RECURRING CRUX CONFIGURATIONS 3

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Triangles whose angles satisfy $2B = C + A$

Because the angles of a triangle $ABC$ sum to $180^\circ$, they are in arithmetic progression if and only if the intermediate angle measures $60^\circ$. I found nearly two dozen problems in CRUX with MAYHEM that deal with triangles having a $60^\circ$ angle; there was considerable overlap, so for this month's column instead of listing those problems, I will simply list and discuss the properties that readers have discovered, and provide references to where proofs can be found. As usual, $A$, $B$, and $C$ will represent either the vertices of a triangle or the measure of its angles, depending on the context; $a$, $b$, and $c$ represent either the opposite sides or their lengths; $s = (a + b + c)/2$ is the semiperimeter, while $H$, $I$, $O$, and $G$ are the orthocentre, incentre, circumcentre, and centroid, respectively. We shall use $2B = C + A$ and $B = 60^\circ$ interchangeably.

The first eight properties came, in part, from Problem 724 [1982 : 78; 1983 : 92-94] (proposed by Hayo Ahlburg) and the comments found there.

**Property 1.** $\angle B = 60^\circ$ if and only if $\sin(A - B) = \sin A - \sin C$.

The simple proof of only if is on page 92; for the if part note that $\sin C = \sin(A + B)$ and expand that and $\sin(A - B)$ to get an equation that reduces to $\cos B = \frac{1}{2}$.

**Property 2.** $\angle B = 60^\circ$ if and only if $a^2 - b^2 = c(a - c)$.

Property 2 [1983 : 93] is just the cosine law. It forms the basis of an olympiad problem proposal of Murray Klamkin that was never used: One of the angles of a triangle is $60^\circ$ if and only if the square of the side opposite that angle equals the sum of the cubes of the sides divided by the perimeter; that is,

$$\angle B = 60^\circ$$ if and only if $b^2 = \frac{a^3 + b^3 + c^3}{a + b + c}$.

A more substantial use of Property 2 came in [3] to obtain a characterization of integer-sided triangles having an angle of $60^\circ$: Let $p$ and $q$ be (i) relatively prime integers with (ii) one of them odd, the other even, and (iii) $p$ not a multiple of 3; use $p$ and $q$ to define the integers $x = |p^2 - 3q^2|$ and $y = 2pq$. If $x > y$ we set $b = p^2 + 3q^2$, $a = x + y$ and $c = 2y, x - y$ (there are two values of $c$ for each $a$ and $b$ because the quadratic equation of Property 2 will have two integer zeros); if $x < y$ we use the same $b$ but set $a = 2y$ and $c = y \pm x$. Then in either case with either value of $c$, $ABC$ is a triangle with $\angle B = 60^\circ$ and side lengths $a > b > c$ that are relatively prime integers; conversely, for any such triangle there exists a pair of integers $p$ and $q$ that produce that triangle using the given recipe. For example, $p = 1, q = 2$ determines $x = 11, y = 4$ and triangles with sides 15, 13, 8 as well as 15, 13, 7; for $p = 2, q = 1$, the parameters are $x = 1, y = 4$, and
the resulting triangles have sides 8, 7, 5, and 8, 7, 3. Similar results and further references can be found in [4].

**Property 3.** If $\angle B = 60^\circ$ then the points $A, C, O, I, H$ lie on a circle that also contains the excentre $I_b$ opposite vertex $B$; its radius equals the circumradius of $\Delta ABC$ and its centre $O'$ is where the angle bisector $BI$ again meets the circumcircle.

The solution to Problem M1046 from the 1987 U.S.S.R. journal *Kvant* [1988 : 165; 1990 : 103] follows easily: If $\angle B = 60^\circ$ then one of the bisectors of the angle between the altitudes from $A$ and $C$ passes through $O$. One should take care with the converse of Property 3, which is the topic of problem 1521 [1990 : 74; 1991 : 126-127] (proposed by J.T. Groenman). If either $A, I, O, C$ or $A, H, I, C$ are concyclic, then it follows that $\angle B = 60^\circ$ and all five points are concyclic. Such is not the case, however, with $A, H, O, C$ concyclic because then $\angle B = 120^\circ$ is also possible (which can be seen by changing the roles of the points $B$ with $H$ and $O$ with $O'$). This property is clearly equivalent to the following result, which is the subject of Problem 998 [1984 : 319; 1986 : 65] (proposed by Andrew P. Guinand):

If one angle of a triangle is either $60^\circ$ or $120^\circ$, then the image of the orthocentre under inversion with respect to the circumcircle lies on the side (possibly extended) opposite that angle.

![Figure 1: The angles of $\Delta ABC$ satisfy $2B = C + A$.](image)

**Property 4.** The circle of Property 3, with radius $R$ and centre $O'$, intersects the lines $BA$ and $BC$ at points $A'$ and $C'$ for which $AA' = CC' = |c - a|$. (Proof is on page [1983 : 93].)

**Property 5.** In any triangle, if $N$ is the centre of its nine-point circle (and, therefore, the midpoint of $OH$), and $P$ is the projection of the incentre $I$ onto the Euler line $OGNH$, then $P$ lies between $G$ and $H$; furthermore, $P = N$ if and only if one angle of the triangle has measure $60^\circ$.

This is Problem 260 [1977 : 155; 1978 : 58-60] (Proposed by W.J. Blundon). A variant of this property became Problem 5 on the 2007 Indian Team Selection
For triangles that are not equilateral, the common tangent to the incircle and the nine-point circle is parallel to the Euler line if and only if the angles of the triangle are in arithmetic progression. This, of course, is because in all triangles that common tangent is perpendicular to \( AI \).

**Property 6.** If \( \angle B = 60^\circ \) then \( N \) lies on the bisector of that angle; conversely, if the nine-point centre of a triangle lies on the interior bisector of \( \angle CBA \), then the vertex \( B \) lies on the perpendicular bisector of \( AC \) or \( \angle B = 60^\circ \).

This is Problem 2855 [2003 : 316; 2004 : 308-309] (Proposed by Andreas P. Hatzopolakis and Paul Yiu); the claim that \( BN \) bisects \( \angle B \) when \( \angle B = 60^\circ \) is also proved as a part of Problem 724 [1983 : 94].

**Property 7.** If \( B \) is the intermediate angle of \( \triangle ABC \), then \( \angle B = 60^\circ \) if and only if \( OI = IH \), if and only if \( OI_b = I_bH \).

The equivalence of \( \angle B = 60^\circ \) and \( OI = IH \) is proved as part of Problem 260 (Property 5 above). It was proved yet again as part of Problem 1521 (see Property 3 above). This result also appeared as Problem 739 [1982 : 107; 1983 : 153-154, 210-211] (proposed by G.C. Giri), where there is another proof and references to textbooks where it appears as an exercise. There is also a reference to a stronger result \([2]\): If the angles of triangle \( ABC \) are labeled so that \( A \leq B \leq C \) then

\[
\begin{align*}
\angle B & > 60^\circ \Rightarrow 0 < \frac{HI}{IO} < 1, \\
\angle B & = 60^\circ \Rightarrow HI = IO, \\
\angle B & < 60^\circ \Rightarrow 1 < \frac{HI}{IO} < 2.
\end{align*}
\]

The proof of the result for \( OI_b = I_bH \) can be found in \([1]\).

**Property 8.** \( \angle B = 60^\circ \) if and only if \( s = \sqrt{3}(R + r) \).

The proof is another part of the solution to Problem 260 [1978 : 58-60].

**Property 9.** \( \angle B = 60^\circ \) if and only if the bisector \( BO' \) of \( \angle B \) is perpendicular to the Euler line \( OH \); when these properties hold, then \( N \) is the common midpoint of \( BO' \) and \( OH \). (Recall that \( O' \) was defined in Property 3 to be where the angle bisector again intersects the circumcircle. Note that in any triangle, \( BH \) and \( OO' \) are both perpendicular to \( AC \).

The claims follow from the proof in Problem 1521 (see Property 3). Three more proofs can be found in \([5]\). An immediate consequence is Problem 3 of Round 2 of the 2006-2007 British Mathematical Olympiad [2010: 154; 2011: 165]: If the Euler line \( OH \) meets \( BA \) at \( P \) and \( BC \) at \( Q \), then \( \angle B = 60^\circ \) implies that \( OQ = HP \).

See \([2011: 165]\) for an independent proof (although the problem and proof found there were unnecessarily restricted to acute-angled triangles). Another immediate consequence of Problem 1521 is Problem 1673 [1991 : 237; 1992 : 218-219] (proposed by D.J. Smeenk):
Given \( \Delta ABC \) let \( P \) be an arbitrary point of the line \( BA \) and \( Q \) be on \( BC \), neither point coinciding with a vertex. If \( \angle B = 60^\circ \) then the Euler lines of \( \Delta ABC \) and \( \Delta PBQ \) are parallel; moreover, if the two Euler lines coincide then the circumcircle \( PQR \) contains \( O' \).

Jordi Dou added a proof that the sides \( PQ \) of those triangles whose Euler lines coincide with that of the given triangle are tangent to the parabola with focus \( O' \) and directrix \( OH \).

**Property 10.** \( \angle B = 60^\circ \) or \( 120^\circ \) if and only if \( BH = BO \).

This is case (i) of Problem 1518 [1990 : 44; 1991 : 122] (proposed by K.R.S. Sastry). Compare Property 9; Problem 1232(b) in [5] says that \( BO'||OH \) if and only if \( \angle B = 120^\circ \).

**Property 11.** \( \angle B = 60^\circ \) or \( 120^\circ \) if and only if its internal bisector divides an altitude in the ratio \( 1 : 2 \).


We devote the remainder of this compilation to properties that were discovered over the past 20 years by Toshio Seimiya. What a pity that we failed to invite him to write this article for us! For most of these properties we denote by \( D \) and \( E \) the points where the interior bisectors of angles \( A \) and \( C \) meet the opposite sides. Properties 12 through 16 are quite closely related.

**Property 12.** If \( \angle B = 60^\circ \) then the points \( D \) and \( E \) are two vertices of an equilateral triangle whose third vertex lies on \( AC \) and whose incentre is \( I \).

In other words, the bisectors of angles \( A \) and \( C \) meet the opposite sides at the centres of two circles with common radius \( DE \) that intersect on \( AC \). This is Seimiya’s counterexample to the incorrect claim made by the proposer of Problem 1446(c) [1989 : 148; 1990 : 217-219] (namely, that the existence of this inscribed equilateral triangle implied that the original triangle \( ABC \) was necessarily equilateral).

**Property 13.** Define \( P \) to be the point where the line perpendicular to \( DE \)
meets $AC$, and $Q$ to be where it meets $DE$. Then $IP = 2IQ$ if and only if $\angle B = 60^\circ$. (Problem 2011 [1995 : 52; 1996 : 80])

**Property 14.** Define $P$ to be the point where the bisector of $\angle AIC$ meets $AC$, and $Q$ to be where it meets $DE$. Then $IP = 2IQ$ if and only if $\angle B = 60^\circ$. (Problem 2939 [2004 : 229, 232; 2005 : 243-244])

**Property 15.** $\angle ADE = 30^\circ$ if and only if $\angle B = 60^\circ$ or $\angle C = 120^\circ$. (Problem 2263 [1997 : 364; 1998 : 432-433])

**Property 16.** Call $F$ the point where $DE$ intersects $AC$. If $\triangle ABC$ has $BC > BA$ and $\angle DFC = \frac{1}{2}(\angle DAC - \angle ECA)$, then $\angle B = 60^\circ$. (Problem 1692 [1991 : 301; 1992 : 284-285])

**Property 17.** An acute-angled triangle $ABC$ is given, and equilateral triangles $ABP$ and $BCQ$ are drawn outwardly on the sides $AB$ and $BC$. Suppose that $AQ$ and $CP$ meet $BC$ and $AB$ at $R$ and $T$, respectively, and that $AQ$ and $CP$ intersect at $S$. If the area of the quadrilateral $BRST$ is equal to the area of the triangle $ASC$, then $\angle B = 60^\circ$. (Problem 2304 [1998 : 46; 1999 : 56-57])

In addition to his many problems, Semiya also wrote an article for CRUX with MAYHEM entitled “On Some Examples of Geometric Fallacies” [29:6 (October 2003) 393-396]. He began with a theorem and proposed two attempted converses, both of which came with very convincing arguments; he then pointed out the subtle but critical errors in the arguments, and provided counterexamples to show that those converses were indeed false. It is the theorem that is relevant here:

**Theorem.** Let $\triangle ABC$ be a triangle with $\angle B = 60^\circ$. Let $D$ be the point on $BC$ produced beyond $C$ such that $CD = CA$, and let $E$ be the point on $BA$ produced beyond $A$ such that $CA = AE$. Then $\angle DCA = 2\angle AED$, $\angle CAE = 2\angle EDC$, and $\angle EDA = 30^\circ$.

**References**


