MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The interim Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School, Orleans, ON). The Assistant Mayhem Editor is Lynn Miller (Cairine Wilson Secondary School, Orleans, ON). The other staff members are Ann Arden (Osgoode Township District High School, Osgoode, ON), Nicole Diotte (Windsor, ON), Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Daphne Shani (Bell High School, Nepean, ON).

Editorial

Shawn Godin

Hello ***Mayhem*** readers. The CMS and the editors of ***CRUX with MAYHEM*** have been working on some changes for the journal. Over the last few months, we have set up a page on Facebook to communicate with our readers. We have surveyed the readers of ***CRUX with MAYHEM*** and have listened to their comments. One change that we are planning directly impacts the readers of ***Mayhem***.

As of volume 38, ***Mathematical Mayhem*** will no longer be part of ***Crux Mathematicorum***. It will continue to exist, but only on the web. The numbering and dates of the issues will revert back to the days before it joined ***Crux Mathematicorum***. ***Mathematical Mayhem*** will appear on line 5 times a year: September, November, January, March and May. The last volume of ***Mathematical Mayhem*** to appear as a stand alone was volume 8 [1995-1996]. The next stand alone volume starting in September 2012 will be volume 24 [2012-2013]. We will also be expanding and adding features that will be of interest to mathematics students and teachers at the high school level. The first issue will be here before you know it, so get ready!

Also, this issue marks the last installment of the Problem of the Month by Ian VanderBurgh. Ian has been writing this column since September 2004 [2004 : 264-265] (57 columns!). We will miss Ian's column, it has been a great part of ***Mayhem***. All the best Ian and thanks for sharing your passion of mathematical problem-solving with us.

Shawn Godin

Mayhem Problems

*Please send your solutions to the problems in this edition by **1 August 2012**. Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the problems from English into French.

M495. *Proposed by the Mayhem Staff.*

All possible lines are drawn through the point $(0,0)$ and the points (x,y), where x and y are whole numbers with $1 \leq x, y \leq 10$. How many distinct lines are drawn?

M496. *Proposed by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.*

Show that if we write the numbers from 1 to n around a circle, in any order, then, for all $x = 1, 2, \ldots, n$, we are guaranteed to find a block of x consecutive numbers that add up to at least $\left\lceil \dfrac{x(n+1)}{2} \right\rceil$. Here $\lceil y \rceil$ is the ceiling function, that is, the least integer greater than or equal to y. So $\lceil 6.2 \rceil = 7$, $\lceil \pi \rceil = 4$, $\lceil -8.3 \rceil = -8$ and $\lceil 10 \rceil = 10$.

M497. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

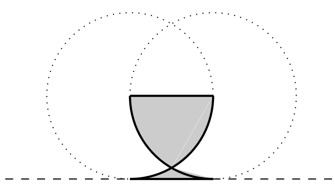
Find all integers a, b, c where c is a prime number such that $a^b + c$ and $a^b - c$ are both perfect squares.

M498. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Right triangle ABC has its right angle at C. The two sides CB and CA are of integer length. Determine the condition for the radius of the incircle of triangle ABC to be a rational number.

M499. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Two circles of radius 1 are drawn so that each circle passes through the centre of the other circle. Find the area of the goblet like region contained between the common radius, the circumferences and one of the common tangents as shown in the diagram to the right.



M500. *Proposed by Edward T.H. Wang and Dexter S.Y. Wei, Wilfrid Laurier University, Waterloo, ON.*

Let $\mathbb{N}$ denote the set of natural numbers.

(a) Show that if $n \in \mathbb{N}$, there do not exist $a, b \in \mathbb{N}$ such that $\dfrac{[a,b]}{a+b} = n$, where $[a,b]$ denotes the least common multiple of a and b.

(b) Show that for any $n \in \mathbb{N}$, there exists infinitely many triples (a,b,c) of natural numbers such that $\dfrac{[a,b,c]}{a+b+c} = n$, where $[a,b,c]$ denotes the least common multiple of a, b and c.

. . . .

M495. *Proposé par l'Équipe de Mayhem.*

Toutes les droites possibles sont tracées à partir du point $(0,0)$ et des points (x,y), où x et y sont des entiers tels que $1 \leq x, y \leq 10$. Combien de droites distinctes ont été tracées ?

M496. *Proposé par Sally Li, Institut collégial Marc Garneau, Toronto ON.*

Démontrer que si on place tous les entiers de 1 à n autour d'un cercle dans un ordre quelconque alors, pour tout $x = 1, 2, \ldots, n$, on pourra certainement trouver un bloc de x nombres consécutifs dont la somme sera d'au moins $\left\lceil \dfrac{x(n+1)}{2} \right\rceil$, $\lceil y \rceil$ désignant la partie "plafond" de y, c'est-à-dire le plus petit entier plus grand ou égal à y. Ainsi $\lceil 6.2 \rceil = 7$, $\lceil \pi \rceil = 4$, $\lceil -8.3 \rceil = -8$ and $\lceil 10 \rceil = 10$.

M497. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

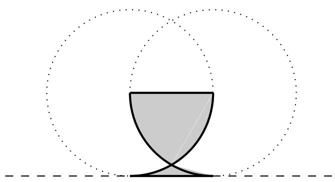
Déterminer tous les entiers a, b et c où c est un nombre premier tel que $a^b + c$ et $a^b - c$ sont tous les deux des carrés d'entiers.

M498. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Le triangle rectangle ABC a l'angle droit à C ; les deux côtés CB et CA sont de longueurs entières. Déterminer la condition pour que le rayon du cercle inscrit du triangle ABC soit un nombre rationnel.

M499. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Deux cercles de rayon 1 sont tracés de façon à ce que chacun passe par le centre de l'autre. Déterminer la surface de la région en forme de gobelet qui se trouve entre le rayon commun, les circonférences et une des tangentes communes, tel qu'illustré à droite.



M500. *Proposé par Edward T.H. Wang et Dexter S.Y. Wei, Université Wilfrid Laurier, Waterloo, ON.*

Soit $\mathbb{N}$ l'ensemble des nombres naturels.

(a) Démontrer que si $n \in \mathbb{N}$ alors il n'existe aucun $a, b \in \mathbb{N}$ tels que $\dfrac{[a,b]}{a+b} = n$, où $[a, b]$ dénote le plus petit commun multiple de a et b.

(b) Démontrer que si $n \in \mathbb{N}$ alors il existe un nombre infini de triplets (a, b, c) d'entiers naturels tels que $\dfrac{[a,b,c]}{a+b+c} = n$, où $[a, b, c]$ dénote le plus petit commun multiple de a, b et c.

Mayhem Solutions

M457. *Proposed by the Mayhem Staff.*

Suppose that A is a digit between 0 and 9, inclusive, and that the tens digit of the product of $2A7$ and 39 is 9. Determine the digit A.

Solution by Florencio Cano Vargas, Inca, Spain.

We write $2A7 = 2 \cdot 10^2 + A \cdot 10 + 7$ and $39 = 3 \cdot 10 + 9$. Multiplying both numbers and grouping we get:

$$2A7 \cdot 39 = 8 \cdot 10^3 + 3A \cdot 10^2 + (9A + 7) \cdot 10 + 3\,.$$

The condition stated in the problem implies that $9A + 7 \equiv 9 \pmod{10}$ which implies that $9A \equiv 2 \pmod{10}$. Hence, the solution is $A = 8$.

Also solved by JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; LUIZ ERNESTO LEITÃO, Pará, Brazil; TRAVIS B. LITTLE, students, Angelo State University, San Angelo, TX, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia;

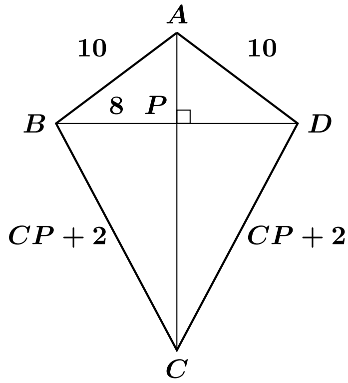
M458. *Proposed by the Mayhem Staff.*

Convex quadrilateral $ABCD$ has $AB = AD = 10$ and $BC = CD$. Also, AC is perpendicular to BD, with AC and BD intersecting at P. If $BP = 8$ and $CD = CP + 2$, determine the area of quadrilateral $ABCD$.

Solution by Ingesti Bilkis Zulpatina, student, SMPN 8, Yogyakarta, Indonesia.

From the properties which are written above, $ABCD$ is surely a kite since $AB = AD$, $BC = CD$, and $AC \perp BD$.

Using the Pythagorean theorem:



$$\begin{aligned} AP^2 &= AB^2 - BP^2 \\ AP^2 &= 36 \\ \therefore AP &= 6 \end{aligned}$$

and

$$\begin{aligned} BP^2 + CP^2 &= CB^2 \\ 64 + CP^2 &= CP^2 + 4CP + 4 \\ 60 &= 4CP \\ CP &= 15 \end{aligned}$$

Hence $AC = AP + PC = 6 + 15 = 21$.

Thus the area of quadrilateral $ABCD$ is

$$[ABCD] = \frac{AC \times BD}{2} = \frac{21 \times 16}{2} = 168$$

square units.

Also solved by SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia.

M459. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine whether or not it is possible to create a collection of ten distinct subsets of $S = \{1, 2, 3, 4, 5, 6\}$ so that each subset contains three elements, each element of S appears in five subsets, and each pair of elements from S appears in two subsets.

Solution by Jaclyn Chang, student, University of Calgary, Calgary, AB.

It is possible to create ten distinct subsets of $S = \{1, 2, 3, 4, 5, 6\}$ such that each subset contains three elements, each element of S appears in five subsets, and each pair of elements from S appears in two subsets.

Each of the following distinct subsets contains three elements of S:

$S_1 = \{1,2,3\}$, $S_2 = \{1,2,4\}$, $S_3 = \{1,3,5\}$, $S_4 = \{1,4,6\}$,
$S_5 = \{1,5,6\}$, $S_6 = \{2,3,6\}$, $S_7 = \{2,4,5\}$, $S_8 = \{2,5,6\}$,
$S_9 = \{3,4,5\}$, $S_{10} = \{3,4,6\}$.

Each element of S appears in five subsets of S:

Element 1 in S_1, S_2, S_3, S_4 ,S_5; Element 2 in S_1, S_2, S_6, S_7, S_8;
Element 3 in S_1, S_3, S_6, S_9, S_{10}; Element 4 in S_2, S_4, S_7, S_9, S_{10};
Element 5 in S_3, S_5, S_7, S_8, S_9; Element 6 in S_4, S_5, S_6, S_8, S_{10}.

Each pair of elements from S appears in two subsets of S:

$\{1,2\}$ in S_1, S_2; $\{2,3\}$ in S_1, S_6; $\{3,5\}$ in S_3, S_9; $\{1,3\}$ in S_1, S_3;
$\{2,4\}$ in S_2, S_7; $\{3,6\}$ in S_6, S_{10}; $\{1,4\}$ in S_2, S_4; $\{2,5\}$ in S_7, S_8;
$\{4,5\}$ in S_7, S_9; $\{1,5\}$ in S_3, S_5; $\{2,6\}$ in S_6, S_8; $\{4,6\}$ in S_4, S_{10};
$\{1,6\}$ in S_4, S_5; $\{3,4\}$ in S_9, S_{10}; $\{5,6\}$ in S_5, S_8.

Also solved by ALEX SONG, Detroit Country Day School, Detroit, MI, USA, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. One incomplete solution was received.

M460. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let a and b be positive real numbers. Define $A = \dfrac{a+b}{2}$, $G = \sqrt{ab}$, and $K = \sqrt{\dfrac{a^2+b^2}{2}}$. Prove that (a) $G^2 + K^2 = 2A^2$, (b) $A^2 \geq KG$, (c) $G + K \leq 2A$, and (d) $G^4 + K^4 \geq 2A^4$.

Solution by Jaclyn Chang, student, University of Calgary, Calgary, AB.

(a) By direct computation we get

$$\begin{aligned} G^2 + K^2 &= ab + \frac{a^2+b^2}{2} = \frac{a^2+2ab+b^2}{2} \\ &= \frac{(a+b)^2}{2} = 2\left(\frac{a+b}{2}\right)^2 = 2A^2\,. \end{aligned}$$

(b) Since $(a-b)^4 \geq 0$ we get

$$\begin{aligned} & a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0 \\ \Rightarrow\ & a^4 + 4a^3b + 6a^2b^2 + 4ab^2 + b^4 \geq 8a^3b + 8ab^2 \\ \Rightarrow\ & \frac{(a+b)^4}{16} \geq \frac{a^3b+ab^3}{2} \Rightarrow \left(\frac{a+b}{2}\right)^4 \geq \frac{ab(a^2+b^2)}{2} \\ \Rightarrow\ & \left(\frac{a+b}{2}\right)^2 \geq \left(\sqrt{ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}}\right) \Rightarrow A^2 \geq KG\,. \end{aligned}$$

[*Ed.: Note that from the AM-GM inequality* $\frac{G^2+K^2}{2} \geq GK$. *Thus, using the result from (a) we get* $A^2 = \frac{G^2+K^2}{2} \geq GK$.]

(c) We have$(G+K)^2 = G^2 + 2KG + K^2$, but from (b) we know that $2KG \leq 2A^2$, thus $(G+K)^2 \leq G^2 + K^2 + 2A^2$. Using part (a) we can deduce $(G+K)^2 \leq 4A^2 = (2A)^2$ and therefore $G + K \leq 2A$.

(d) Since $(a-b)^4 \geq 0$ we have $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0$, hence

$$\begin{aligned} & a^4 + 6a^2b^2 + b^4 \geq 4a^3b + 4ab^3 \Rightarrow \frac{a^4+6a^2b^2+b^4}{8} \geq \frac{4a^3b+4ab^3}{8} \\ \Rightarrow\ & \frac{2a^4+12a^2b^2+2b^4}{8} \geq \frac{a^4+4a^3b+6a^2b^2+4ab^3+b^4}{8} \\ \Rightarrow\ & a^2b^2 + \frac{a^4+2a^2b^2+b^4}{4} \geq 2\left(\frac{a+b}{2}\right)^4 \Rightarrow G^4 + K^4 \geq 2A^4\,. \end{aligned}$$

[*Ed.: Note that from (a) and (b), we have* $G^4 + K^4 = (G^2 + K^2)^2 - 2K^2G^2 = \left(2A^2\right)^2 - 2K^2G^2 \geq 4A^4 - 2A^4 = 2A^4$.]

Also solved by MIHÁLY BENCZE, Brasov, Romania; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA(parts a, b, c); FLORENCIO CANO VARGAS, Inca, Spain; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia(parts a, c); LUIZ ERNESTO LEITÃO, Pará, Brazil(part a); RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia; and the proposer.

M461. *Proposed by Landelino Arboniés, Colegio Marcelino Champagnat, Santo Dominigo, Dominican Republic.*

A *Champagnat* number is equal to the sum of all the digits in a set of consecutive positive integers, one of which is the number itself. Thus, 42 is a Champagnat number, since 42 is the sum of all of the digits of 39, 40, 41, 42, 43, 44. Prove that there exist infinitely many Champagnat numbers.

Solution by the proposer.

We prove that for any $n > 6$ there is at least one Champagnat number with $n+1$ digits. Indeed, consider the number 10^n and suppose if is not a Champagnat number. Let k_n be the greatest number such that the digital sum of the numbers $10^n, 10^n+1, 10^n+2, \ldots, 10^n+k_n$ is less than 10^n. Consider now the number N equal to the digital sum of all the integers from 10^n to 10^n+k_n+1 inclusive. Now, since k_n is at least $\frac{10^n}{9(n+1)}$ (since each of the numbers is less than $10^{n+1}-1$ which has a digital sum of $9(n+1)$) and N is at most $10^n + 9(n+1)$ (only if $k_n + 1 = 10^{n+1} - 1$), then (if $n > 6$) N is one of the numbers between 10^n and $10^n + k_n + 1$ inclusive, and hence it is a Champagnat number being the sum of a set of consecutive numbers, one of which is itself.

No other solutions were received.

M462. *Proposed by Alex Song, Detroit Country Day School, Detroit, MI, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x and let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. For example, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor -1.4 \rfloor = -2$, and $\lceil -1.4 \rceil = -1$. Determine all real numbers x for which $\lfloor x \rfloor \lceil x \rceil = x^2$.

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

If $x = n \in \mathbb{Z}$, then $\lfloor x \rfloor = n$ and $\lceil x \rceil = n$. Hence, $\lfloor x \rfloor \lceil x \rceil = n^2 = x^2$ for all $x \in \mathbb{Z}$. If $x \notin \mathbb{Z}$ and $x > 0$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $n < x < n+1$. We can then conclude that $\lfloor x \rfloor = n$ and $\lceil x \rceil = n+1$. Consequently, $\lfloor x \rfloor \lceil x \rceil = n(n+1) = x^2$, hence $x = \sqrt{n(n+1)}$. If $x \notin \mathbb{Z}$ and $x < 0$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $-(n+1) < x < -n$. We can then conclude that $\lfloor x \rfloor = -(n+1)$ and $\lceil x \rceil = -n$. Consequently,

$\lfloor x \rfloor \lceil x \rceil = n(n+1) = x^2$, hence $x = -\sqrt{n(n+1)}$. Thus, the set of all real numbers for which $\lfloor x \rfloor \lceil x \rceil = x^2$ is $x = \pm n$ or $x = \pm\sqrt{n(n+1)}$, $n \in \mathbb{N} \cup \{0\}$.

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposers. Three incomplete solutions were received.

Problem of the Month

Ian VanderBurgh

Many problems that appear on contests are word problems. Here is a problem that appeared last year on a Scottish competition:

Problem (2010-2011 Scottish Mathematical Challenge) Katie had a collection of red, green and blue beads. She noticed that the number of beads of each colour was a prime number and that the numbers were all different. She also observed that if she multiplied the number of red beads by the total number of red and green beads she obtained a number exactly 120 greater than the number of blue beads. How many beads of each colour did she have?

Often, the first step with a word problem is to translate the words into mathematics. Since this problem is dealing with the numbers of red, green and blue beads, let's assign a variable to each of these numbers – say, r, g and b, respectively. (We'll write this up nicely in a minute.) These seem to be the relevant quantities.

We are next told that each of these quantities is a prime number. Let's make a mental note to come back to this, and keep reading. The fact that the product of the number of red beads with the sum of the numbers of red and green beads is 120 more than the number of blue beads translates into the equation $r(r+g) = 120 + b$.

Now, I seem to remember that usually when we have three variables, one equation is not enough to determine the values of the variables. (Often, we need three equations.) This is mildly concerning, but let's persevere to see what happens.

What information haven't we used? We haven't used the fact that each of r, g and b is a prime number. How can we use this information? Again, let's back up half a step. What do we know about prime numbers? It's good to check the definition first: a prime number is a positive integer larger than 1 (remember, 1 is not prime) that has no positive divisors other than 1 and itself. Is there a "formula" for prime numbers? There isn't a good one that we know. However, there are lots and lots of properties of prime numbers: all prime numbers other than 2 are odd, there are infinitely many prime numbers, every prime number greater than $\mathbf{3}$ is either one more or one less than a multiple of $\mathbf{6}$... The list goes on! Many mathematicians spend much of their professional lives investigating properties of prime numbers.

Given such a vast number to choose from, how do we know what properties to use? Therein lies the essence of problem solving! Figuring this out is not always easy.

Here's a solution to the problem.

Solution Suppose that $\boldsymbol{r}$, $\boldsymbol{g}$ and $\boldsymbol{b}$ are the numbers of red, green and blue beads, respectively. We are told that each of $\boldsymbol{r}$, $\boldsymbol{g}$ and $\boldsymbol{b}$ is a prime number and that $\boldsymbol{r(r+g) = 120 + b}$.

Let's focus on the fact that the only even prime number is $\mathbf{2}$ and on the parity of the two sides of the equation. (Remember, checking parity means checking to see if an integer is even or odd.) If both $\boldsymbol{r}$ and $\boldsymbol{g}$ are odd, then $\boldsymbol{r+g}$ is even, so the left side of the equation is even, which means the right side is even. If $\boldsymbol{120+b}$ is even, then $\boldsymbol{b}$ is even, which means that $\boldsymbol{b = 2}$. In this case, $\boldsymbol{r(r+g) = 122}$. Since $\boldsymbol{122 = 2 \times 61}$ and each of $\mathbf{2}$ and $\mathbf{61}$ is prime, then we must have $\boldsymbol{r = 2}$ or $\boldsymbol{r+g = 2}$. Neither of these is possible, since $\boldsymbol{r}$ cannot equal $\boldsymbol{b}$ (since $\boldsymbol{r}$, $\boldsymbol{b}$ and $\boldsymbol{g}$ are all different) and since $\boldsymbol{r+g}$ is at least $\mathbf{4}$.

Also, since $\mathbf{2}$ is the only even prime number, then $\boldsymbol{r}$ and $\boldsymbol{g}$ can't both be even, since then they would both be $\mathbf{2}$, which would contradict the given hypothesis that $\boldsymbol{r}$, $\boldsymbol{b}$ and $\boldsymbol{g}$ are all different.

Therefore, $\boldsymbol{r}$ and $\boldsymbol{g}$ are even and odd in some order. In other words, one of $\boldsymbol{r}$ and $\boldsymbol{g}$ equals 2 and the other is an odd prime number. Which is which? If $\boldsymbol{r = 2}$, then the equation becomes $\boldsymbol{2(2+g) = 120 + b}$. Since the left side is even, then the right side is even too, so again $\boldsymbol{b = 2}$ which is impossible since our assumption is that $\boldsymbol{r = 2}$.

Thus, it must be the case that $\boldsymbol{g = 2}$ and $\boldsymbol{r}$ is an odd prime. This gives $\boldsymbol{r(r+2) = 120 + b}$.

So we've got one value ($\boldsymbol{g = 2}$), but still have one equation and two unknowns. What to do?

Let's try solving for $\boldsymbol{b}$, which gives $\boldsymbol{b = r^2 + 2r - 120}$. At this point, it might occur to try to factor the right side to obtain $\boldsymbol{b = (r+12)(r-10)}$.

How does this help? Since $\boldsymbol{b}$ is a prime number, then it can't be factored in many ways! Aha – that is probably useful. If $\boldsymbol{b}$ is a prime number that is written as the product of two integers, then one of the factors is either $\mathbf{1}$ or $\mathbf{-1}$. This gives us four possibilities to check ($\boldsymbol{r+12}$ equals $\mathbf{1}$ or $\mathbf{-1}$ and $\boldsymbol{r-10}$ equals $\mathbf{1}$ or $\mathbf{-1}$). The only one that yields a positive value of $\boldsymbol{r}$ that is a prime number is $\boldsymbol{r - 10 = 1}$, giving $\boldsymbol{r = 11}$. In this case, $\boldsymbol{b = (11+12)(11-10) = 23}$, which is (thankfully) a prime number.

Therefore, $\boldsymbol{g = 2}$, $\boldsymbol{r = 11}$ and $\boldsymbol{g = 23}$. We can check that these satisfy the original hypotheses. □