Summations according to Gauss

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A well-known anecdote relates that when Carl Friedrich Gauss (1777–1855) was only ten years old, his school teacher wanted to keep the pupils busy and asked them to add up all the integers from 1 to 100. Almost immediately Gauss placed his slate on the table and said "There it is." The slate just contained the number 5050 without further calculations. When the teacher finally checked the results, Gauss's slate was the only one with the correct answer.

If there is any truth in this anecdote, then the young Gauss must have paired up the integers in the following (or some closely related) way. In his mind, he wrote down the summation twice: once in the standard fashion from left to right, and once he flipped it around and wrote it from right to left.

$$\begin{array}{ccccccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & \cdots\cdots & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & 97 & + & \cdots\cdots & + & 2 & + & 1 \end{array}$$

This yielded 100 vertically aligned pairs, where the numbers in each pair added up to 101. Hence the sum of all listed numbers was $100 \cdot 101$, and as each number was listed twice the answer desired by Gauss's teacher was $\frac{1}{2} \cdot 100 \cdot 101 = 5050$.

In this article we will discuss several related problems that all can be settled by this *"write it once down left-to-right and once right-to-left"* approach of Gauss. For warming up, the reader may want to generalize the above calculation to an arbitrary number of terms.

Problem 1 *Determine a closed form expression for* $1 + 2 + 3 + 4 + \cdots + n$.

Our next problem is a standard textbook exercise in the manipulation of sums of binomial coefficients.

Problem 2 *Determine a closed form expression for the following sum* S*:*

$$S \;=\; 1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 3 \cdot \binom{n}{2} + \;\cdots\; + (n+1) \cdot \binom{n}{n} \qquad (2)$$

We present a solution that is based on the trick of Gauss. We write down the sum once again, but in reversed order with its terms taken from right to left:

$$S \;=\; (n+1) \cdot \binom{n}{n} + n \cdot \binom{n}{n-1} + (n-1) \cdot \binom{n}{n-2} + \;\cdots\; + 1 \cdot \binom{n}{0} \quad (3)$$

This yields $n + 1$ vertically aligned pairs, where the kth pairs consists of the kth term in (2) and the kth term in (3). By applying the well-known relation

$\binom{n}{\ell} = \binom{n}{n-\ell}$ with $\ell = k-1$, we derive that the terms in the kth pair add up to

$$k \cdot \binom{n}{k-1} + (n-k+2) \cdot \binom{n}{n-k+1} \;=\; (n+2) \cdot \binom{n}{k-1}. \tag{4}$$

Although the resulting value in the right hand side of (4) still depends on the parameter k, we have made substantial progress. By using (4), we see that the sum of all terms listed in (2) and (3) is

$$2S \;=\; (n+2) \cdot \left\{ \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \right\} \;=\; (n+2) \cdot 2^n.$$

Here we used the fact that the binomial coefficients in the nth row of Pascal's triangle add up to 2^n. Hence the answer to this problem is $S = (n+2) \cdot 2^{n-1}$.

The following problem can be settled by a very similar argument. The reader is encouraged to verify that the answer is $T_n = n$.

Problem 3 *For* $n \geq 1$ *evaluate* $T_n = \sum_{k=0}^{2n} k \cdot \cos\left(\frac{k\pi}{2n}\right)$.

Now let us turn to a summation problem from the 2000 Asian Pacific Mathematical Olympiad (APMO'2000).

Problem 4 *Compute the sum* $\sum_{k=0}^{101} \frac{x_k^3}{3x_k^2 - 3x_k + 1}$ *with* $x_k = k/101$ *for* $k = 0, \ldots, 101$.

Every single term in this summation is bulky, and there are lots of bulky terms that must be added up. Computing this sum by hand is certainly not a good idea. Let us follow Gauss and let us pair up the terms in the given sum with the terms in reversed order: term 0 is paired with term 101, term 1 with term 100, term 2 with term 99, and so on. Observe that in every resulting pair we have $x_k + x_{101-k} = k/101 + (101-k)/101 = 1$, which implies

$$\begin{aligned} 3x_k^2 - 3x_k + 1 &= (1-x_k)^3 + x_k^3 \\ &= x_{101-k}^3 + (1-x_{101-k})^3 \;=\; 3x_{101-k}^2 - 3x_{101-k} + 1. \end{aligned}$$

Adding up the term for x_k and the term for x_{101-k} then yields

$$\begin{aligned} &\frac{x_k^3}{3x_k^2 - 3x_k + 1} + \frac{x_{101-k}^3}{3x_{101-k}^2 - 3x_{101-k} + 1} \;= \\ &\quad = \frac{x_k^3}{(1-x_k)^3 + x_k^3} + \frac{x_{101-k}^3}{(1-x_k)^3 + x_k^3} \;=\; \frac{x_k^3 + (1-x_k)^3}{(1-x_k)^3 + x_k^3} \;=\; 1. \end{aligned}$$

Since altogether there are 102 pairs, we see that twice the value of the sum equals 102, and that hence the answer to the APMO problem is 51. The bulky summation can actually be performed in a routine fashion!

The following problem was posed as problem A3 on the 1980 William Lowell Putnam Mathematics Competition.

Problem 5 *Evaluate* $I = \int_{x=0}^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$.

The integrand in this problem looks horrible. A little bit of playing around soon confirms our impression that it is hopeless to search for an antiderivative in closed form. But remember that a definite integral is really just some kind of fancy summation, as it is obtained by adding up lots of very small numbers. Hence let us try to apply the trick of young Gauss. We write the integral down once again, but we flip it over so that this time we integrate from right to left. This flipping over operation corresponds to the substitution $y = \pi/2 - x$ with $dy = -dx$. We derive:

$$I = \int_{y=\pi/2}^{0} \frac{-dy}{1+(\tan(\pi/2-y))^{\sqrt{2}}} = \int_{y=0}^{\pi/2} \frac{dy}{1+(\tan(\pi/2-y))^{\sqrt{2}}}. \qquad (5)$$

We recall the trigonometric identity $\tan(\pi/2-\alpha) = \cot(\alpha)$, and with its help we rewrite the integrand in (5) as

$$\frac{1}{1+(\tan(\pi/2-y))^{\sqrt{2}}} = \frac{1}{1+(\cot y)^{\sqrt{2}}} = \frac{(\tan y)^{\sqrt{2}}}{(\tan y)^{\sqrt{2}}+1}. \qquad (6)$$

Now (5) and (6) imply

$$\begin{aligned} 2I &= \int_{x=0}^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}} + \int_{y=0}^{\pi/2} \frac{(\tan y)^{\sqrt{2}}}{(\tan y)^{\sqrt{2}}+1}\, dy \\ &= \int_{x=0}^{\pi/2} \frac{1+(\tan x)^{\sqrt{2}}}{1+(\tan x)^{\sqrt{2}}}\, dx = \int_{x=0}^{\pi/2} dx = \pi/2. \end{aligned}$$

Therefore the answer to this Putnam problem is $I = \pi/4$. Note that the exponent $\sqrt{2}$ does not play any special role in our calculations. If we replace it by an arbitrary positive real number, the answer will still remain $\pi/4$.

The same integration theme resurfaced seven years later as problem B1 on the 1987 Putnam exam. The reader should have little difficulty in finding the right substitution for the following problem, which leads to the answer $J = 1$.

Problem 6 *Evaluate* $J = \int\limits_{x=2}^{4} \frac{\sqrt{\ln(9-x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$.

The occurrence of the function $\sqrt{\ln x}$ in this problem is purely artificial. The solution virtually remains the same, if $\sqrt{\ln x}$ is replaced by any integrable function $f(x)$ for which $f(9-x) + f(x+3) \neq 0$ for $2 \leq x \leq 4$.

Finally we want to discuss problem A4 from the 1999 Putnam exam.

Problem 7 *Sum the series* $S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m\,(n3^m + m3^n)}$.

Let us flip over the summation so that m becomes n and simultaneously n becomes m. Of course this does not change the value of the sum, and (similarly as before) we get

$$2S \;=\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m\,(n3^m + m3^n)} \;+\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2 m}{3^n\,(m3^n + n3^m)}. \qquad (7)$$

We pair up terms (similarly as before), and by some simple algebra derive

$$\frac{m^2 n}{3^m\,(n3^m + m3^n)} + \frac{n^2 m}{3^n\,(m3^n + n3^m)} \;=\; \frac{m\,n}{3^m\,3^n}. \qquad (8)$$

The right hand side of (8) indicates that it might be useful to investigate the auxiliary sum $T = \sum_{m=1}^{\infty} m/3^m$. Since

$$3T \;=\; \sum_{m=1}^{\infty} \frac{m}{3^{m-1}} \;=\; \sum_{m=0}^{\infty} \frac{m+1}{3^m} \;=\; T + \sum_{m=0}^{\infty} \frac{1}{3^m} \;=\; T + \frac{3}{2},$$

we conclude $T = 3/4$. By combining this with (7) and (8) we derive (similarly as before)

$$2S \;=\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m\,n}{3^m\,3^n} \;=\; \sum_{m=1}^{\infty} \frac{m}{3^m} \cdot \sum_{n=1}^{\infty} \frac{n}{3^n} \;=\; T^2 \;=\; \frac{9}{16}.$$

Thus the final answer to our final problem is $S = 9/32$.

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