A nest of Euler Inequalities

Luo Qi

Abstract

For any given $\triangle ABC$, we define the *antipodal triangle*. Repeating this construction gives a sequence of triangles with circumradii R_n and inradii r_n obeying a generalized form of Euler's inequality

$$2^n R_n \geq \cdots \geq 2^2 R_2 \geq 2R_1 \geq R_0 \geq 2r_0 \geq 2^2 r_1 \geq \cdots \geq 2^{n+1} r_n,$$

$(n = 1, 2, \cdots)$, with equalities iff $\triangle ABC$ is equilateral.

Key words: Euler inequality; antipodal triangle

Let R, r be the radius of circumcircle and inscribed circle of a triangle; then $R \geq 2r$, with equalities iff the triangle is equilateral ([1], p.50). This is the famous Euler inequality. In this note, we are going to build a nest of Euler inequalities for a certain family of related triangle.

Definition 1 *If a vertex A of a triangle ABC and another point A' on the perimeter divide the perimeter into two equal parts (that is, $|AB| + |BA'| = |AC| + |CA'|$) we call $|A'|$ the* antipode *of $|A|$, and the triangle $\triangle A'B'C'$ of which three vertices are antipodes of A, B, C respectively the* antipodal triangle *of $\triangle ABC$.*

Note that A' is necessarily on the (non-extended) edge BC and in fact it is the point where that edge touches the appropriate escribed circle.[2] Thus, we can easily find a way to draw an antipodal triangle $\triangle A'B'C'$ of a given triangle $\triangle ABC$.

Lemma 1 *Denote by $a, b, c, a_1, b_1, c_1, s, s_1, \mathrm{A}, \mathrm{A}_1$ the sides, semiperimeters, and areas of $\triangle ABC$ and its antipodal triangle $\triangle A_1B_1C_1$, and let R and r be the circumradius and inradius of $\triangle ABC$. Then*

1. $|AB_1| = |BA_1| = s - c, |AC_1| = |CA_1| = s - b, |BC_1| = |CB_1| = s - a$;
2. $\dfrac{\mathrm{A}_1}{\mathrm{A}} = \dfrac{r}{2R}$;
3. $\dfrac{a_1 b_1 c_1}{abc} \geq \dfrac{r}{4R}$ *(with equality iff $\triangle ABC$ is equilateral);*
4. $2s_1 \geq s$*(with equality iff $\triangle ABC$ is equilateral).*

Proof (See the figure 1)
(1) We have

$$|AB_1| = \frac{1}{2}(|AB| + |BC| + |CA|) - |AB|,$$
$$|BA_1| = \frac{1}{2}(|AB| + |BC| + |CA|) - |AB|$$

and so $|AB_1| = |BA_1| = s - c.$ In the same way, we have

$$|AC_1| = |CA_1| = s - b,$$
$$|BC_1| = |CB_1| = s - a.$$

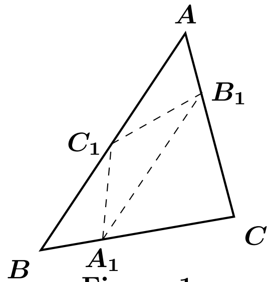

Figure 1

(2) Denote by $\mathbf{A}_{AB_1C_1}$, $\mathbf{A}_{BA_1C_1}$, $\mathbf{A}_{CA_1B_1}$, the areas of $\triangle AB_1C_1$, $\triangle BA_1C_1$, $\triangle CA_1B_1$.

Because

$$\frac{\mathbf{A}_{AB_1C_1}}{\mathbf{A}} = \frac{|AB_1| \cdot |AC_1|}{|AB| \cdot |AC|} = \frac{(s-c)(s-b)}{c \cdot b},$$
$$\frac{\mathbf{A}_{BA_1C_1}}{\mathbf{A}} = \frac{|BA_1| \cdot |BC_1|}{|BA| \cdot |BC|} = \frac{(s-c)(s-a)}{c \cdot a},$$
$$\frac{\mathbf{A}_{CA_1B_1}}{\mathbf{A}} = \frac{|CA_1| \cdot |CB_1|}{|CA| \cdot |CB|} = \frac{(s-b)(s-a)}{b \cdot a}$$

we have

$$\begin{aligned}
\frac{\mathbf{A}_1}{\mathbf{A}} &= \frac{\mathbf{A} - \mathbf{A}_{AB_1C_1} - \mathbf{A}_{BA_1C_1} - \mathbf{A}_{CA_1B_1}}{\mathbf{A}} \\
&= 1 - \frac{\mathbf{A}_{AB_1C_1}}{\mathbf{A}} - \frac{\mathbf{A}_{BA_1C_1}}{\mathbf{A}} - \frac{\mathbf{A}_{CA_1B_1}}{\mathbf{A}} \\
&= 1 - \frac{(s-c)(s-b)}{c \cdot b} - \frac{(s-c)(s-a)}{c \cdot a} - \frac{(s-b)(s-a)}{b \cdot a} \\
&= \frac{2(s-a)(s-b)(s-c)}{a \cdot b \cdot c}
\end{aligned}$$

By Heron's and other well-known formulas,

$$\mathrm{A} = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = sr$$

Thus

$$\frac{\mathrm{A}_1}{\mathrm{A}} = \frac{2(s-a)(s-b)(s-c)}{a\cdot b\cdot c} = \frac{2\mathrm{A}^2}{s\cdot 4\mathrm{A}\cdot R} = \frac{\mathrm{A}}{2sR} = \frac{r}{2R}$$

(3) Using the law of sines on $\triangle AB_1C_1$, we have

$$\begin{aligned}
\frac{a_1}{\sin A} &= \frac{s-b}{\sin\angle AB_1C_1} = \frac{s-c}{\sin\angle AC_1B_1} = \frac{s-b+s-c}{\sin\angle AB_1C_1 + \sin\angle AC_1B_1} \\
&= \frac{a}{2\sin\frac{\angle AB_1C_1+\angle AC_1B_1}{2}\cos\frac{\angle AB_1C_1-\angle AC_1B_1}{2}} \\
&\geq \frac{a}{2\sin\frac{\pi-A}{2}} = \frac{a}{2\cos\frac{A}{2}}
\end{aligned}$$

(with equality iff $\angle AB_1C_1 = \angle AC_1B_1$).

Therefore we have

$$\frac{a_1}{a} \geq \frac{\sin A}{2\cos\frac{A}{2}} = \sin\frac{A}{2}$$

(with equality iff $\angle AB_1C_1 = \angle AC_1B_1$).

In the same way, we have

$$\frac{b_1}{b} \geq \sin\frac{B}{2}$$

(with equality iff $\angle BA_1C_1 = \angle BC_1A_1$), and

$$\frac{c_1}{c} \geq \sin\frac{C}{2}$$

(with equality iff $\angle CB_1A_1 = \angle CA_1B_1$).

Multiplying these three inequalities, we obtain

$$\frac{a_1b_1c_1}{abc} \geq \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(with equality iff $\triangle ABC$ is equilateral).

Also, because

$$\mathrm{A} = \frac{1}{2}ab\sin C = \frac{abc}{4R} = \frac{r(a+b+c)}{2}$$

we have

$$\frac{(2R)^3\sin A\sin B\sin C}{4R} = \frac{2Rr(\sin A+\sin B+\sin C)}{2}.$$

So

$$\begin{aligned}
\frac{r}{R} &= \frac{2\sin A\sin B\sin C}{\sin A+\sin B+\sin C} \\
&= \frac{16\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}} \\
&= \frac{16\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\cos\frac{C}{2}(\cos\frac{A-B}{2}+\sin\frac{C}{2})} \\
&= \frac{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{A+B}{2}} \\
&= \frac{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}} \\
&= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}
\end{aligned}$$

Thus

$$\frac{a_1b_1c_1}{abc} \geq \frac{r}{4R}$$

(with equality iff $\triangle ABC$ is equilateral).

(4) Construct perpendiculars B_1E and C_1D to BC at E and D, respectively (See Figure 2). Then $a_1 \geq |DE|$ (with equality iff $BC \parallel B_1C_1$.) But

$$|DE| = a - |BD| - |CE| = a - (s-a)\cos B - (s-a)\cos C\ ,$$

so

$$a_1 \geq a - (s-a)(\cos B + \cos C) \text{ with equality iff } BC \parallel B_1C_1\ .$$

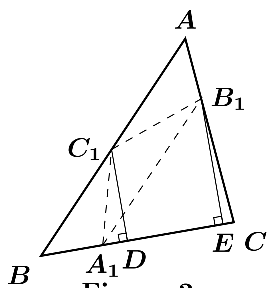

Figure 2

Similarly, we have:

$$b_1 \geq b - (s-b)(\cos C + \cos A) \text{ with equality iff } CA \parallel C_1A_1\ ,$$

$$c_1 \geq c - (s-c)(\cos A + \cos B) \text{ with equality iff } AB \parallel A_1B_1\ .$$

Adding up these three inequalities yields

$$\begin{aligned} 2s_1 &\geq 2s - (a\cos A + b\cos B + c\cos C) \\ &\geq 2s - \frac{1}{2}(a\cos B + a\cos C + b\cos A + b\cos C + c\cos A + c\cos B) \\ &= 2s - \frac{1}{2}(a+b+c) = s \end{aligned}$$

Therefore $2s_1 \geq s$ with equality iff $\triangle ABC$ is equilateral.

Studying these two triangles we can also find other interesting properties. The reader may verify that the antipodal triangle is "less equilateral" in that the ratio between longest and shortest side is always greater than in the original triangle.

Theorem 1 *Denote by R, R_1, r, r_1, the circumradii and inradii of $\triangle ABC$ and its antipodal triangle $\triangle A_1B_1C_1$. Then $2R_1 \geq R \geq 2r \geq 4r_1$, with equalities iff $\triangle ABC$ is equilateral.*

Proof By (2) and (3) of lemma 1, we have

$$\frac{r}{2R} = \frac{\mathrm{A}_1}{\mathrm{A}} = \frac{a_1b_1c_1/4R_1}{abc/4R} = \frac{Ra_1b_1c_1}{R_1abc} \geq \frac{R}{R_1}\cdot\frac{r}{4R} = \frac{r}{4R_1}$$

So $2R_1 \geq R$, with equality iff $\triangle ABC$ is equilateral.

By (2) and (4) of lemma 1, we have

$$\frac{1}{4} \geq \frac{r}{2R} = \frac{\mathrm{A}_1}{\mathrm{A}} = \frac{r_1s_1}{rs} \geq \frac{r_1s_1}{r\cdot 2s_1} = \frac{r_1}{2r}$$

and so $2r \geq 4r_1$ with equality iff $\triangle ABC$ is equilateral. Hence, we get $2R_1 \geq R \geq 2r \geq 4r_1$, again with equalities iff $\triangle ABC$ is equilateral.

Using mathematical induction and the theorem we immediately get:

Corollary 1 *(See Figure 3) Let $\triangle A_0B_0C_0$ be given, and let $R_0, R_1, \cdots, R_n$; $r_0, r_1, \cdots, r_n$ denote the circumradii and inradii of $\triangle A_0B_0C_0$, $\triangle A_1B_1C_1, \cdots$, $\triangle A_nB_nC_n$ respectively, and $\triangle A_iB_iC_i$ is the antipodal triangle of $\triangle A_{i-1}B_{i-1}C_{i-1}$, $(i = 1, 2, \cdots)$. Then*

$$2^nR_n \geq \cdots \geq 2^2R_2 \geq 2R_1 \geq R_0 \geq 2r_0 \geq 2^2r_1 \geq \cdots \geq 2^{n+1}r_n,$$

with equalities iff $\triangle A_0B_0C_0$ is equilateral.

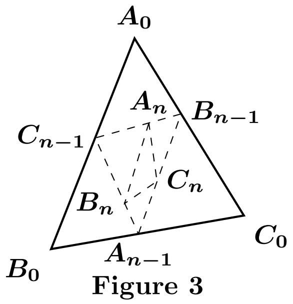

Figure 3

So, we build a nest of Euler inequalities.

Open question: In [3] Yang derives Euler-type inequalities for tetrahedra in 3-dimensional space. Can we define antipodal points for a tetrahedron in such a way that the results of this paper generalize?

References

[1] O. Bottema et al. *Geometric Inequalities*, Wolter-Noordhoff, Groningen, 1969.

[2] Weisstein, Eric W. "Semiperimeter." From*MathWord–A Wolfram Web Resource*. http://mathworld.wolfram.com/Semiperimeter.html. Accessed 10 January 2012.

[3] Yang Shiguo. Several improvements of tetrahedral Euler inequality. China collection of studies of elementary mathematics [M]. Zhengzhou province: Henan education press,1992.6.

Mathematics Department, Guilin Normal College,
Guilin,

Luo Qi
Mathematics Department
Guilin Normal College
Guilin, Guangxi
China
luoqi67@163.com