SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3224. [2007 : 112, 115; 2008 : 121- 124] *Proposed by J. Chris Fisher and Harley Weston, University of Regina, Regina, SK.*

Let $A_0B_0C_0$ be an isosceles triangle whose apex angle A_0 is not 120°. We define a sequence of triangles $A_nB_nC_n$ in which $\triangle A_{i+1}B_{i+1}C_{i+1}$ is obtained from $\triangle A_iB_iC_i$ by reflecting each vertex in the opposite side (that is, B_iC_i is the perpendicular bisector of A_iA_{i+1}, and so forth). Prove that all three angles approach 60° as $n \to \infty$.

[*Ed:* This problem is a special case of an open problem described by Judah Schwartz in "Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible?", *International Journal of Computers for Mathematical Learning*, **4** (1999), pp. 99–119.]

II. Solution by Grégoire Nicollier, University of Applied Sciences of Western Switzerland, Sion, Switzerland.

In his paper [1] Nicollier provides another solution to our problem (which is restricted to isosceles triangles). In addition he resolves Schwartz's open problem by describing those triangles for which iterating the reflection map produces a sequence of triangles whose limit is equilateral, whose limit is degenerate, and whose limit is neither equilateral nor degenerate.

References

[1] Grégoire Nicollier, Reflection triangles and their iterates, **Forum Geometricorum**, 12 (2012) 83–128.

3551. [2010 : 314, 316] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $p \geq 2$ be an integer. Find the product

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{\lfloor \sqrt[p]{n} \rfloor}\right)^{(-1)^{n-1}},$$

where $\lfloor a \rfloor$ is the greatest integer not exceeding a.

Solution by Joel Schlosberg, Bayside, NY, USA.

There are an equal number of odd and even integers in the interval $[m^p + 1, (m+1)^p - 1]$, so $\sum\limits_{n=m^p}^{(m+1)^p-1} (-1)^{n-1} = (-1)^{m^p-1} = (-1)^{m-1}$.

Therefore,

$$\prod_{n=1}^{N}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}}$$
$$=\prod_{m=1}^{\lfloor\sqrt[p]{N}\rfloor-1}\ \prod_{n=m^p}^{(m+1)^p-1}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}}\cdot\prod_{n=\lfloor\sqrt[p]{N}\rfloor^p}^{N}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}}$$
$$=\prod_{m=1}^{\lfloor\sqrt[p]{N}\rfloor-1}\left(1+\frac{1}{m}\right)^{\sum_{n=m^p}^{(m+1)^p-1}(-1)^{n-1}}\cdot\left(1+\frac{1}{\lfloor\sqrt[p]{N}\rfloor}\right)^{\sum_{n=\lfloor\sqrt[p]{N}\rfloor^p}^{N}(-1)^{n-1}}$$
$$=\prod_{m=1}^{\lfloor\sqrt[p]{N}\rfloor-1}\left(\frac{m+1}{m}\right)^{(-1)^{m-1}}\cdot\left(1+O\left(\frac{1}{\lfloor\sqrt[p]{N}\rfloor}\right)\right).$$

Letting $N\to\infty$ and using Wallis' product,

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}}=\prod_{m=1}^{\infty}\left(\frac{m+1}{m}\right)^{(-1)^{m-1}}$$
$$=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots=\frac{\pi}{2}.$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; NEVEN JURIČ, Zagreb, Croatia; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incorrect solution was submitted.

3552. [2010 : 314, 316] *Proposed by N. Javier Buitrago Aza, Universidad Nacional de Colombia, Bogota, Colombia.*

Let θ be a real number. Prove that

$$\sum_{k=0}^{n-1}\frac{\sin\left(\frac{2k\pi}{n}-\theta\right)}{3+2\cos\left(\frac{2k\pi}{n}-\theta\right)}=\frac{(-1)^n n\sin(n\theta)}{5F_n^2+4(-1)^n\sin^2\left(\frac{n\theta}{2}\right)},$$

where F_n denotes the n^{th} Fibonacci number.

Solution by Michel Bataille, Rouen, France, modified and expanded by the editor.

First, we have, by the well known Binet's formula that $\sqrt{5}F_n=\alpha^n-\beta^n$ where $\alpha=\frac{1}{2}(1+\sqrt{5})$ and $\beta=\frac{1}{2}(1-\sqrt{5})$. Note that $\alpha\beta=-1$ so $5F_n^2=\alpha^{2n}+\beta^{2n}-2(-1)^n$. Also, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=3$.

Now, for real x, let $z = e^{-ix}$. Then it is readily checked that

$$\begin{aligned}\frac{\sin x}{3+2\cos x} &= \frac{1}{2i}\cdot\frac{e^{ix}-e^{-ix}}{3+e^{ix}+e^{-ix}} = \frac{1}{2i}\cdot\frac{z^{-1}-z}{3+z^{-1}+z} = \frac{1}{2i}\cdot\frac{1-z^2}{z^2+3z+1}\\ &= \frac{1}{2i}\left(\frac{3z+2}{z^2+3z+1}-1\right) = \frac{1}{2i}\left(\frac{2+(\alpha^2+\beta^2)z}{z^2+(\alpha^2+\beta^2)z+1}-1\right)\\ &= \frac{1}{2i}\left(\frac{1}{1+\alpha^2 z}+\frac{1}{1+\beta^2 z}-1\right).\end{aligned}$$

Letting $w = e^{-\frac{2\pi i}{n}}$ and replacing x with $\frac{2k\pi}{n}-\theta$, $k = 0, 1, 2, \ldots, n-1$, we have $z = e^{-i\left(\frac{2k\pi}{n}-\theta\right)} = e^{i\theta}\cdot e^{-\frac{2k\pi i}{n}} = e^{i\theta}\cdot w^k$. Hence,

$$\frac{\sin x}{3+2\cos x} = \frac{1}{2i}\left(\frac{1}{1+\alpha^2 e^{i\theta}w^k}+\frac{1}{1+\beta^2 e^{i\theta}w^k}-1\right). \tag{1}$$

The identity below is known and can be verified easily by the method of partial fractions decomposition:

$$\frac{1}{t^n-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{w^k t-1}. \tag{2}$$

From (1) and (2) we have

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{\sin\left(\frac{2k\pi}{n}-\theta\right)}{3+2\cos\left(\frac{2k\pi}{n}-\theta\right)}\\ &= \frac{-1}{2i}\sum_{k=0}^{n-1}\left(\frac{1}{(-\alpha^2e^{i\theta})w^k-1}+\frac{1}{(-\beta^2e^{i\theta})w^k-1}+1\right)\\ &= \frac{n}{2i}\left(\frac{1}{1-(-1)^n\alpha^{2n}e^{in\theta}}+\frac{1}{1-(-1)^n\beta^{2n}e^{in\theta}}-1\right)\\ &= \frac{n}{2i}\left(\frac{1-e^{2in\theta}}{1-(-1)^n(\alpha^{2n}+\beta^{2n})e^{in\theta}+e^{2in\theta}}\right)\\ &= \frac{n}{2i}\cdot\frac{e^{-in\theta}-e^{in\theta}}{e^{-in\theta}-(-1)^n(\alpha^{2n}+\beta^{2n})+e^{in\theta}}\\ &= \frac{(-1)^n n\sin(n\theta)}{\alpha^{2n}+\beta^{2n}-2(-1)^n\cos(n\theta)}.\end{aligned}$$

The result now follows by substituting $\alpha^{2n}+\beta^{2n} = 2(-1)^n+5F_n^2$ and $\cos(n\theta) = 1-2\sin^2\left(\frac{n\theta}{2}\right)$.

Also solved by PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Both Bataille and Geupel pointed out that this problem (proposed by the same person) has appeared as problem U173 in Mathematical Reflections, 2010, issue 5. The solution featured above is different from the one that was published in issue 6.

3553. [2010 : 314, 317] *Proposed by Michel Bataille, Rouen, France.*

Let A, B, and C be the angles of a triangle. Prove that

$$\sum_{\text{cyclic}} \left(\sin A \cos \frac{B}{2} \cos \frac{C}{2}\right)^2 \ \leq \ \sum_{\text{cyclic}} \cos^6 \frac{A}{2}\,.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

Using basic trigonometric identities and the fact that $A + B + C = \pi$, we obtain

$$\begin{aligned}
&\sin A \cos \frac{B}{2} \cos \frac{C}{2} \\
&\quad= 2 \sin \frac{A}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \cos \frac{A}{2} \cos \frac{C}{2} \left[\sin \frac{A+B}{2} + \sin \frac{A-B}{2}\right] \\
&\quad= \frac{1}{2} \cos \frac{A}{2} \left[\sin \frac{A+B+C}{2} + \sin \frac{A+B-C}{2} + \sin \frac{A-B+C}{2}\right. \\
&\qquad\qquad\left. + \sin \frac{A-B-C}{2}\right] \\
&\quad= \frac{1}{2} \cos \frac{A}{2} \left[\sin \frac{\pi}{2} + \sin\left(\frac{\pi}{2} - C\right) + \sin\left(\frac{\pi}{2} - B\right) - \sin\left(\frac{\pi}{2} - A\right)\right] \\
&\quad= \frac{1}{2} \cos \frac{A}{2} [1 + \cos C + \cos B - \cos A] \\
&\quad= \frac{1}{2} \cos \frac{A}{2} \left[2\cos^2 \frac{C}{2} + 2\cos^2 \frac{B}{2} - 2\cos^2 \frac{A}{2}\right] \\
&\quad= \cos \frac{A}{2} \left[\cos^2 \frac{C}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{A}{2}\right] \ .
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(\sin A \cos \frac{B}{2} \cos \frac{C}{2}\right)^2 = \cos^6 \frac{A}{2} + \cos^2 \frac{A}{2} \Big[&\cos^4 \frac{B}{2} + \cos^4 \frac{C}{2} \\
&+ 2\cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 2\cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \\
&- 2\cos^2 \frac{A}{2} \cos^2 \frac{C}{2}\Big] \ ,
\end{aligned}$$

with similar expressions for $\left(\sin B \cos \frac{C}{2} \cos \frac{A}{2}\right)^2$ and $\left(\sin C \cos \frac{A}{2} \cos \frac{B}{2}\right)^2$.

Therefore,

$$\begin{aligned}
&\sum_{\text{cyclic}} \left(\sin A \cos\frac{B}{2}\cos\frac{C}{2}\right)^2 \\
&= \sum_{\text{cyclic}} \cos^6\frac{A}{2} \\
&- \left[\cos^2\frac{A}{2}\cos^4\frac{B}{2} - 2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{A}{2}\cos^4\frac{C}{2}\right] \\
&- \left[\cos^2\frac{B}{2}\cos^4\frac{C}{2} - 2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{B}{2}\cos^4\frac{A}{2}\right] \\
&- \left[\cos^2\frac{C}{2}\cos^4\frac{A}{2} - 2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{C}{2}\cos^4\frac{B}{2}\right] \\
&= \sum_{\text{cyclic}} \cos^6\frac{A}{2} - \sum_{\text{cyclic}} \cos^2\frac{A}{2}\left[\cos^2\frac{B}{2} - \cos^2\frac{C}{2}\right]^2 \\
&\le \sum_{\text{cyclic}} \cos^6\frac{A}{2}\,.
\end{aligned}$$

Furthermore, since $0 < \frac{A}{2}, \frac{B}{2}, \frac{C}{2} < \frac{\pi}{2}$, equality is attained if and only if $\cos^2\frac{A}{2} = \cos^2\frac{B}{2} = \cos^2\frac{C}{2}$; that is, if and only if $A = B = C = \frac{\pi}{3}$, so that the given triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3554. [2010 : 314, 317] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{\sqrt{a^2+bc}}{b+c} + \frac{\sqrt{b^2+ca}}{c+a} + \frac{\sqrt{c^2+ab}}{a+b} \ \ge\ \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}}\,.$$

Solution by Joe Howard, Portales, NM, USA.

By symmetry we can assume that $a \ge b \ge c > 0$. Then $(a-c)(b-c) \ge 0$ and thus

$$c^2 + ab \ge c(a+b)\ .$$

Hence

$$\frac{\sqrt{c^2+ab}}{a+b} \ge \frac{\sqrt{c(a+b)}}{a+b} = \sqrt{\frac{c}{a+b}}\ .$$

To complete the inequality, we will prove that

$$\left(\frac{\sqrt{a^2+bc}}{b+c}+\frac{\sqrt{b^2+ac}}{a+c}\right)^2 \geq \left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{a+b}}\right)^2 .$$

First we will show that

$$\frac{\sqrt{a^2+bc}}{b+c}\frac{\sqrt{b^2+ac}}{a+c} \geq \sqrt{\frac{a}{b+c}}\sqrt{\frac{b}{a+b}} \tag{1}$$

This simplifies to $c(a^3+b^3) \geq abc(a+b)$. Dividing by $c(a+b)$ this inequality is equivalent to

$$a^2-ab+b^2 \geq ab ,$$

or

$$(a-b)^2 \geq 0 .$$

Thus, (1) holds.
Now we also show that

$$\frac{a^2+bc}{(b+c)^2}+\frac{b^2+ac}{(a+c)^2} \geq \frac{a}{b+c}+\frac{b}{a+b} \tag{2}$$

This simplifies to

$$a^4+b^4+a^3c+b^3c+2abc^2 \geq a^3b+b^3a+a^2c^2+b^2c^2+a^2bc+ab^2c ,$$

or

$$(a-b)^2(a^2+ab+b^2)+c(a-b)[a(a-c)-b(b-c)] \geq 0 .$$

This last inequality is an immediate consequence of $a \geq b \geq c > 0$.
This completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer .

3555. [2010 : 315, 317] *Proposed by Vahagn Aslanyan, Yerevan, Armenia.*

Let a and b be positive integers, $1 < a < b$, such that a does not divide b. Prove that there exists an integer x such that $1 < x \leq a$ and both a and b divide $x^{\phi(b)+1} - x$, where ϕ is Euler's totient function.

Similar solutions by Oliver Geupel, Brühl, NRW, Germany and John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA. We give Geupel the write up.

We write $a = a_1 a_2$; $b = b_1 b_2$ where

$$a_1 = \prod_{i=1}^{m} p_i^{\alpha_i} \; ; \; a_2 = \prod_{i=m+1}^{n} p_i^{\alpha_i} \; ; \; b_1 = \prod_{i=1}^{m} p_i^{\beta_i} \; ; \; b_2 = \prod_{i=m+1}^{n} p_i^{\beta_i} \; ,$$

with distinct primes p_i and nonnegative integers α_i, β_i satisfying $\alpha_i \geq \beta_i$ for all $1 \leq i \leq m$ and $\alpha_i < \beta_i$ for all $i > m$. That is a_1 is the product of prime powers that occur with at least the same exponent in a as in b, and b_1 is the product of the corresponding powers in b.

We prove that $x = a_1$ works.

The hypothesis $a \nmid b$ yields $x \neq 1$.

We know that $\gcd(b_1, b_2) = 1$, thus $\phi(b) = \phi(b_1)\phi(b_2)$. Since $\gcd(x, b_2)$=1, by Euler Theorem we have

$$x^{\phi(b)} \equiv \left(x^{\phi(b_2)}\right)^{\phi(b_1)} \equiv (1)^{\phi(b_1)} \equiv 1 \pmod{b_2} \; .$$

Hence

$$x^{\phi(b)+1} \equiv x \pmod{a_1 b_2} \; .$$

From the definition of a_1, a_2, b_1, b_2, it follows that $b_1 | a_1$ and $a_2 | b_2$ and thus, both a, b divide $a_1 b_2$.

Thus, both a, b divide $x^{\phi(b)+1} - x$ which completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer . There was also an incorrect solution.

Three solvers mentioned that this problem appeared simultaneously as Problem O170 in Mathematical Reflections 5 (2010). Geupel pointed that the solution featured in that journal is wrong, since it uses the incorrect fact that $\frac{a}{\gcd(a,b)}$ and b are relatively prime [ED: $a = 4, b = 6$ is a counterexample].

3557. [2010 : 315, 317] *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers with $\sum\limits_{k=1}^{\infty} a_k = 1$ and $a_{k+1} \le \dfrac{a_k}{1-a_k}$. Let $S_n^{(p)} = \left(\sum\limits_{k=1}^{n} a_k^p\right)^{1/p}$, and for $p \ge 1$ prove that

$$\lim_{n\to\infty} \sum_{k=n+1}^{2n} \frac{k}{2} \left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j} a_j}{S_{k+j}^{(p)}}\right)^{1/n} \;=\; 0\,.$$

[*Ed.: The upper limit of summation is now correctly stated as* $2n$*; our apologies for this error.*]

Solution by Albert Stadler, Herrliberg, Switzerland.

We will only use the condition that $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive numbers such that $\sum\limits_{k\ge 1} a_k$ converges to a positive number c. The conditions that $c = 1$ and $a_{k+1} \le \dfrac{a_k}{1-a_k}$ are superfluous.

We first note that by the AM–GM Inequality

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n} \le \frac{c}{n}.$$

It follows that

$$\sum_{k=n+1}^{2n} \frac{k}{n} \left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j} a_j}{S_{k+j}^{(p)}}\right)^{1/n} \le 2n\cdot\frac{c}{n} \max_{n<k\le 2n} \left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j}}{S_{k+j}^{(p)}}\right)^{1/n}\,. \quad (1)$$

Put

$$T_n^{(p)} = \sum_{k=1}^{n} a_k^p\,,$$

$$M_n^{(p)} = \max_{n<k\le 2n} \left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j}}{S_{k+j}^{(p)}}\right)^{p/n} = \max_{n<k\le 2n} \left(\prod_{j=1}^{n} \frac{j a_{k+j}^p}{T_{k+j}^{(p)}}\right)^{1/n}.$$

We also have

$$\left(\prod_{j=1}^{n} j\right)^{1/n} \le \left(\prod_{j=1}^{n} n\right)^{1/n} = n\,, \quad (2)$$

$$\left(\prod_{j=1}^{n} a_{k+j}^p\right)^{1/n} \le \frac{1}{n} \sum_{j=1}^{n} a_{k+j}^p\,, \quad (3)$$

$$T_{k+n}^{(p)} \ge T_{k+n-1}^{(p)} \ge \cdots \ge T_{k+1}^{(p)} \ge T_k^{(p)}\,. \quad (4)$$

where (3) holds by the AM–GM Inequality.

Using (2), (3), and (4) we obtain

$$M_n^{(p)} \le \max_{n<k\le 2n} \left(\frac{\sum\limits_{j=1}^{n} a_{k+j}^p}{T_k^{(p)}} \right) .$$

By assumption, $\sum\limits_{k\ge 1} a_k$ converges, so there are only finitely many a_k for which $a_k \ge 1$ and the other summands satisfy $0 \le a_k^p \le a_k < 1$. Thus, $\sum\limits_{k\ge 1} a_k^p$ converges, say to σ. Put $\epsilon_k = \sigma - \sum\limits_{j=1}^{k} a_j^p = \sum\limits_{j=k+1}^{\infty} a_j^p$. It then follows that

$$M_n^{(p)} \le \max_{n<k\le 2n} \left(\frac{\sum\limits_{j=1}^{n} a_{k+j}^p}{T_k^{(p)}} \right) \le \max_{n<k\le 2n} \left(\frac{\epsilon_k}{\sigma - \epsilon_k} \right) \to 0 \quad \text{as} \quad n \to \infty .$$

Finally, we conclude from (1) that

$$\sum_{k=n+1}^{2n} \frac{k}{n} \left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j} a_j}{S_{k+j}^{(p)}} \right)^{1/n} \to 0 \quad \text{as} \quad n \to \infty ,$$

since the sum is nonnegative and bounded above by $2c\sqrt[p]{M_n^{(p)}}$, which vanishes as n tends to infinity.

Also solved by Oliver Geupel, Brühl, NRW, Germany, and the proposer.
Geupel also corrected the problem and removed the hypothesis $a_{k+1} \le a_k/(1-a_k)$.

3558. [2010 : 315, 317] *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Given two distinct positive integers a and b, prove that there exists a positive integer n such that an and bn have different numbers of digits.

Solution by Skidmore College Problem Solving Group, Skidmore College, Saratoga Springs, NY, USA(abbreviated by editor).

Without loss of generality we assume $0 < a < b$. For each positive integer j let I_j denote the interval (ja, jb). It will be sufficient to show that there exist positive integers n and N such that $10^N \in I_n$, for then na will require at most N digits and nb more than N digits in their decimal representations.

It is observed that I_m and I_{m+1} will overlap for all sufficiently large m. Indeed, for any $m > M = \lceil \frac{a}{b-a} \rceil$ it is easy to see that $m(b-a) > a$, therefore

$(m+1)a < mb$ and thus $I_m \cap I_{m+1} = ((m+1)a, mb) \neq \emptyset$. Consequently

$$\bigcup_{m>M} I_m = \Big((M+1)a, \lim_{m\to\infty} mb\Big) = ((M+1)a, \infty)$$

.

Now take any integer N such that $10^N \in ((M+1)a, \infty)$ and we will have that $10^N \in I_n$ for some $n > M$ as claimed.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon(2 solutions); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer who provided several solutions.

3559★. [2010 : 315, 318] *Proposed by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*

Let ABC be a triangle with side lengths a, b, c, inradius r, circumradius R, and semiperimeter s. Prove that

$$\frac{(b+c)^2}{4bc} \leq \frac{s^2}{3r(4R+r)}.$$

Solution I by Albert Stadler, Herrliberg, Switzerland, expanded slightly by the editor.

Let F be the area of the triangle. We have the formulae

$$F = \frac{abc}{4R} = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

So

$$\begin{aligned}\frac{s^2}{3r(4R+r)} = \frac{s^4}{3F(4Rs+F)} &= \frac{s^3}{3abcs + 3(s-a)(s-b)(s-c)} \\ &= \frac{1}{3}\cdot\frac{s^2}{s^2-(a+b+c)s+ab+bc+ca} \\ &= \frac{1}{3}\cdot\frac{(a+b+c)^2}{(a+b+c)^2-2(a+b+c)^2+4(ab+bc+ca)} \\ &= \frac{1}{3}\cdot\frac{a^2+b^2+c^2+2ab+2bc+2ca}{-a^2-b^2-c^2+2ab+2bc+2ca}.\end{aligned}$$

Hence, the given inequality is equivalent in succession to

$$\begin{aligned} 3(b+c)^2(-a^2-b^2-c^2+2ab+2bc+2ca) \\ \le 4bc(a^2+b^2+c^2+2ab+2bc+2ca) \\ 0 \le (3b^2+10bc+3c^2)a^2-2(b+c)(3b^2+2bc+3c^2)a \\ +(b+c)^2(3b^2-2bc+3c^2)\,. \qquad (1) \end{aligned}$$

Let $f(a)$ be the quadratic polynomial in a on the right hand side of (1) and let Δ denote its discriminant.

Since the leading coefficient of $f(a)$ is clearly positive, to conclude that $f(a) \ge 0$ it suffices to show that $\Delta \le 0$. Letting $d = 3b^2+3c^2$ we have

$$\begin{aligned} \Delta &= 4(b+c)^2[(d+2bc)^2-(d-2bc)(d+10bc)] \\ &= 4(b+c)^2(4bcd+4b^2c^2-8bcd+20b^2c^2) \\ &= 4(b+c)^2(-4bcd+24b^2c^2) \\ &= -48bc(b+c)^2(b^2+c^2-2bc) = -48bc(b+c)^2(b-c)^2 \le 0 \end{aligned}$$

so the proof is now complete.

Solution II by Joe Howard, Portales, NM, USA.

We prove the following extension:

$$\frac{(b+c)^2}{4bc} \le \frac{(a+b+c)^3}{27abc} \le \frac{s^2}{3r(4R+r)}\,.$$

Let A_n and G_n denote the arithmetic and geometric means of n positive numbers, respectively. Then it is known [1] that $\left(\frac{A_n}{G_n}\right)$ is monotonically increasing. The left inequality follows from $\frac{A_2}{G_2} \le \frac{A_3}{G_3}$.

To establish the right inequality we use the known formulae: $a+b+c = 2s$ and $abc = 4Rrs$. Then $\frac{(a+b+c)^3}{27abc} \le \frac{s^2}{3r(4R+r)}$ is equivalent in succession to

$$\begin{aligned} \frac{8s^3}{27(4Rrs)} &\le \frac{s^2}{12Rs+3r^2} \\ 24Rr+6r^2 &\le 27Rr \\ 2r &\le R \end{aligned}$$

which is the famous Euler's Inequality.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Arslanagić, Geupel and Malikić all pointed out that equality holds if and only if the triangle is equilateral.

The proposer remarked that the proposed inequality sharpens the result $3r(4R+r) \le s^2$ *by G. Colombier and T. Doucet (see item 5.5 on p.49 of [2]).*

References

[1] B. Arbed, *From Tricks to Strategies for Problem Solving*, Int. J. Math., Edu. Sci. Tech. 21(3), 1990; pp. 429 – 438

[2] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

3560. [2010 : 315, 318] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x and y be real numbers such that $x^2 + y^2 = 1$. Find the maximum value of

$$f(x,y) = |x-y| + \left|x^3 - y^3\right| .$$

Solution by Joel Schlosberg, Bayside, NY, USA.

Since $|xy| \le \frac{1}{2}(x^2+y^2) = \frac{1}{2}$,

$$x^2 + xy + y^2 \ge \frac{1}{2} \ge 0, \quad 1 - 2xy \ge 0, \quad \text{and} \quad 2 + xy \ge 0 .$$

Noting that $|x^3 - y^3| = |x-y||x^2+xy+y^2|$ and using the AM-GM inequality we see that

$$\begin{aligned} (f(x,y))^2 &= |x-y|^2\left(1 + |x^2+xy+y^2|\right)^2 \\ &= (x^2 - 2xy + y^2)(1 + x^2 + xy + y^2)^2 \\ &= (1-2xy)(2+xy)^2 \\ &\le \left(\frac{(1-2xy) + 2(2+xy)}{3}\right)^3 = \left(\frac{5}{3}\right)^3 ; \end{aligned}$$

that is, $f(x,y) \le \left(\frac{5}{3}\right)^{3/2}$. Since $x^2+y^2 = 1$, and $f(x,y) = \left(\frac{5}{3}\right)^{3/2}$ for

$$\{x,y\} = \left\{\frac{1+\sqrt{5}}{2\sqrt{3}}, \frac{1-\sqrt{5}}{2\sqrt{3}}\right\} \quad \text{or} \quad \{x,y\} = \left\{-\frac{1+\sqrt{5}}{2\sqrt{3}}, -\frac{1-\sqrt{5}}{2\sqrt{3}}\right\} ,$$

the maximum value of $f(x,y)$ is $\left(\frac{5}{3}\right)^{3/2}$.

Also solved by ARKADY ALT, San Jose, CA, USA(2 solutions); GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; NEVEN JURIČ, Zagreb, Croatia; KEE-WAI LAU, Hong Kong, China; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; CRISTINEL MORTICI, Valahia University of Târgovişte, *Romania; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA(3 solutions); DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.*

3561. [2010 : 316, 318] *Proposed by Mihály Bencze, Brasov, Romania.*

An n-sided polygon has perimeter k with $k^2 < 2n^2$. Prove that some three consecutive vertices along the polygon form a triangle with area less than 1 unit.

Solution by George Apostolopoulos, Messolonghi, Greece.

If $s_1, \ldots, s_n, s_{n+1}$ are the side lengths in cyclic order (with $s_{n+1} = s_1$), then $s_i > 0$ and $k = \sum_{i=1}^{n} s_i$. A triangle formed by three consecutive vertices that determine an angle θ_i has an area F_i that satisfies

$$F_i = \frac{s_i s_{i+1} \sin\theta_i}{2} \le \frac{s_i s_{i+1}}{2}.$$

From this inequality together with the AM-GM inequality and $k^2 < 2n^2$ (given) we get

$$\begin{aligned}\prod_{i=1}^{n} F_i &\le \prod_{i=1}^{n} \frac{s_i s_{i+1}}{2} = \frac{1}{2^n}\left(\prod_{i=1}^{n} s_i\right)^2 \\ &\le \frac{1}{2^n}\left(\left(\frac{\sum_{i=1}^{n} s_i}{n}\right)^n\right)^2 \\ &= \frac{1}{2^n}\left(\frac{k}{n}\right)^{2n} = \left(\frac{k^2}{2n^2}\right)^n < 1.\end{aligned}$$

We deduce that at least one F_i is less than 1, as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.

All the submitted solutions were quite similar, although most used an indirect argument.

3562. [2010 : 316, 318] *Proposed by Vahagn Aslanyan, Yerevan, Armenia.*

Let p be a prime number. Prove that there exists a prime number q such that $p \mid (q-1)$ and with the property that $q^k \mid \left(a^p - b^p\right)$ whenever $q^k \mid \left(a^{p^m} - b^{p^m}\right)$ for positive integers a, b, m, k with a and b not divisible by q.

Solution by Joel Schlosberg, Bayside, NY, USA, modified slightly by the editor.

Since $(p^2, p+1) = 1$, by Dirichlet's Theorem, there exists infinitely many primes in the arithmetic progression $p + 1 + np^2$, $n = 0, 1, 2, \ldots$. Thus, there exists a prime q such that $q = p + 1 + np^2$ or $q - 1 = p + np^2$ for some n. Hence $p \mid q - 1$ but $p^2 \nmid q - 1$.

Suppose that a, b, m, k are positive integers such that $q \nmid a$, $q \nmid b$ and $q^k \mid a^{p^m} - b^{p^m}$. Then

$$(ab^{-1})^{p^m} \equiv 1 \pmod{q^k} \tag{1}$$

where b^{-1} denotes the multiplicitive inverse of b modulo q^k. Since $bb^{-1} \equiv 1 \pmod{q^k}$, $q^k \nmid b^{-1}$ so $(q^k, ab^{-1}) = 1$. Hence, by Euler's theorem, we have

$$(ab^{-1})^{(q-1)q^{k-1}} = (ab^{-1})^{\phi(q^k)} \equiv 1 \pmod{q^k} \tag{2}$$

where ϕ denotes Euler's totient function.

By (1) and (2), the multiplicative order of ab^{-1} modulo q^k divides $(p^m, (q-1)q^{k-1})$ which equals p since $p \nmid q$, $p \mid q - 1$ and $p^2 \nmid q - 1$. Therefore, $(ab^{-1})^p \equiv 1 \pmod{q^k}$, from which $a^p \equiv b^p \pmod{q^k}$ follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Geupel remarked that this problem is closely related to problem 6 of the IMO 2003. [Ed: The IMO problems asked to show that for each prime p there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.]