)\) and by repeatedly moving to successors, we can reach every possible solution of equation (2). Figure 1 lists the first few solutions of the equation.

Three shiny examples

Next, we want to discuss three concrete problems from mathematical competitions where that old root flipping trick of Andrey Andreyevich Markov serves as a crucial
tool. Our first problem was posed as problem B4 on the 1978 William Lowell Putnam Mathematics Competition.

**Problem 1** Prove that for every positive integer \(N\), the equation

\[
a^2 + b^2 + c^2 + d^2 = abc + abd + acd + bcd
\]

has a solution in integers \(a, b, c, d \geq N\).

Consider an arbitrary positive integer solution \((a, b, c, d)\), and assume without loss of generality that \(a\) with \(a \leq b, c, d\) is the smallest coordinate. For applying Markov’s root flipping trick, we introduce the quadratic equation

\[
x^2 - (bc + bd + cd) x + (b^2 + c^2 + d^2 - bcd) = 0.
\]

One root of this equation (5) is \(a\) itself, and the other root is \(a' = bc + bd + cd - a\). Since we selected \(a\) as the smallest coordinate in the quadruple, the second root satisfies

\[a' \geq 3a^2 - a > a.\]

Hence flipping the smallest coordinate will automatically increase the value of that coordinate! The rest is easy. We start from the trivial solution \((1, 1, 1, 1)\), and repeatedly flip the smallest coordinate. Eventually this must bring all coordinates above any fixed bound \(N\). This fully settles the Putnam question.

It is instructive to see the actual sequence of solutions. Starting from \((1, 1, 1, 1)\) we get successively \((2, 1, 1, 1), (2, 4, 1, 1), (2, 4, 13, 1), (2, 4, 13, 85)\). One can show — and we encourage the reader to do so — that from this point onwards the four coordinates are always pairwise distinct. Furthermore if the coordinates are ordered as \(a < b < c < d\), then \(2a \leq b, 2b \leq c, 2c \leq d\), and \(2d \leq a'\).

In the year 1988, Markov’s old root flipping trick made its first appearance at the International Mathematical Olympiad. Only eleven students managed to find the solution to the following problem.

**Problem 2** Let \(a\) and \(b\) be two positive integers such that \(q = \frac{a^2 + b^2}{ab + 1}\) is integer. Show that \(q\) is a perfect square.

We choose a pair \((a, b)\) with the smallest possible sum \(a + b\) among all pairs of positive integers that yield this ratio \(q\) and hence satisfy

\[
a^2 + b^2 - abq - q = 0.
\]

Without loss of generality we assume \(a \geq b\). The next step is already routine for us: in order to apply Markov’s root flipping trick, we introduce the quadratic equation

\[
x^2 - bq \cdot x + (b^2 - q) = 0.
\]

The first root of (7) is \(a\), and the second root is \(a' = bq - a\). Clearly \(a'\) is integer, but is it positive or negative? For settling this question we will distinguish three cases on the sign of \(b^2 - q\).
In the first case we deal with \( b^2 - q > 0 \). Then the root \( a' = (b^2 - q)/a \) is indeed positive. Furthermore \( a \geq b \) implies \( a' = (b^2 - q)/a < (a^2 - q)/a < a \).

The resulting contradiction \( a' + b < a + b \) shows that this first case cannot occur.

In the second case we deal with \( b^2 - q < 0 \). Then (6) yields \( a(a - bq) = q - b^2 > 0 \), which implies \( a > bq \). Since (6) also yields \( q = b^2 + a(a - bq) > a > bq \), we get the contradiction \( 1 > b \). Hence also the second case cannot occur.

All in all, the only remaining possibility is the third case with \( b^2 - q = 0 \).

But then \( q = b^2 \) indeed is a perfect square, and we have arrived at the desired conclusion.

Our next problem is the central piece of a slightly more difficult question from the 2007 International Mathematical Olympiad.

**Problem 3** Let \( a \) and \( b \) be positive integers such that \( 4ab - 1 \) divides \((a - b)^2\).
Show that \( a = b \).

Once again Markov’s root flipping trick applies. Suppose that for some integer \( q \geq 0 \) there exists a pair \((a, b)\) of positive integers with

\[
a^2 - 2ab + b^2 = (4ab - 1)q. \tag{8}
\]

Among all such pairs \((a, b)\) we pick one that minimizes the value \( \max\{a, b\} \), and without loss of generality we furthermore assume \( a \geq b \). The quadratic equation

\[
x^2 - (2b + 4bq) \cdot x + (b^2 + q) = 0.
\]

has \( a \) and \( a' = (b^2 + q)/a \) as positive integer roots. Since we considered a pair with smallest possible value \( \max\{a, b\} \), we conclude \( a' \geq a \) and hence \( b^2 + q \geq a^2 \). By plugging this into (8) we derive

\[
(a - b)^2 = (4ab - 1)q \geq q \geq a^2 - b^2,
\]

which implies \( b \geq a \). Since we started from the assumption \( a \geq b \), we arrive at the desired conclusion \( a = b \). This implies \( q = 0 \) and completes the proof.

**Homework exercises**

We challenge the reader to settle the following five problems along the lines indicated above.

**Problem 4** Show that there are infinitely many quadruplets of positive integers \((a, b, c, d)\) that satisfy the equation

\[
a^2 + b^2 + c^2 + d^2 = abcd.
\]

One possible approach to this problem first guesses a solution with \( a = b = c = d \) and then repeatedly applies Markov’s root flipping trick.

**Problem 5** For \( q \geq 4 \), show that the following equation has no solution over the integers:

\[
a^2 + b^2 + c^2 = q \cdot abc.
\]
Note that for $q = 3$ the given equation coincides with the Markov equation (2) that we have discussed in detail. For $q \geq 4$ one can actually recycle the machinery from the $q = 3$ case: every solution triple with pairwise distinct coordinates generates another solution triple with strictly smaller weight. But this time the final step and the conclusion of the argument are different, as the chain of predecessors does never terminate! This leads to an infinite descent argument and shows that there are no solutions.

**Problem 6** Determine all positive integers $q$ for which the following equation has a solution $(a, b)$ over the positive integers.

(a) $a^2 + b^2 + 1 = q \cdot ab$
(b) $a^2 + b^2 + 6 = q \cdot ab$
(c) $a^2 + b^2 - 1 = q \cdot ab$

In parts (a) and (b) you will detect all possible values for $q$ by setting $a = b = 1$. For showing that these are the only possible values, you should evoke the root flipping trick. Part (c) is a trick question, and you may want to understand what’s going on for $a = q$ and $b = q^2 - 1$.

**Problem 7** Determine all positive integers $q$ for which the following equation has a solution $(a, b)$ over the positive integers.

(a) $a^2 + b^2 = q \cdot (ab - 1)$
(b) $a^2 + b^2 + ab = q \cdot (ab - 1)$

Part (a) of this problem is from the 2002 USA team selection test for the International Mathematical Olympiad, and part (b) is from the 2004 Romanian team selection test for the International Mathematical Olympiad. It is not difficult to guess that the answer to part (a) is $q = 5$, and that the answer to part (b) is $q = 4$ and $q = 7$. Both parts can be attacked by Markov’s trick, and in both parts the root flipping process gets stuck as soon as $a' \geq a \geq b$ holds (where as usually $a$ denotes the old root and $a'$ denotes the new root). All effort then goes into understanding and characterizing these terminal situations.

**Problem 8** Show that there are infinitely many pairs of positive integers $(a, b)$ for which $\frac{a + 1}{b} + \frac{b + 1}{a}$ is a positive integer.

This problem is taken from the second round of the 2007 British Mathematical Olympiad. The pair $(1, 1)$ yields the integer value $4$, and then once again Markov’s root flipping trick does the job.

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