M500. *Proposé par Edward T.H. Wang et Dexter S.Y. Wei, Université Wilfrid Laurier, Waterloo, ON.*

Soit $\mathbb{N}$ l'ensemble des nombres naturels.

(a) Démontrer que si $n \in \mathbb{N}$ alors il n'existe aucun $a, b \in \mathbb{N}$ tels que $\dfrac{[a,b]}{a+b} = n$, où $[a,b]$ dénote le plus petit commun multiple de a et b.

(b) Démontrer que si $n \in \mathbb{N}$ alors il existe un nombre infini de triplets (a,b,c) d'entiers naturels tels que $\dfrac{[a,b,c]}{a+b+c} = n$, où $[a,b,c]$ dénote le plus petit commun multiple de a, b et c.

Mayhem Solutions

M457. *Proposed by the Mayhem Staff.*

Suppose that A is a digit between 0 and 9, inclusive, and that the tens digit of the product of $2A7$ and 39 is 9. Determine the digit A.

Solution by Florencio Cano Vargas, Inca, Spain.

We write $2A7 = 2 \cdot 10^2 + A \cdot 10 + 7$ and $39 = 3 \cdot 10 + 9$. Multiplying both numbers and grouping we get:

$$2A7 \cdot 39 = 8 \cdot 10^3 + 3A \cdot 10^2 + (9A + 7) \cdot 10 + 3\,.$$

The condition stated in the problem implies that $9A + 7 \equiv 9 \pmod{10}$ which implies that $9A \equiv 2 \pmod{10}$. Hence, the solution is $A = 8$.

Also solved by JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; LUIZ ERNESTO LEITÃO, Pará, Brazil; TRAVIS B. LITTLE, students, Angelo State University, San Angelo, TX, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia;

M458. *Proposed by the Mayhem Staff.*

Convex quadrilateral $ABCD$ has $AB = AD = 10$ and $BC = CD$. Also, AC is perpendicular to BD, with AC and BD intersecting at P. If $BP = 8$ and $CD = CP + 2$, determine the area of quadrilateral $ABCD$.

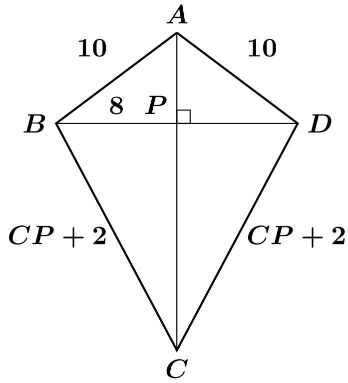
Solution by Ingesti Bilkis Zulpatina, student, SMPN 8, Yogyakarta, Indonesia.

From the properties which are written above, $ABCD$ is surely a kite since $AB = AD$, $BC = CD$, and $AC \perp BD$.

Using the Pythagorean theorem:

$$AP^2 = AB^2 - BP^2$$
$$AP^2 = 36$$
$$\therefore AP = 6$$

and



$$BP^2 + CP^2 = CB^2$$
$$64 + CP^2 = CP^2 + 4CP + 4$$
$$60 = 4CP$$
$$CP = 15$$

Hence $AC = AP + PC = 6 + 15 = 21$.

Thus the area of quadrilateral $ABCD$ is

$$[ABCD] = \frac{AC \times BD}{2} = \frac{21 \times 16}{2} = 168$$

square units.

Also solved by SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia.

M459. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine whether or not it is possible to create a collection of ten distinct subsets of $S = \{1, 2, 3, 4, 5, 6\}$ so that each subset contains three elements, each element of S appears in five subsets, and each pair of elements from S appears in two subsets.

Solution by Jaclyn Chang, student, University of Calgary, Calgary, AB.

It is possible to create ten distinct subsets of $S = \{1, 2, 3, 4, 5, 6\}$ such that each subset contains three elements, each element of S appears in five subsets, and each pair of elements from S appears in two subsets.

Each of the following distinct subsets contains three elements of S:

$S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2, 4\}$, $S_3 = \{1, 3, 5\}$, $S_4 = \{1, 4, 6\}$,
$S_5 = \{1, 5, 6\}$, $S_6 = \{2, 3, 6\}$, $S_7 = \{2, 4, 5\}$, $S_8 = \{2, 5, 6\}$,
$S_9 = \{3, 4, 5\}$, $S_{10} = \{3, 4, 6\}$.

Each element of S appears in five subsets of S:

Element 1 in S_1, S_2, S_3, S_4 ,S_5; Element 2 in S_1, S_2, S_6, S_7, S_8;
Element 3 in S_1, S_3, S_6, S_9, S_{10}; Element 4 in S_2, S_4, S_7, S_9, S_{10};
Element 5 in S_3, S_5, S_7, S_8, S_9; Element 6 in S_4, S_5, S_6, S_8, S_{10}.

Each pair of elements from S appears in two subsets of S:

$\{1,2\}$ in S_1, S_2; $\{2,3\}$ in S_1, S_6; $\{3,5\}$ in S_3, S_9; $\{1,3\}$ in S_1, S_3;
$\{2,4\}$ in S_2, S_7; $\{3,6\}$ in S_6, S_{10}; $\{1,4\}$ in S_2, S_4; $\{2,5\}$ in S_7, S_8;
$\{4,5\}$ in S_7, S_9; $\{1,5\}$ in S_3, S_5; $\{2,6\}$ in S_6, S_8; $\{4,6\}$ in S_4, S_{10};
$\{1,6\}$ in S_4, S_5; $\{3,4\}$ in S_9, S_{10}; $\{5,6\}$ in S_5, S_8.

Also solved by ALEX SONG, Detroit Country Day School, Detroit, MI, USA, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. One incomplete solution was received.

M460. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let a and b be positive real numbers. Define $A = \dfrac{a+b}{2}$, $G = \sqrt{ab}$, and $K = \sqrt{\dfrac{a^2+b^2}{2}}$. Prove that (a) $G^2 + K^2 = 2A^2$, (b) $A^2 \geq KG$, (c) $G + K \leq 2A$, and (d) $G^4 + K^4 \geq 2A^4$.

Solution by Jaclyn Chang, student, University of Calgary, Calgary, AB.

(a) By direct computation we get

$$G^2 + K^2 = ab + \frac{a^2+b^2}{2} = \frac{a^2+2ab+b^2}{2}$$
$$= \frac{(a+b)^2}{2} = 2\left(\frac{a+b}{2}\right)^2 = 2A^2\,.$$

(b) Since $(a-b)^4 \geq 0$ we get

$$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0$$
$$\Rightarrow a^4 + 4a^3b + 6a^2b^2 + 4ab^2 + b^4 \geq 8a^3b + 8ab^2$$
$$\Rightarrow \frac{(a+b)^4}{16} \geq \frac{a^3b+ab^3}{2} \Rightarrow \left(\frac{a+b}{2}\right)^4 \geq \frac{ab(a^2+b^2)}{2}$$
$$\Rightarrow \left(\frac{a+b}{2}\right)^2 \geq \left(\sqrt{ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}}\right) \Rightarrow A^2 \geq KG\,.$$

[*Ed.: Note that from the AM-GM inequality* $\frac{G^2+K^2}{2} \geq GK$. *Thus, using the result from (a) we get* $A^2 = \frac{G^2+K^2}{2} \geq GK$.]

(c) We have$(G+K)^2 = G^2 + 2KG + K^2$, but from (b) we know that $2KG \leq 2A^2$, thus $(G+K)^2 \leq G^2 + K^2 + 2A^2$. Using part (a) we can deduce $(G+K)^2 \leq 4A^2 = (2A)^2$ and therefore $G + K \leq 2A$.

(d) Since $(a-b)^4 \geq 0$ we have $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0$, hence

$$a^4 + 6a^2b^2 + b^4 \geq 4a^3b + 4ab^3 \Rightarrow \frac{a^4+6a^2b^2+b^4}{8} \geq \frac{4a^3b+4ab^3}{8}$$
$$\Rightarrow \frac{2a^4+12a^2b^2+2b^4}{8} \geq \frac{a^4+4a^3b+6a^2b^2+4ab^3+b^4}{8}$$
$$\Rightarrow a^2b^2 + \frac{a^4+2a^2b^2+b^4}{4} \geq 2\left(\frac{a+b}{2}\right)^4 \Rightarrow G^4 + K^4 \geq 2A^4\,.$$

[*Ed.: Note that from (a) and (b), we have* $G^4 + K^4 = (G^2 + K^2)^2 - 2K^2G^2 = \left(2A^2\right)^2 - 2K^2G^2 \geq 4A^4 - 2A^4 = 2A^4$.]

Also solved by MIHÁLY BENCZE, Brasov, Romania; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA(parts a, b, c); FLORENCIO CANO VARGAS, Inca, Spain; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia(parts a, c); LUIZ ERNESTO LEITÃO, Pará, Brazil(part a); RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia; and the proposer.

M461. *Proposed by Landelino Arboniés, Colegio Marcelino Champagnat, Santo Dominigo, Dominican Republic.*

A *Champagnat* number is equal to the sum of all the digits in a set of consecutive positive integers, one of which is the number itself. Thus, 42 is a Champagnat number, since 42 is the sum of all of the digits of 39, 40, 41, 42, 43, 44. Prove that there exist infinitely many Champagnat numbers.

Solution by the proposer.

We prove that for any $n > 6$ there is at least one Champagnat number with $n+1$ digits. Indeed, consider the number 10^n and suppose if is not a Champagnat number. Let k_n be the greatest number such that the digital sum of the numbers $10^n, 10^n+1, 10^n+2, \ldots, 10^n+k_n$ is less than 10^n. Consider now the number N equal to the digital sum of all the integers from 10^n to 10^n+k_n+1 inclusive. Now, since k_n is at least $\frac{10^n}{9(n+1)}$ (since each of the numbers is less than $10^{n+1}-1$ which has a digital sum of $9(n+1)$) and N is at most $10^n + 9(n+1)$ (only if $k_n + 1 = 10^{n+1} - 1$), then (if $n > 6$) N is one of the numbers between 10^n and $10^n + k_n + 1$ inclusive, and hence it is a Champagnat number being the sum of a set of consecutive numbers, one of which is itself.

No other solutions were received.

M462. *Proposed by Alex Song, Detroit Country Day School, Detroit, MI, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x and let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. For example, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor -1.4 \rfloor = -2$, and $\lceil -1.4 \rceil = -1$. Determine all real numbers x for which $\lfloor x \rfloor \lceil x \rceil = x^2$.

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

If $x = n \in \mathbb{Z}$, then $\lfloor x \rfloor = n$ and $\lceil x \rceil = n$. Hence, $\lfloor x \rfloor \lceil x \rceil = n^2 = x^2$ for all $x \in \mathbb{Z}$. If $x \notin \mathbb{Z}$ and $x > 0$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $n < x < n+1$. We can then conclude that $\lfloor x \rfloor = n$ and $\lceil x \rceil = n+1$. Consequently, $\lfloor x \rfloor \lceil x \rceil = n(n+1) = x^2$, hence $x = \sqrt{n(n+1)}$. If $x \notin \mathbb{Z}$ and $x < 0$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $-(n+1) < x < -n$. We can then conclude that $\lfloor x \rfloor = -(n+1)$ and $\lceil x \rceil = -n$. Consequently,

$\lfloor x \rfloor \lceil x \rceil = n(n+1) = x^2$, hence $x = -\sqrt{n(n+1)}$. Thus, the set of all real numbers for which $\lfloor x \rfloor \lceil x \rceil = x^2$ is $x = \pm n$ or $x = \pm\sqrt{n(n+1)}$, $n \in \mathbb{N} \cup \{0\}$.

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposers. Three incomplete solutions were received.

Problem of the Month

Ian VanderBurgh

Many problems that appear on contests are word problems. Here is a problem that appeared last year on a Scottish competition:

Problem (2010-2011 Scottish Mathematical Challenge) Katie had a collection of red, green and blue beads. She noticed that the number of beads of each colour was a prime number and that the numbers were all different. She also observed that if she multiplied the number of red beads by the total number of red and green beads she obtained a number exactly 120 greater than the number of blue beads. How many beads of each colour did she have?

Often, the first step with a word problem is to translate the words into mathematics. Since this problem is dealing with the numbers of red, green and blue beads, let's assign a variable to each of these numbers – say, r, g and b, respectively. (We'll write this up nicely in a minute.) These seem to be the relevant quantities.

We are next told that each of these quantities is a prime number. Let's make a mental note to come back to this, and keep reading. The fact that the product of the number of red beads with the sum of the numbers of red and green beads is 120 more than the number of blue beads translates into the equation $r(r+g) = 120 + b$.

Now, I seem to remember that usually when we have three variables, one equation is not enough to determine the values of the variables. (Often, we need three equations.) This is mildly concerning, but let's persevere to see what happens.

What information haven't we used? We haven't used the fact that each of r, g and b is a prime number. How can we use this information? Again, let's back up half a step. What do we know about prime numbers? It's good to check the definition first: a prime number is a positive integer larger than 1 (remember, 1 is not prime) that has no positive divisors other than 1 and itself. Is there a "formula" for prime numbers? There isn't a good one that we know. However, there are lots and lots of properties of prime numbers: all prime numbers other than 2 are odd, there are infinitely many prime numbers, every prime number