SOLUTIONS

A filing error took place and as a result a few solvers were not acknowledged in the past few issues. The editors would like to recognize the following correct solutions: George Apostolopoulos, Messolonghi, Greece (3754); Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3566, 3570, 3572, 3574); Michel Bataille, Rouen, France (3566, 3570, 3572); John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA (3558); Oliver Geupel, Brühl, NRW, Germany (3564); Dragoljub Milošević, Gornji Milanovac, Serbia (3457); Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (3564); Henry Ricardo, Tappan, NY, USA (3558); Joel Schlosberg, Bayside, NY, USA (3556, 3563, 3566); Digby Smith, Mount Royal University, Calgary, AB (3571); and Titu Zvonaru, Comănești, Romania (3572). If any other errors or omissions occur, please send an email to crux-editors@cms.math.ca.

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


Let \( n \) be a nonnegative integer and let \( a_k \) be the coefficient of \( z^k \) in the McLaurin expansion of \( (z - 1)^n \ln(1 - z) \). Prove that

\[
a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{and} \quad a_k = \frac{-1}{(n + 1)(\binom{n}{k+1})}, \quad k > n.
\]

I. Solution by George Apostolopoulos, Messolonghi, Greece.

It is well known that \( (z - 1)^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} z^i \) and

\[
\ln(1 - z) = -\sum_{i=1}^{\infty} \frac{z^i}{i},
\]

thus

\[
(z - 1)^n \ln(1 - z) = \left( \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} z^i \right) \left( -\sum_{i=1}^{\infty} \frac{z^i}{i} \right).
\]

From (1) we deduce that

\[
a_n = \sum_{i=0}^{n-1} (-1)^{n-i} \binom{n}{i} \left( -\frac{1}{n - i} \right) = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} / j.
\]
Let \( g(n) = a_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \). Recall that \( \binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1} \)
so
\[
g(n) = \sum_{i=1}^{n} (-1)^{i+1} \frac{n-1}{i} + \sum_{i=1}^{n} (-1)^{i+1} \frac{n-1}{i-1}.
\]
(2)

Now, since \( \binom{n}{i} = 0 \) for \( i > n \) or \( i < 0 \), then
\[
\sum_{i=1}^{n} (-1)^{i+1} \frac{n-1}{i} = \sum_{i=1}^{n-1} (-1)^{i+1} \frac{n-1}{i} = g(n-1).
\]
(3)

It is easy to verify that \( \binom{n-1}{i-1} \cdot \frac{1}{i} = \binom{n}{i} \cdot \frac{1}{n} \), hence
\[
\sum_{i=1}^{n} (-1)^{i+1} \frac{(n-1)}{i} = \sum_{i=1}^{n} (-1)^{i+1} \frac{n}{i} = -\frac{1}{n} \sum_{i=1}^{n} (-1)^{i} \binom{n}{i}
= -\frac{1}{n} \left[ (1-1)^n - (-1)^0 \binom{n}{0} \right] = \frac{1}{n}.
\]
(4)

Thus (2), (3) and (4) yield \( g(n) = g(n-1) + \frac{1}{n} \). Since \( g(1) = (-1)^2 \binom{1}{1} = 1 \), we deduce that \( g(n) = \sum_{i=1}^{n} \frac{1}{i} \), that is, \( a_n = \sum_{i=1}^{n} \frac{1}{i} \).

From (1) we deduce that for \( k > n \)
\[
a_k = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \left( -\frac{1}{k-i} \right).
\]

Let \( f(n, k) = \sum_{i=0}^{n} (-1)^{n-i} \frac{n}{k-i} \), then
\[
f(n, k) = \sum_{i=0}^{n} (-1)^{n-i} \frac{n-1}{k-i} + \sum_{i=0}^{n} (-1)^{n-i} \frac{(n-1)}{k-i}
= -\sum_{i=0}^{n-1} (-1)^{n-1-i} \frac{n-1}{k-i} + \sum_{j=0}^{n-1} (-1)^{n-1-j} \frac{(n-1)}{k-1-j}
= f(n-1, k) - f(n-1, k).
\]

Now \( f(0, k) = \frac{1}{k} = \frac{1}{(0+1)} \binom{k}{0+1} \) for all positive integer values of \( k \), and if \( f(n-1, k) = \frac{1}{n} \binom{k}{n} \) holds for all \( k > n - 1 \), then \( f(n, k) = f(n-1, k-1) - f(n-1, k) \) for all \( k - 1 > n - 1 \), i.e. \( k > n \). Thus
\[ f(n, k) = \frac{1}{n^{(k-1)/n}} - \frac{1}{n^{(k)/n}} = \frac{1}{(n+1)^{(k)/n+1}} \]

for \( k > n \) so \( a_k = \frac{-1}{(n+1)^{(k)/n+1}} \) for \( k > n \).

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

Considering \((z - 1)^n \ln(1 - z) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} z^j \left( -\sum_{j=1}^{\infty} \frac{1}{j} z^j \right)\), we have to prove that

\[ \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} \binom{n}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \]

(1)

and

\[ \sum_{j=0}^{n} \frac{(-1)^j}{k - n + j} \binom{n}{j} = \frac{1}{(n+1)^{(k)/n+1}}, \quad k > n. \]

(2)

For a proof of identity (1) we refer to [1]. We only prove (2).

We have \(x^{k-n-1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^j = x^{k-n-1}(1 - x)^n\); hence

\[ \sum_{j=0}^{n} \frac{(-1)^j}{k - n + j} \binom{n}{j} = \int_0^1 x^{k-n-1}(1 - x)^n dx. \]

Integration by parts for nonnegative \( m, n \) yields

\[ \int_0^1 x^m(1 - x)^n dx = \left[ \frac{(1 - x)^{n+1}x^{m+1}}{m+1} \right]_0^1 + \frac{n}{m+1} \int_0^1 x^{m+1}(1 - x)^{n-1} dx \]

\[ = \frac{n}{m+1} \int_0^1 x^{m+1}(1 - x)^{n-1} dx. \]

By repeated application of this formula, we obtain

\[ \int_0^1 x^{k-n-1}(1 - x)^n dx = \frac{n(n-1) \cdots 1}{(k-n)(k-n+1) \cdots (k-1)} \int_0^1 x^{k-1} dx \]

\[ = \frac{n(n-1) \cdots 1}{(k-n)(k-n+1) \cdots (k-1)} \frac{1}{k} \]

\[ = \frac{1}{(n+1)^{(k)/n+1}}, \]

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.
Let \((2n+1)!! = 1 \cdot 3 \cdots (2n+1)\) be the double factorial, so (for example) \(7!! = 105\). Make the convention that \(0!! = (-1)!! = 1\). Prove that for any nonnegative integer \(n\),

\[
\sum_{i,j,k \geq 0 \atop i+j+k=n} \binom{n}{i,j,k}(2i-1)!!(2j-1)!!(2k-1)!! = (2n+1)!!.
\]

**Solution by Oliver Geupel, Brühl, NRW, Germany.**

From the identity

\[
(2m-1)!! = 1 \cdot 3 \cdots (2m-1) = \frac{(2m)!}{2^m m!} = \frac{m!}{2^m} \binom{2m}{m}
\]

we deduce the formal power series

\[
(1 - 4z)^{-\frac{1}{2}} = \sum_{m} \binom{-\frac{1}{2}}{m} (-4)^m z^m
\]

\[
= \sum_{m} \frac{(-4)^m m!}{m!} \left(\frac{-1}{2}\right) \left(\frac{-3}{2}\right) \cdots \left(\frac{2m-1}{2}\right) z^m
\]

\[
= \sum_{m} \frac{2^m(2m-1)!!}{m!} z^m = \sum_{m} \binom{2m}{m} z^m.
\]

Hence, the number \(\sum_{i+j+k=n \atop i,j,k \geq 0} \binom{2i}{i} \binom{2j}{j} \binom{2k}{k}\) is the coefficient of \(z^n\) in the series

\[
(1 - 4z)^{-\frac{3}{2}} = \sum_{n} \binom{\frac{3}{2}}{n} (-4)^n z^n
\]

\[
= \sum_{n} \frac{(-4)^n n!}{n!} \left(\frac{-3}{2}\right) \left(\frac{-5}{2}\right) \cdots \left(\frac{-2n+1}{2}\right) z^n
\]

\[
= \sum_{n} \frac{2^n(2n+1)!!}{n!} z^n = \sum_{n} \frac{n+1}{2} \binom{2n+2}{n+1} z^n.
\]
We conclude

\[
\sum_{\substack{i+j+k=n \\
i,j,k \geq 0}} \binom{n}{i,j,k} (2i - 1)!!(2j - 1)!!(2k - 1)!!
\]

\[
= \sum_{\substack{i+j+k=n \\
i,j,k \geq 0}} \binom{n}{i,j,k} \frac{n!}{i!j!k!} \cdot \frac{i!}{2^i} \cdot \frac{j!}{2^j} \cdot \frac{k!}{2^k} \cdot (2i)(2j)(2k)
\]

\[
= \frac{n!}{2^n} \sum_{\substack{i+j+k=n \\
i,j,k \geq 0}} \binom{2i}{i} \binom{2j}{j} \binom{2k}{k}
\]

\[
= \frac{(n+1)!}{2^{n+1}} \binom{2n+2}{n+1} = (2n+1)!!
\]

which completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.


Let \(ABC\) be a triangle with interior points \(D, E, F\) such that \(\angle FAB = \angle EAC, \angle FBA = \angle DBC, \angle DCF = \angle ECA, AF = AE, BF = BD,\) and \(CD = CE\). If \(R\) is the circumradius of \(ABC\), \(r\) is the circumradius of \(EDF\), and \(s\) is the semiperimeter of \(ABC\), prove that the area of triangle \(EDF\) is \(sr^2/2R\).

Solution by John G. Heuver, Grande Prairie, AB.

Let \(I_a, I_b, \) and \(I_c\) be the excentres of \(\Delta ABC\). The definition of points \(E\) and \(F\) implies that the lines \(AE\) and \(AF\) are symmetric by reflection in the internal bisector \(AI_a\) of \(\angle BAC\); that is, \(AI_a\) is the perpendicular bisector of \(EF\). But \(AI_a\) is also perpendicular to \(I_bI_c\) (which is the external bisector of \(\angle BAC\)), whence \(EF||I_bI_c\). Analogous statements hold for \(FD\) and \(DE\). We deduce first that the sides of \(\Delta DEF\) are parallel to the corresponding sides of \(\Delta I_aI_bI_c\), so that the two triangles are similar, and second that \(I\) is the circumcentre of \(\Delta DEF\). Furthermore, \(\angle EDF = \angle I_bI_aI_c = \frac{1}{2}(\angle B + \angle C)\), and, because \(I\) is the circumcentre of \(\Delta DEF\), \(\angle EIF = 2\angle EDF = \angle B + \angle C\). Because the circumradius of \(\Delta DEF\) is given to be \(r\), similar reasoning for the angles at \(E\)
and $F$ allows us to deduce that

$$\text{Area}(DEF) = \text{Area}(EIF) + \text{Area}(FID) + \text{Area}(DIE)$$

$$= \frac{1}{2} r^2 (\sin(B + C) + \sin(C + A) + \sin(A + B))$$

$$= \frac{1}{2} r^2 (\sin A + \sin B + \sin C) = \frac{1}{2} r^2 \left( \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \right)$$

$$= \frac{r^2}{2} \left( \frac{s}{R} \right).$$

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; JOEL SCHLOSBERG, Bayside, NY, USA; MIHAI STOÈNESCU, Bischwiller, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Our featured solution proves only that if $\Delta DEF$ were to exist, its area would equal the predicted value. Bataille and Schlosberg both addressed the question of existence: our featured solution makes clear that reflection in $AI_a$ takes $E$ to $F$, in $BI_b$ takes $F$ to $D$, and in $CI_c$ takes $D$ back to $E$. Because the product of three reflections is an opposite isometry, while $I$ and $E$ are points that are fixed by this product, this isometry must be a reflection in $IE$. Define $\ell$ to be the line that makes a directed angle with $CI_c$ equal to the directed angle from $AI_a$ to $BI_b$. We conclude that it is both necessary and sufficient that $E$ be a point of $\ell$ inside $\Delta ABC$ different from $I$, for which its reflections in $AI_a$ and $CI_c$, namely $F$ and $D$, also lie inside that triangle.


Let $H$ be the orthocentre of the acute triangle $ABC$ with $A'$ on the ray $HA$ and such that $A'A = BC$. Define $B'$, $C'$ similarly. Prove that

$$\text{Area}(A'B'C') = 4\text{Area}(ABC) + \frac{a^2 + b^2 + c^2}{2}.$$

Solution by Joel Schlosberg, Bayside, NY, USA.

Editor’s comment. The statement of the problem is somewhat flawed. What Schlosberg proves here is the theorem,

Let $H$ be the orthocentre of the arbitrary triangle $ABC$; define $A'$ to be the unique point satisfying $AA' = BC$ that lies on the half-line which starts at $A$ and extends along the line $HA$ in the direction that misses the line $BC$. Define $B'$, $C'$ similarly. Then $\text{Area}(A'B'C') = 4\text{Area}(ABC) + \frac{a^2 + b^2 + c^2}{2}$.

We have modified Schlosberg’s proof to make use of directed angles. Recall that $\angle XYZ$ as a directed angle denotes that angle (whose measure ranges from $0^\circ$ to $180^\circ$) through which the line $XY$ must be rotated about $Y$ in the positive direction in order to coincide with $YZ$. 

Because corresponding sides of $\angle BHC$ and $\angle CAB$ are perpendicular, while $\angle ABH$ and $\angle HCA$ are complements of $\angle CAB$ in right triangles, we have 

$$\angle BHC = \angle CAB = 90^\circ - \angle ABH = 90^\circ - \angle HCA.$$ 

It follows that

$$\text{Area}(HB'C') = \frac{1}{2} HB' \cdot HC' \sin \angle BHC$$

$$= \frac{1}{2} (HB + b)(HC + c) \sin \angle BHC$$

$$= \frac{1}{2} HB \cdot HC \sin \angle BHC + \frac{1}{2} cHB \cos \angle ABH$$

$$+ \frac{1}{2} bHC \cos \angle HCA + \frac{1}{2} bc \sin \angle CAB$$

$$= \text{Area}(HBC) + \frac{1}{2} BA \cdot BH + \frac{1}{2} CA \cdot CH + \text{Area}(ABC).$$

Similarly,

$$\text{Area}(HC'A') = \text{Area}(HCA) + \frac{1}{2} CB \cdot CH + \frac{1}{2} AB \cdot AH + \text{Area}(ABC),$$

and

$$\text{Area}(HA'B') = \text{Area}(HAB) + \frac{1}{2} AC \cdot AH + \frac{1}{2} BC \cdot BH + \text{Area}(ABC).$$

Consequently,

$$\text{Area}(A'B'C') = \text{Area}(HB'C') + \text{Area}(HC'A') + \text{Area}(HA'B')$$

$$= \text{Area}(HBC) + \text{Area}(HCA) + \text{Area}(HAB) + 3\text{Area}(ABC)$$

$$+ \frac{1}{2} BC \cdot (BH + HC) + \frac{1}{2} AC \cdot (CH + HA)$$

$$+ \frac{1}{2} AB \cdot (AH + HB)$$

$$= 4\text{Area}(ABC) + \frac{1}{2} BC \cdot BC + \frac{1}{2} AC \cdot CA + \frac{1}{2} AB \cdot AB$$

$$= 4\text{Area}(ABC) + \frac{a^2 + b^2 + c^2}{2}.$$

Let \( a > 0 \) and \( b > 1 \) be real numbers and let \( f : [0, 1] \to \mathbb{R} \) be a continuous function. Find

\[
\lim_{n \to \infty} n^{a/b} \int_0^1 \frac{f(x)}{1 + n^a x^b} \, dx.
\]

Solution by Mohammed Aassila, Strasbourg, France (expanded slightly by the editor).

We show that the required limit is \( \frac{\pi f(0)}{b \sin \left( \frac{\pi}{b} \right)} \).

Let \( L \) denote the given limit. Using the substitution \( y = n^{a/b} x \), we have

\[
n^{a/b} \int_0^1 \frac{f(x)}{1 + n^a x^b} \, dx = \int_0^1 \frac{f(x)}{1 + (n^{a/b} x)^b} \cdot n^{a/b} \, dx = \int_0^1 \frac{f\left( n^{a/b} y \right)}{1 + y^b} \, dy.
\]

Hence, \( L = f(0) \int_0^\infty \frac{1}{1 + y^b} \, dy \).

Let \( u = y^b \) so \( y = u^{1/b} \) and \( dy = \frac{1}{b} u^{1/b - 1} \, du \). Then we have

\[
L = \frac{f(0)}{b} \int_0^\infty \frac{u^{1/b}}{u(1 + u)} \, du.
\]  

Let \( \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \), \( x > 0 \) and \( B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt \), \( x > 0 \), \( y > 0 \) denote the Gamma function and the Beta function, respectively. The following formulae are well known [1]:

\[
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \quad \text{and} \quad \Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}.
\]

Let \( u = \frac{t}{1 - t} \). Then \( du = \frac{1}{1 - t^2} \, dt \) and \( u(1 + u) = \frac{t}{(1 - t)^2} \). Hence, from (1), (2) and the obvious fact that \( \Gamma(1) = 1 \), we have

\[
L = \frac{f(0)}{b} \int_0^1 t^{b-1} (1 - t)^{-b} \, dt = \frac{f(0)}{b} B \left( \frac{1}{b}, 1 - \frac{1}{b} \right) = \frac{f(0)}{b} \cdot \frac{\Gamma\left( \frac{1}{b} \right) \Gamma\left( 1 - \frac{1}{b} \right)}{\Gamma(1)} = \frac{f(0)}{b} \cdot \frac{\pi}{\sin\left( \frac{\pi}{b} \right)}
\]

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

There were three incomplete solutions all of which gave \( f(0) \int_0^\infty \frac{at}{1 + at} \) or its equivalent form, as the final answer.

Stadler remarked that it suffices to assume that \( f(x) \) is a bounded integrable function which is continuous from the right at 0.
Let $\alpha = \frac{\pi}{13}$ and
\[
x_1 = \tan(4\alpha) + 4 \sin(\alpha) = -\tan(\alpha) + 4 \sin(3\alpha),
\]
\[
x_2 = \tan(\alpha) + 4 \sin(\alpha) = -\tan(4\alpha) + 4 \sin(3\alpha),
\]
\[
x_3 = \tan(6\alpha) - 4 \sin(6\alpha) = \tan(2\alpha) + 4 \sin(5\alpha).
\]

Prove that the length $x_1$ can be constructed with compass and straightedge and determine whether or not the same is true for $x_2$ and $x_3$.

**Solution by Stan Wagon, Macalester College, St. Paul, MN, USA.**

Applying trigonometric expansion with the help of Mathematica, we find that $x_1 = \sqrt{13 - 2\sqrt{13}}$ which is clearly constructible since it contains only square roots. [Ed.: The proposer remarked that $x_1 = \sqrt{13 - 2\sqrt{13}}$ can be proved from the solution to problem # 3305.]

On the other hand, using trigonometric expansion and an algorithm to deduce the minimal polynomial satisfied by an algebraic number, we learn that each of $x_2$ and $x_3$ is a root of the irreducible polynomial $x^{12} - 78x^{10} + 1963x^8 - 20020x^6 + 81991x^4 - 138398x^2 + 81133$. Since it is a classical result that an algebraic number is constructible if and only if the degree of its minimal polynomial is a power of $2$, we conclude that $x_2$ and $x_3$ are not constructible.

The proposer gave a partial answer by showing that $x_1$ is constructible. No other solutions were received.

**References**

Let $I$ denote the integral. The substitution $u = 1 - x$ yields

$$I = \int_1^0 \left\{ \frac{1}{(1-u)^k} - \frac{1}{u^k} \right\} u^m(1-u)^m(-du)$$

$$= \int_0^1 \left\{ \frac{1}{(1-x)^k} - \frac{1}{x^k} \right\} x^m(1-x)^m dx .$$

Thus

$$2I = \int_0^1 [\{ \phi(x) \} + \{-\phi(x)\}] x^m(1-x)^m dx ,$$

where $\phi(x) := \frac{1}{(1-x)^x} - \frac{1}{x^x}$.

Since $\phi(x)$ is continuous, strictly increasing on $(0, 1)$, and $\lim_{x \to 0^+} \phi(x) = -\infty$; $\lim_{x \to 1^-} \phi(x) = +\infty$, it follows that $\phi$ is a bijection from $(0, 1)$ onto $\mathbb{R}$. Thus, for each integer $p \in \mathbb{Z}$, the equation $\phi(x) = p$ has exactly one solution $a_p \in (0, 1)$.

It is easy to see that

$$\{a\} + \{-a\} = \begin{cases} 0 & \text{if } a \in \mathbb{Z} \\ 1 & \text{if } a \notin \mathbb{Z} \end{cases}$$

Thus

$$[\{ \phi(x) \} + \{-\phi(x)\}] x^m(1-x)^m = x^m(1-x)^m$$

outside the countable set $\{a_p | p \in \mathbb{Z}\}$.

Hence

$$I = \frac{1}{2} \int_0^1 [\{ \phi(x) \} + \{-\phi(x)\}] x^m(1-x)^m dx$$

$$= \frac{1}{2} \int_0^1 x^m(1-x)^m dx = \frac{(\Gamma(m+1))^2}{2\Gamma(2m+2)} = \frac{(m!)^2}{2(2m+1)!} .$$

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.


Let $\Gamma_1$, $\Gamma_2$ be circles of radius $r$ with centres $A$, $B$ (respectively), let $\{C, D\} = \Gamma_1 \cap \Gamma_2$, and suppose that $\angle BCA = 90^\circ$. A line through $C$ intersects $\Gamma_1$ and $\Gamma_2$ again at $E$ and $F$, respectively. The circle $\Gamma$ with centre $O$ and radius $R$ passes through points $E$ and $F$. A second line passes through $C$, is perpendicular to the segment $EF$, and intersects the circle $\Gamma$ in $G$ and $H$. Prove that $CH^2 + CG^2 = 4(R^2 - r^2)$. 
Let $M$ be the midpoint of $EC$, $N$ the midpoint of $CF$, $a = AM$, and $b = BN$. Because corresponding sides of $\triangle AMC$ and $\triangle CNB$ are perpendicular, while $BC = AC = r$, the triangles are congruent. In particular, $a = CN = NF$ and $b = MC = EM$, so that $EF = 2(b + a)$ and

$$r^2 = a^2 + b^2.$$

Furthermore, if $P$ is the midpoint of $EF$, then $EP = b + a$ so that $MP = a$ and $PC = b - a$.

Next, let $Q$ be the midpoint of $GH$ and $h = PO (= CQ)$. Then $OQ = PC = b - a$, whence

$$QH^2 = R^2 - (b - a)^2.$$

Moreover,

$$CH^2 + CG^2 = (h + QH)^2 + (h - QH)^2 = 2(QH^2 + h^2).$$

But $R^2 = OE^2 = h^2 + (b + a)^2$, hence

$$h^2 = R^2 - (b + a)^2.$$

Assembling the pieces, we conclude that

$$CH^2 + CG^2 = 2 \left[ (R^2 - (b - a)^2) + (R^2 - (b + a)^2) \right]$$

$$= 2 \left[ 2R^2 - 2(b^2 + a^2) \right] = 4R^2 - 4r^2,$$

as claimed.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.
along the lines $EF$ and $GH$ (so that $XY = -YX$ for points $X$ and $Y$ both on one of these lines or on a line parallel to one of them), then Woo’s notation has been modified by the editor so that his argument deals with both cases simultaneously. Otherwise, one must observe that when $C$ is not between $E$ and $F$ (and therefore not between $G$ and $H$) and the diagram is labeled so that $b \geq a$, then $PE = b - a$ and $PC = b + a$, and the featured argument goes through without difficulty.

3583. [2010: 460, 463] Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Let $\alpha$ and $\beta$ be nonnegative real numbers and define

$$a_n = (n + \ln(n + 1)) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k + 1) + \ln(k + 1)},$$

$$p_n = (\alpha + n + 1 + \ln(n + 1)) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k + 1) + \ln(k + 1)}.$$

Find those nonnegative real numbers $\alpha$ and $\beta$ for which $\sum_{n=1}^{\infty} a_n$ converges, and determine the relation between $\alpha$ and $\beta$ that ensures that

$$\sum_{n=1}^{\infty} \left( a_n - p_n \ln \left( 1 + \frac{1}{n + 1} \right) \right) = (\alpha + 1)(\alpha + 2 + \ln 2) - \frac{(\alpha + 1)^2}{2}.$$

Solution by the proposer.

We claim that the series converges if and only if $\beta > \alpha + 1$.

Let’s start by observing that

$$\frac{\alpha + n + 1 + \ln(n + 1)}{\beta + n + 2 + \ln(n + 2)} = 1 + \frac{\alpha - \beta - 1 + \ln \left( 1 - \frac{1}{n+2} \right)}{\beta + 2 + n + \ln(n + 2)}$$

and

$$\frac{n + 1 + \ln(n + 2)}{n + \ln(n + 1)} = 1 + \frac{1 + \ln \left( 1 + \frac{1}{n+1} \right)}{n + \ln(n + 1)}.$$

Hence

$$\frac{a_{n+1}}{a_n} = \frac{\alpha + n + 1 + \ln(n + 1)}{\beta + n + 2 + \ln(n + 2)} \cdot \frac{n + 1 + \ln(n + 2)}{n + \ln(n + 1)}$$

$$= \left( 1 + \frac{\alpha - \beta - 1 + \ln \left( 1 - \frac{1}{n+2} \right)}{\beta + 2 + n + \ln(n + 2)} \right) \cdot \left( 1 + \frac{1 + \ln \left( 1 + \frac{1}{n+1} \right)}{n + \ln(n + 1)} \right)$$

$$= 1 + \frac{\alpha - \beta - 1 + \ln \left( 1 - \frac{1}{n+2} \right)}{\beta + 2 + n + \ln(n + 2)} + \frac{1 + \ln \left( 1 + \frac{1}{n+1} \right)}{n + \ln(n + 1)}$$

$$+ \frac{\alpha - \beta - 1 + \ln \left( 1 - \frac{1}{n+2} \right)}{\beta + 2 + n + \ln(n + 2)} \cdot \frac{1 + \ln \left( 1 + \frac{1}{n+1} \right)}{n + \ln(n + 1)}.$$
In particular

\[
\frac{a_{n+1}}{a_n} = \left(1 + \frac{\alpha - \beta}{n}\right)
\]

\[
= \frac{\alpha - \beta - 1 + \ln \left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} - \frac{\alpha - \beta - 1}{n} + \frac{1 + \ln \left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)} - \frac{1}{n}
\]

\[
+ \frac{\alpha - \beta - 1 + \ln \left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} \cdot \frac{1 + \ln \left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}
\]

\[
= (\alpha - \beta - 1) \left(\frac{\frac{1}{\beta + 2 + n + \ln(n+2)} - \frac{1}{n}}{n + \ln(n+1)} + \frac{1}{n + \ln(n+1)} - \frac{1}{n}\right)
\]

\[
+ \frac{\alpha - \beta - 1 + \ln \left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} \cdot \frac{1 + \ln \left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}
\]

\[
+ \frac{\ln \left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} + \frac{\ln \left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}
\]

\[
= (\alpha - \beta - 1) \left(\frac{-\beta - 2 - \ln(n+2)}{n^2 + \beta n + 2n + n \ln(n+2)} - \frac{\ln(n+1)}{n^2 + n \ln(n+1)} \right)
\]

\[
+ \frac{\alpha - \beta - 1 + \ln \left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} \cdot \frac{1 + \ln \left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}
\]

\[
+ \frac{\ln \left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} + \frac{\ln \left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}
\]

\[
= O \left(\frac{\ln(n)}{n^2}\right)
\]

Thus, we get

\[
\frac{a_{n+1}}{a_n} = 1 + \frac{\alpha - \beta}{n} + O \left(\frac{\ln(n)}{n^2}\right).
\]

(1)

Convergence for \(\beta > \alpha + 1\).

For all \(n \geq 2\) set \(b_n = \frac{1}{n \ln^2(n)}\). Then

\[
\frac{b_{n+1}}{b_n} = \frac{n \ln^2(n)}{(n+1) \ln^2(n+1)} = 1 - \frac{\ln^2(n+1) - n \ln^2(n)}{(n+1) \ln^2(n+1)}
\]

\[
= 1 - \frac{(n+1) \ln^2(n+1) - n \ln^2(n+1) - n \ln^2(n+1) - n \ln^2(n)}{(n+1) \ln^2(n+1)}
\]

\[
= 1 - \frac{1}{(n+1)} - \frac{(n \ln(n+1) - n \ln(n)) (\ln(n+1) + \ln(n))}{(n+1) \ln^2(n+1)}
\]

\[
= 1 - \frac{1}{(n+1)} - \frac{\ln \left(1 + \frac{1}{n}\right)^n (\ln(n+1) + \ln(n))}{(n+1) \ln^2(n+1)}.
\]
so

\[
\frac{b_{n+1}}{b_n} = 1 - \frac{1}{n} + \frac{1}{n(n+1)} - \frac{\ln \left( 1 + \frac{1}{n} \right)^n (\ln(n+1) + \ln(n))}{(n+1) \ln^2(n+1)}
\]

\[
= 1 - \frac{1}{n} + \frac{1}{n(n+1)} - \frac{2}{(n+1) \ln(n+1)}
\]

\[
= 1 - \frac{1}{n} + \frac{1}{n(n+1)} - \frac{\ln \left( 1 + \frac{1}{n} \right)^n (\ln(n+1) + \ln(n))}{(n+1) \ln^2(n+1)}
\]

\[
= 1 - \frac{1}{n} - \frac{2}{(n+1) \ln(n+1)} + O \left( \frac{\ln(n)}{n^2} \right) .
\]  \hspace{1cm} (2)

Since \( \beta > \alpha + 1 \), by combining (1) and (2) we get that there exists some \( N \) so that for all \( n > N \) we have

\[
0 < \frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}.
\]

Hence, for all \( n > N \) we have

\[
\frac{a_{n+1}}{b_{n+1}} < \frac{a_n}{b_n}.
\]

It follows that the sequence \( \frac{a_n}{b_n} \) is eventually decreasing, and hence bounded from above. Hence, there exists an \( \bar{M} \) so that, for all \( n \) we have

\[
0 < a_n < \bar{M}b_n .
\]

Since the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)} \) is convergent by the integral test, by the Comparison Test we get that \( \sum_{n} a_n \) is convergent.

**Divergence for \( \beta \leq \alpha + 1. \)**

For all \( n \geq 2 \) set \( c_n = \frac{1}{n \ln^2(n)} \). Exactly as in (2) one can show that

\[
\frac{c_{n+1}}{c_n} = 1 - \frac{1}{n} - \frac{1}{(n+1) \ln(n+1)} + O \left( \frac{\ln(n)}{n^2} \right) .
\]

Thus, there exists some \( N \) so that, for all \( n > N \) we have

\[
\frac{a_{n+1}}{a_n} \geq \frac{c_{n+1}}{c_n} > 0.
\]
Hence, for all $n > N$ we have

$$\frac{a_{n+1}}{c_{n+1}} > \frac{a_n}{c_n} > 0.$$ 

It follows that the sequence $\frac{a_n}{c_n}$ is eventually increasing. Hence, there exists an $m$ so that, for all $n$ we have

$$a_n > mc_n.$$ 

Moreover, $m$ can be chosen as the smallest term of $\frac{a_n}{c_n}$, and hence it can be chosen so that $m > 0$.

Since the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent, by the Comparison Test we get that $\sum_n a_n$ is also divergent.

This completes the first part of the problem.

For the second part, let’s observe first that

$$a_n = (\alpha + n + 1 + \ln(n + 1)) \prod_{k=1}^{n} \frac{\alpha + k + \ln(k)}{\beta + k + 1 + \ln(k + 1)}$$

$$- (\alpha + 1) \prod_{k=1}^{n} \frac{\alpha + k + \ln(k)}{\beta + k + 1 + \ln(k + 1)}.$$ 

Define

$$q_n := \prod_{k=1}^{n} \frac{\alpha + k + \ln(k)}{\beta + k + 1 + \ln(k + 1)}.$$ 

Then

$$a_n = p_n - (\alpha + 1)q_n.$$ 

Since by the definition of $q_n$ we have $0 < q_n \leq a_n$, it follows that $\sum_n p_n$ and $\sum_n q_n$ are absolutely convergent.

Let’s note that

$$\frac{q_{n+1}}{q_n} = \frac{\alpha + n + 1 + \ln(n + 1)}{\beta + n + 2 + \ln(n + 2)}.$$ 

In particular

$$\beta(q_{n+1} - q_n) + (\beta - \alpha)q_n + (n + 2)q_{n+1} - q_n(n + 1)$$

$$= q_n \ln(n + 1) - q_{n+1} \ln(n + 2).$$ 

Since $\sum a_n$ is convergent, we are in the case $\beta > \alpha + 1$. Thus

$$\frac{q_{n+1}}{q_n} < \frac{\beta + n + \ln(n + 1)}{\beta + n + 2 + \ln(n + 2)} < 1,$$

and hence $q_n$ is decreasing.

We will employ the following well known lemma:
Lemma: Let $q_n$ be monotonic and positive. If $\sum q_n$ is convergent, then

$$\lim_{n} nq_n = 0.$$ 

Let's denote $\sum q_n = U$. By summing (4) from 1 to $N$ we get

$$\beta(q_{N+1} - q_1) + (\beta - \alpha) \sum_{n=1}^{N} q_n + (N + 2)q_{N+1} - 2q_1(n + 1)$$

$$= q_1 \ln(2) - q_{N+1} \ln(N + 2).$$

Using the above Lemma and letting $N \to \infty$ we get:

$$-\beta q_1 + (\beta - \alpha)U - 2q_1 = q_1 \ln(2).$$

Since $q_1 = \frac{\alpha + 1}{\beta + 2 + \ln 2}$ we get

$$U = \frac{(\beta + 2 + \ln(2))q_1}{\beta - \alpha} = \frac{\alpha + 1}{\beta - \alpha}.$$

Now, let's observe that

$$\frac{p_{n+1}}{p_n} = \frac{\alpha + n + 2 + \ln(n + 2)}{\beta + n + 2 + \ln(n + 2)},$$

and hence

$$\beta(p_{n+1} - p_n) + (\beta - \alpha)p_n + [(n + 2)p_{n+1} - (n + 1)p_n] - p_n$$

$$= [p_n \ln(n + 1) - p_{n+1} \ln(n + 2)] + [p_n \ln(n + 2) - p_n \ln(n + 1)].$$

By summing we get

$$\beta(p_{N+1} - p_1) + (\beta - \alpha) \sum_{n=1}^{N} p_n + [(N + 2)p_{N+1} - 2p_1] - \sum_{n=1}^{N} p_n$$

$$= [p_1 \ln(2) - p_{N+1} \ln(N + 2)] + \left[ \sum_{n=1}^{N} p_n \ln \left(1 + \frac{1}{n + 1}\right) \right].$$

Again using the Lemma, and letting $N \to \infty$ we get:

$$-\beta p_1 - 2p_1 + (\beta - \alpha) \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} p_n = p_1 \ln(2) + \sum_{n=1}^{N} p_n \ln \left(1 + \frac{1}{n + 1}\right).$$

Since

$$p_1 = (\alpha + 2 + \ln(2)) \frac{\alpha + 1}{\beta + 2 + \ln(2)},$$
we get
\[
(\beta - \alpha) \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{N} p_n \ln\left(1 + \frac{1}{n+1}\right) = (\alpha + 2 + \ln(2))(\alpha + 1),
\]
and hence
\[
\sum_{n=1}^{\infty} p_n = \frac{1}{\beta - \alpha - 1} \sum_{n=1}^{N} p_n \ln\left(1 + \frac{1}{n+1}\right) + \frac{(\alpha + 2 + \ln(2))(\alpha + 1)}{\beta - \alpha - 1}.
\]
This shows that
\[
\sum_{n=1}^{\infty} \left(a_n - p_n \ln\left(1 + \frac{1}{n+1}\right)\right)
= \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} (\alpha + 1)q_n - \sum_{n=1}^{\infty} p_n \ln\left(1 + \frac{1}{n+1}\right)
= \frac{(\alpha + 2 + \ln(2))(\alpha + 1)}{\beta - \alpha - 1} + \frac{\beta - \alpha - 2}{\beta - \alpha - 1} \sum_{n=1}^{\infty} p_n \ln\left(1 + \frac{1}{n+1}\right) - \frac{1}{\beta - \alpha}.
\]
Thus, if $\beta = \alpha + 2$, the second formula holds.

No other solution was received.

3584. [2010: 460, 463] Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Let $ABC$ be a triangle with inradius $r$, side lengths $a, b, c$ and medians $m_a, m_b, m_c$. Prove that $\frac{c}{m_a^2m_b^2} + \frac{a}{m_b^2m_c^2} + \frac{b}{m_c^2m_a^2} \leq \frac{3\sqrt{3}}{27r^2}$.

Solution by Michel Bataille, Rouen, France.

We first observe that
\[
m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4} \geq \frac{b^2 + 2bc + c^2 - a^2}{4} = \frac{(b + c)^2 - a^2}{4} = p(p - a),
\]
where $p = \frac{a + b + c}{2}$ is the semiperimeter of the triangle $ABC$. Similarly we get
\[
m_b^2 \geq p(p - b) ; m_c^2 \geq p(p - c).
\]
Thus, we get
\[
\frac{c}{m_a^2 m_b^2} + \frac{a}{m_b^2 m_c^2} + \frac{b}{m_a^2 m_c^2} \leq \frac{c}{p^2 (p-a)(p-b)} + \frac{a}{p^2 (p-b)(p-c)} + \frac{b}{p^2 (p-a)(p-c)}
\]
\[
= \frac{c(p-c) + a(p-a) + b(p-b)}{p^2 (p-a)(p-b)(p-c)}
\]

Let \( S \) denote the area of the triangle \( ABC \). Since \( p(p-a)(p-b)(p-c) = S^2 = r^2 p^2 \) and
\[
a(p-a) + b(p-b) + c(p-c) = \frac{2[abc + (p-a)(p-b)(p-c)]}{p},
\]
to complete the proof we need to show that
\[
abc + (p-a)(p-b)(p-c) \leq \frac{\sqrt{3}p^4}{27r}
\] (1)

By applying the AM-GM inequality to both terms on the left side we get
\[
abc + (p-a)(p-b)(p-c) \leq \left( \frac{a+b+c}{3} \right)^3 + \left( \frac{(p-a)+(p-b)+(p-c)}{3} \right)^3
\]
\[
= \frac{8p^3}{27} + \frac{p^3}{27} = \frac{p^3}{3}
\]

Also
\[
r^2 p^2 = p(p-a)(p-b)(p-c) \leq p \left( \frac{(p-a)(p-b)(p-c)}{3} \right)^3 \leq \frac{p^4}{27},
\]
and hence
\[
3\sqrt{3}r \leq p.
\]
Thus
\[
abc + (p-a)(p-b)(p-c) \leq \frac{p^3}{3} = \frac{p^4}{3p} \leq \frac{p^4}{9\sqrt{3}r},
\]
which proves (1).

Also solved by ARKADY ALT, San Jose, CA, USA; KEE-WAI LAU, Hong Kong, China; DRAGOLJUB MILOSEVIC, Gornji Milanovac, Serbia; and the proposer.
Let \( T_n(x) \) be the Chebyshev polynomial of the first kind defined by the recurrence 
\[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \]
for \( n \geq 1 \) and the initial conditions 
\[ T_0(x) = 1 \text{ and } T_1(x) = x. \]
Find all positive integers \( n \) such that 
\[ T_n(x) \leq (2^{n-2} + 1)x^n - 2^{n-2}x^{n-1}, \quad x \in [1, \infty). \]

Solution by Albert Stadler, Herrliberg, Switzerland.

The given recurrence defining \( T_n(x) \) implies that it is a polynomial of degree \( n \), whose leading coefficient is \( 2^{n-1} \) for all \( n \geq 1 \). If the required inequality is to hold for all \( x \geq 1 \) at a particular positive integer \( n \), then necessarily the leading coefficient of \( T_n(x) \) must be at most \( 2^{n-2} + 1 \). Thus \( 2^{n-1} \leq 2^{n-2} + 1 \), which implies that \( 2^{n-2} \leq 1 \), which is true only when \( n = 1 \) or \( n = 2 \).

With \( n = 1 \), the inequality demands that \( x \leq \frac{3}{2}x - \frac{1}{2} \), which clearly holds for all \( x \geq 1 \). However, with \( n = 2 \), the inequality demands that \( 2x^2 - 1 \leq 2x^2 - x \), which clearly fails for all \( x > 1 \). Thus \( n = 1 \) is the only positive integer with the required property.

Also solved by MICHEL BATAILLE, Rouen, France.


For each positive integer \( n \), \( a_n \) is the number of positive divisors of \( n \) of the form \( 4m + 1 \) minus the number of positive divisors of \( n \) of the form \( 4m + 3 \) (so \( a_4 = 1 \), \( a_5 = 2 \), and \( a_6 = 0 \)). Evaluate the sum \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a_n}{n} \).

Solution by the proposer.

We show that 
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a_n}{n} = \frac{\pi \ln 2}{4}. \]

(1)

To prove (1) we first establish the following lemma:

Lemma: Let \( x_k = (-1)^{k+1}k \) and \( y_k = (-1)^{k+1}(2k - 1) \), for each integer \( k \geq 1 \). Then

\[ \lim_{n \to \infty} \sum_{|x_iy_j| \leq n} \frac{1}{x_iy_j} = \left( \sum_{k=1}^{\infty} \frac{1}{x_k} \right) \left( \sum_{k=1}^{\infty} \frac{1}{y_k} \right) = \frac{\pi \ln 2}{4}. \]

Proof: For any fixed \( n \) let \( m = \lfloor \sqrt{n} \rfloor \). Then

\[ \sum_{|x_iy_j| \leq n} \frac{1}{x_iy_j} = \sum_{i=1}^{n} \left( \frac{1}{x_i} \sum_{|x_iy_j| \leq n} \frac{1}{y_j} \right) = \sum_{i=1}^{m} b_i + \sum_{i=m+1}^{n} b_i \]

where \( b_i = \frac{1}{x_i} \sum_{|x_iy_j| \leq n} \frac{1}{y_j}. \)
For each fixed $i = 1, 2, \ldots, m$, let
\[ \sum_{|x_i y_j| \leq n} \frac{1}{y_j} = \frac{\pi}{4} + E_i. \]

Using the fact that
\[ \left| \frac{\pi}{4} - \sum_{j=1}^{k} (-1)^{j+1} \frac{1}{2j - 1} \right| \leq \frac{1}{2k + 1} \]
we have
\[ |E_i| \leq \frac{1}{2k + 1} \quad \text{and} \quad (2k + 1)i > n. \]

Thus,
\[ \sum_{i=1}^{m} b_i - \frac{\pi}{4} \sum_{i=1}^{m} \frac{1}{x_i} \leq \sum_{i=1}^{m} \left| \frac{E_i}{x_i} \right| < \frac{m}{n}. \]

It follows that
\[ \lim_{n \to \infty} \sum_{i=1}^{n} b_i = \frac{\pi}{4} \sum_{i=1}^{\infty} \frac{1}{x_i} = \frac{\pi}{4} \ln 2. \]

Hence, it suffices to show that
\[ \lim_{n \to \infty} \sum_{i=m+1}^{n} b_i = 0. \] (2)

We consider the sums $|b_{m+1} + b_{m+2}|$, $|b_{m+3} + b_{m+4}|$, \ldots. (Note that $|b_n| = \frac{1}{n} \to 0$ as $n \to \infty$.)

Now define
\[ w(i) = \max_{|x_i y_j| \leq n} |y_j| \]
and consider an arbitrary but fixed $i \in \{m+1, m+2, \ldots\}$ where $i \leq n - 1$. If $w(i) = w(i + 1)$, then
\[ |b_i + b_{i+1}| = \left| \sum_{|x_i y_j| \leq n} \frac{1}{y_j} \right| \leq \frac{1}{i(i+1)}. \] (3)

If $w(i) \neq w(i+1)$ then $w(i+1) \leq w(i) - 2$ since clearly $w(i+1) \leq w(i)$ and both are odd integers. We now show that
\[ w(i+1) \geq w(i) - 2. \] (4)

Note first that
\[ iw(i) \leq n \implies (m+1)w(i) \leq n \implies \left(\lfloor \sqrt{n} \rfloor + 1 \right) w(i) \leq n \implies \sqrt{n}w(i) < n \implies w(i) < \sqrt{n} \implies w(i) \leq \lfloor \sqrt{n} \rfloor = m \implies w(i) \leq i - 1 < 2(i+1) \]
so \((i+1)(w(i) - 2) = iw(i) + w(i) - 2(i + 1) < n\).

Returning now to the proof of \((4)\) which clearly holds if \(w(i) = 1\) since \(w(i + 1) \geq 1\). If \(w(i) \geq 3\), then \(w(i) - 2 > 0\) and is odd. Since \(w(i) - 2\) is of the form \(|y_j|\) for some \(j\) and since \(w(i + 1) = \max|x_{i+1}w| \leq n\), we have \((i+1)w(i+1) \geq (i+1)|y_j| = (i+1)(w(i) - 2)\) from which \(w(i+1) \geq w(i) - 2\) follows, establishing \((4)\).

Therefore, \(w(i + 1) = w(i) - 2\). Using \((3)\) we obtain

\[
|b_i + b_{i+1}| \leq \frac{1}{i(i + 1)} + \frac{1}{|x_iw(i)|} = \frac{1}{i(i + 1)} + \frac{1}{iw(i)}
\]

\[
= \frac{1}{i(i + 1)} + \frac{1}{(i + 1)w(i)} \left(1 + \frac{1}{i}\right)
\]

\((5)\)

If \((i + 1)w(i) \leq n\), then \(|x_{i+1}w(i)| \leq n\) would imply that \(|x_{i+1}||y_j| \leq n\) for all \(y_j\) such that \(|x_i||y_j| \leq n\) so \(w(i) < w(i + 1)\), a contradiction.

Hence,

\[
(i + 1)w(i) > n.
\]

\((6)\)

From \((5)\) and \((6)\) we obtain

\[
|b_i + b_{i+1}| < \frac{1}{i(i + 1)} + \frac{2}{n}.
\]

\((7)\)

However, noting that \(w(m + 1) \leq m\) so the number of indices \(i\) for which \(w(i + 1) = w(i) - 2\) is bounded above by \(m\).

Since both \(\frac{1}{(m + 1)(m + 2)} + \frac{1}{(m + 3)(m + 4)} + \cdots\) and \(\frac{m}{n}\) tend to 0 as \(n \to \infty\) we conclude that \(\lim_{n \to \infty} \sum_{i=m+1}^{n} b_i = 0\) establishing \((2)\) and completing the proof.

Also solved by ALBERT STADLER, Herrliberg, Switzerland; who gave a 3-page proof based on the Dirichlet L-function associated with the non-trivial character \((\text{mod } 4)\), and analytic continuation of some complex function defined by Dirichlet series.
Define the prime graph of a set of positive integers as the graph obtained by letting the numbers be the vertices, two of which are joined by an edge if and only if their sum is prime.

(a) Prove that given any tree \( T \) on \( n \) vertices, there is a set of positive integers whose prime graph is isomorphic to \( T \).

(b) For each positive integer \( n \), determine \( t(n) \), the smallest number such that for any tree \( T \) on \( n \) vertices, there is a set of \( n \) positive integers each not greater than \( t(n) \) whose prime graph is isomorphic to \( T \).

No solutions have been received so this problem remains open.