# EDITORIAL

## Shawn Godin

Hello ***CRUX with MAYHEM*** readers, welcome to the September 2011 issue. I must apologize up front for the ongoing delays. The first contract between the CMS and the Ottawa District School Board started in March of 2011 left me with some catching up to do. The contract ran out in January and the negotiations took longer than any of the parties involved thought they would. As a result, for two months I was back to my "day job" full time. Fortunately, things are finally straightened out and we are moving forward with the 2011 volume of ***CRUX with MAYHEM*** .

We have been going through the backlog of proposals and should be up to date with those within a couple of months. With respect to problem proposals, for the last couple of years the majority of the proposals submitted are inequalities. As a result, we have many proposals in the bank. We seem to receive the right amount of geometry problems to keep our problem sets going, so keep those questions coming. Currently we are very short on problems from number theory, algebra, calculus and combinatorics. Please consider sending us your nice proposals from these areas. As always, proposals sent to ***CRUX with MAYHEM*** should be original. If you have sent a proposal to another publication, do not send it to us unless the problem has been rejected, or you have withdrawn it and vice versa.

Currently we construct the problem sets by looking at problems that have been proposed by readers and trying to select a set of problems from a number of different topics that will be appealing to the readers. We have no idea how the readers really feel about these problems (other than the fact that if we receive a number of solutions to a problem from the readers, those readers must have enjoyed it). We have started another series of short surveys where you can give us feedback on the problems we propose in ***Crux*** . Each survey asks you to rate each problem (and it isn't necessary to rate them all) from -5 (meaning you hated the problem) to 5 (meaning you loved the problem). A score of 0 means you are indifferent. You can find the addresses of the surveys for the problems from the 2011 volume on our Facebook page. The survey for the problems from this issue can be found at:

`http://www.surveymonkey.com/s/LK998Z9`

Thank you to the people who have taken the time to answer our recent survey on-line. We have compiled your input and will be looking into making some changes. We are currently searching for columnists to start up some regular columns on problem-solving. This issue contains the first installment of a semi-regular column *Recurring Crux Configurations*. The column explores ideas that have resurfaced in ***Crux*** problems over the years and the first entry was written by editorial board member J. Chris Fisher, I hope you enjoy it.

A major change that we will be making, starting in volume 38, is separating the ***Mathematical Mayhem*** and Skoliad Corner material from ***Crux***. ***Mayhem*** has been a part of ***Crux*** since volume 23 in 1997. It will continue to exist, separately, in electronic format only.

We would like to publish articles more regularly in ***Crux***, but we have been receiving many articles lately that are inappropriate. ***Crux*** is a problem-solving journal at the secondary and university undergraduate levels; as such, articles should reflect the scope of the journal. Some possible categories that articles could fall into:

1. Articles about *problem solving methods or techniques.* An example of an article of this type is *Summations according to Gauss* by Gerhard J. Woeginger which appears in this issue. Articles about a theorem that is known, but maybe not well known to all the readership of ***Crux*** are also acceptable.

2. Articles *inspired by a particular problem.* These could include generalizations of problems that have appeared in ***CRUX with MAYHEM*** or elsewhere. An example of an article of this type is *A Generalization of Mayhem Problem M396 Involving Pythagorean Triangles* [2010 : 540–544] by Konstantine Zelator .

3. Articles that would be of interest to problem solvers. These could include articles surrounding a problem that could have been one of the ***Crux*** numbered problems (although, may have been a bit too involved). An example of an article of this type is *A Nest of Euler Inequalities* by Luo Qi which appears in this issue.

***Crux*** is not a research journal. Articles concerning "serious" mathematics will not be considered for publication (i.e. they will be rejected outright). ***Crux*** is not primarily aimed at teachers (although many of the readership are teachers). Articles concerning pedagogy will not be considered for publication.

We welcome articles from the readers. Articles should be 2 to 8 pages in length. Longer articles may be considered if they can logically be separated into multiple parts. Please send your appropriate articles to our articles editor:

Robert Dawson,
Department of Mathematics and Computing Science,
Saint Mary's University,
Halifax, NS, Canada, B3H 3C3,
or by email to `crux-articles@cms.math.ca`.

Shawn Godin

# SKOLIAD No. 134

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **July 15, 2012**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Baden-Württemberg Mathematics Contest, 2010. Our thanks go to the Landeswettbewerb Mathematik Baden Württemberg for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the contest.

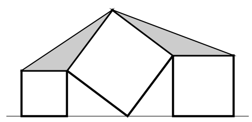
## Baden-Württemberg Mathematics Contest, 2010

**1**. Sonja has nine cards on which the nine smallest two-digit prime numbers are printed. She wants to order these cards in such a way that neighbouring cards always differ by a power of **2**. In how many ways can Sonja order her cards?

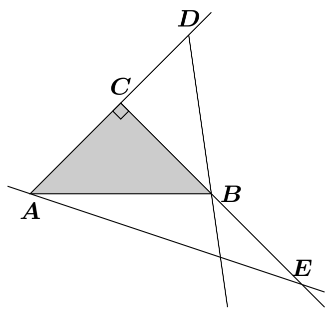
**2**. A **50** cm by **30** cm by **28** cm box contains wooden blocks that all measure **10** cm by **9** cm by **7** cm. At most how many blocks can fit in the box? Explain how to fit that many blocks into the box.

**3**. Five distinct positive numbers are given. Forming all possible sums of two of these numbers you obtain seven different sums. Show that the sum of the five original numbers is divisible by **5**.

**4**. Three squares are arranged as in the figure. Show that the two shaded triangles have the same area.



**5**. Triangle $\triangle ABC$ is isosceles and $\angle ACB = 90^\circ$. The point $D$ is on the line $AC$ beyond $C$, and the point $E$ is on the line $CB$ beyond $B$. Show that $|CD| = |CE|$ if line $BD$ is perpendicular to line $AE$.



**6**. The product of three positive integers is three times as large as their sum. Find all such triples.

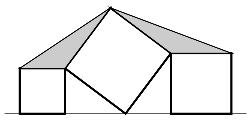
## Concours mathématique Baden-Württemberg 2010

**1**. Sonya dispose de neuf cartes sur lesquelles sont imprimés les neuf plus petits nombres premiers à deux chiffres. Elle voudrait ordonner ses cartes de façon à ce que les cartes voisinantes diffèrent toujours par une puissance de $\mathbf{2}$. De combien de manières est-ce que Sonya peut ordonner ses cartes ?

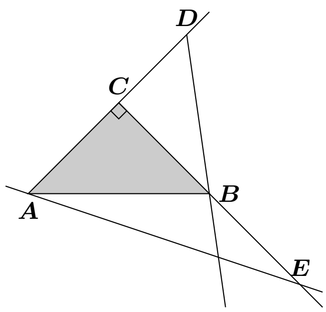
**2**. Une boîte de taille $\mathbf{50}$ cm par $\mathbf{30}$ cm par $\mathbf{28}$ cm contient des blocs en bois, chacun de taille $\mathbf{10}$ cm par $\mathbf{9}$ cm par $\mathbf{7}$ cm. Au plus, combien de tels blocs entrent dans la boîte ? Expliquer comment faire entrer ce nombre de blocs dans la boîte.

**3**. Cinq nombres positifs distincts vous sont donnés. En formant toutes les sommes possibles de deux de ces nombres, on constate qu'on obtient sept sommes distinctes. Démontrer que la somme des cinq nombres originaux est divisible par $\mathbf{5}$.

**4**. Trois carrés sont disposés tel qu'illustré par la figure. Démontrer que les deux triangles ombragés ont la même surface.



**5**. Le triangle $\triangle ABC$ est isocèle et $\angle ACB = 90°$. Le point $D$ se trouve sur la ligne $AC$, au-delà de $C$, et le point $E$ se trouve sur la ligne $CB$, au-delà de $B$. Démontrer que $|CD| = |CE|$ si la ligne $BD$ est perpendiculaire à la ligne $AE$.




**6**. Le produit de trois entiers positifs est trois fois aussi grand que leur somme. Déterminer tout tel triplet.

Next follow solutions to the Mathematics Association of Quebec Contest, Secondary level, 2010, given in Skoliad 128 at [2010:417–419].

**1**. An *alphametic* is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic PAPA + PAPA = MAMAN has the solution P = $\mathbf{7}$, A = $\mathbf{5}$, M = $\mathbf{1}$, and N = $\mathbf{0}$, yielding $\mathbf{7575 + 7575 = 15150}$.

Find the solution to this "reversing" alphametic:

$$\text{NOMBRE} \times \frac{\mathbf{3}}{\mathbf{5}} = \text{ERBMON}\,.$$

*Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.*

If NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ = ERBMON, then NOMBRE $\times\ \mathbf{3}$ = ERBMON $\times\ \mathbf{5}$, so NOMBRE is divisible by $\mathbf{5}$. Therefore E = $\mathbf{0}$ or E = $\mathbf{5}$. However, ERBMON does not begin with a zero, so E = $\mathbf{5}$.

Thus ERBMON $> \mathbf{500\,000}$, so NOMBRE $= \frac{\mathbf{5}}{\mathbf{3}} \times$ ERBMON $> \mathbf{833\,333}$. Therefore N = $\mathbf{8}$ or N = $\mathbf{9}$. When multiplying integers, the ones digit of the result depends only on the ones digits of the factors. Therefore, the ones digit of NOMBRE $\times\ \mathbf{3}$ is the ones digit of E $\times\ \mathbf{3}$ which is $\mathbf{5}$, since E = $\mathbf{5}$. Now, NOMBRE $\times\ \mathbf{3}$ = ERBMON $\times\ \mathbf{5}$, so the ones digit of ERBMON $\times\ \mathbf{5}$ is also $\mathbf{5}$. Therefore, the ones digit of N $\times\ \mathbf{5}$ must be $\mathbf{5}$, so N cannot be $\mathbf{8}$. Thus N = $\mathbf{9}$.

You now know that the ones digit of NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ is $\mathbf{9}$. Note that the ones digit of NOMBRE $\div\ \mathbf{5}$ equals the ones digit of RE $\div\ \mathbf{5}$. Therefore the ones digit of NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ equals the ones digit of RE $\times \frac{\mathbf{3}}{\mathbf{5}}$, which, thus, must be $\mathbf{9}$. Since E = $\mathbf{5}$, either R = $\mathbf{1}$ or R = $\mathbf{6}$. If R = $\mathbf{1}$, then NOMBRE = ERBMON $\times \frac{\mathbf{5}}{\mathbf{3}} \leq \mathbf{519\,999} \times \frac{\mathbf{5}}{\mathbf{3}} = \mathbf{866\,665}$, contradicting the fact that N = $\mathbf{9}$. Thus R = $\mathbf{6}$.

Now, NOMBRE = ERBMON $\times \frac{\mathbf{5}}{\mathbf{3}} \geq \mathbf{560\,000} \times \frac{\mathbf{5}}{\mathbf{3}} > \mathbf{933\,333}$, so O $\geq \mathbf{3}$. Moreover, ERBMON is divisible by $\mathbf{3}$, so E + R + B + M + O + N is divisible by $\mathbf{3}$. Thus NOMBRE is divisible by $\mathbf{3}$, and, hence, ERBMON = NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ is divisible by $\mathbf{9}$. Therefore, N + O + M + B + R + E = $\mathbf{9}$ + O + M + B + $\mathbf{6}$ + $\mathbf{5}$ = O + M + B + $\mathbf{20}$ is divisible by $\mathbf{9}$.

You must now find three digits, (O, M, B), from $\{\mathbf{0,1,2,3,4,7,8}\}$ subject to the two conditions that O $\geq \mathbf{3}$ and that O + M + B + $\mathbf{20}$ is divisible by $\mathbf{9}$. Only ten triples satisfy these conditions: $(\mathbf{3,0,4})$, $(\mathbf{3,4,0})$, $(\mathbf{4,0,3})$, $(\mathbf{4,1,2})$, $(\mathbf{4,2,1})$, $(\mathbf{4,3,0})$, $(\mathbf{7,1,8})$, $(\mathbf{7,8,1})$, $(\mathbf{8,1,7})$, and $(\mathbf{8,7,1})$. Of these only one satisfy the further condition that NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ = ERBMON, namely (O, M, B) = $(\mathbf{3,4,0})$.

$$\mathbf{934\,065} \times \frac{\mathbf{3}}{\mathbf{5}} = \mathbf{560\,439}$$

*The solution* $\mathbf{934\,065} \times \frac{\mathbf{3}}{\mathbf{5}} = \mathbf{560\,439}$ *was also found by ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha*

*Secondary School, Burnaby, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.*

**2**. Find all polynomials of the form $p(x) = x^3 + mx + 6$ whose roots are integers.

*Solution by Billy Suandito, Palembang, Indonesia.*

All integer roots of $p(x)$ must be factors of $6$ (*for the reason why, see the editors' note below*). Thus, the possible integer roots are $1$, $2$, $3$, $6$, $-1$, $-2$, $-3$, and $-6$.

If $1$ is a root, then $0 = p(1) = 1^3 + m + 6 = 7 + m$, so $m = -7$. Thus $p(x) = x^3 - 7x + 6 = (x-1)(x^2+x-6) = (x-1)(x+3)(x-2)$. Thus $p(x)$ has three integer roots if $m = -7$.

If $2$ (or $-3$) is a root, you find the above example again.

If $3$ is a root, then $0 = p(3) = 3^3 + 3m + 6 = 33 + 3m$, so $m = -11$. Thus $p(x) = x^3 - 11x + 6 = (x-3)(x^2+3x-2)$, but the roots of $x^2+3x-2$ are not integers.

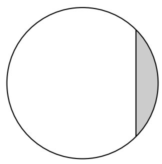
If you continue checking the possible roots, $6$, $-1$, $-2$, and $-6$, in a similar manner, you will find that $p(x)$ fails to have integer roots in each case except $m = -7$. Thus $p(x) = x^3 - 7x + 6$ is the only solution.

*The solution $p(x) = x^3 - 7x + 6$ was also found by WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.*

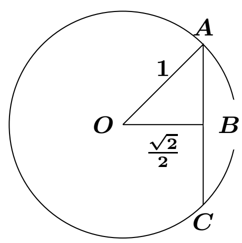
*Alternatively, say the three roots are $a$, $b$, and $c$. Then $x^3 + mx + 6 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$. Thus $a+b+c = 0$ and $abc = -6$. It follows that exactly one of the roots is negative. Since the roots are integers, they must be factors of $6$. The only possibility is, then, that the roots are $1$, $2$, and $-3$. Thus the only solution is $p(x) = (x-1)(x-2)(x+3) = x^3 - 7x + 6$ as above.*

*Note, as in the previous paragraph, that if the leading coefficient of a polynomial is $1$, then the next-to-leading coefficient is the negative of the sum of the roots and the constant term is, apart from a sign, the product of the roots. This observation often comes in handy in contests.*

**3**. A line is located at $\frac{\sqrt{2}}{2}$ units from the centre of a circle of radius $1$, separating it into two parts. What is the area of the smaller part?



*Solution by Lisa Wang, student, Port Moody Secondary School, Port Moody, BC.*



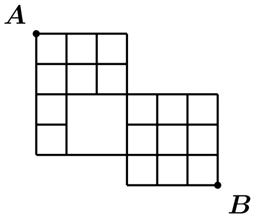

Let $O$ be the centre of the circle and $A$ and $C$ be the endpoints of the chord. Let $B$ be the point on $AC$ that is closest to $O$. Then $|OB| = \frac{\sqrt{2}}{2}$, $\angle OBA = 90^\circ$, and $B$ is the midpoint of $AC$. By the Pythagorean Theorem, $|AB| = \sqrt{1^2 - (\frac{\sqrt{2}}{2})^2} = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{2}}{2}$. Therefore $|AC| = \sqrt{2}$, so the area of $\triangle AOC$ is $\frac{1}{2} \cdot |AC| \cdot |OB| = \frac{1}{2} \cdot \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$.

Moreover, $\triangle AOB$ is isosceles and $\angle AOB = 45^\circ$. By symmetry, $\angle COB = 45^\circ$, so $\angle AOC = 90^\circ$, whence sector $AOC$ is a quarter circle. Thus the area of sector $AOC$ is $\frac{1}{4}\pi 1^2 = \frac{\pi}{4}$.

It follows that the area of the shaded segment is $\frac{\pi}{4} - \frac{1}{2} \approx 0.285$.

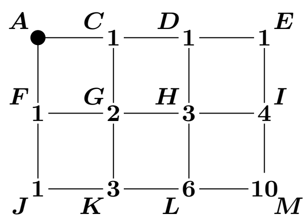
*Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and BILLY SUANDITO, Palembang, Indonesia.*

**4**. The figure shows a map of a city. In how many ways can you travel along the roads of the city from point $A$ to point $B$ if you can only travel east and south (right and down in the figure)?

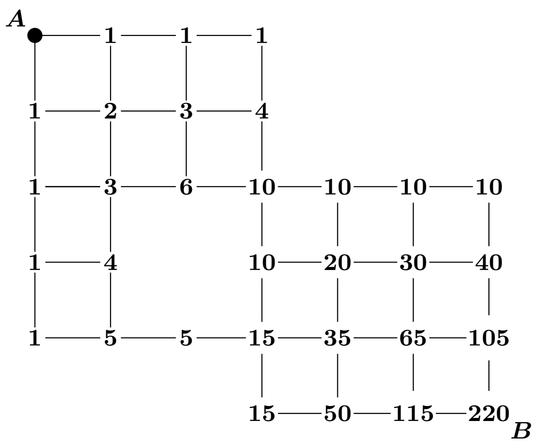



*Solution by Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC.*

Consider this simpler map. Surely, you can arrive at $C$ in only one way, as indicated. To reach $D$, you must come from $C$, so you can arrive at $D$ in only one way. Similarly at $E$ and $F$. You can reach $G$ either from $C$ or from $F$, so you can arrive at $G$ in two ways. You can reach $H$ either from $D$ or from $G$, so you can arrive at $H$ in $1 + 2 = 3$ ways. Now continue in this way: if you can reach $X$ from $Y$ and $Z$, then the number of paths to $X$ is the sum of the number of paths to $Y$ and the number of paths to $Z$.




Using this method of counting paths in the original map yields a total of **220** paths from $A$ to $B$:




*Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha Secondary School, Burnaby, BC; and BILLY SUANDITO, Palembang, Indonesia.*

**5**. (a) How many zeroes are at the right-hand end of the number $\mathbf{1 \times 2 \times 3 \times \cdots \times 52}$?

*Solution by Janice Lew, student, École Alpha Secondary School, Burnaby, BC.*

The number $\mathbf{1 \times 2 \times \cdots \times 52}$ is written more compactly as $\mathbf{52!}$. Let $\boldsymbol{S}$ denote the set $\mathbf{\{1, 2, 3, \ldots, 52\}}$. Half of the numbers in $\boldsymbol{S}$ are even, so $\mathbf{52!}$ is divisible by $\mathbf{2^{26}}$. One quarter of the numbers in $\boldsymbol{S}$ are divisible by $\mathbf{4}$. These contribute an extra $\mathbf{\frac{52}{4} = 13}$ copies of $\mathbf{2}$. Since $\mathbf{\frac{52}{8} = 6.5}$, the numbers in $\boldsymbol{S}$ that are divisible by $\mathbf{8}$ contribute a further $\mathbf{6}$ copies of $\mathbf{2}$. Three of the numbers in $\boldsymbol{S}$ are divisible by $\mathbf{16}$, and one is divisible by $\mathbf{32}$. Thus $\mathbf{52!}$ is divisible by $\mathbf{2^{26+13+6+3+1} = 2^{49}}$ but not by any higher power of $\mathbf{2}$.

Likewise, $\mathbf{52!}$ is divisible by $\mathbf{5^{10+2} = 5^{12}}$ but no higher power of $\mathbf{5}$. Since $\mathbf{10 = 2 \times 5}$, it follows that $\mathbf{52!}$ is divisible by $\mathbf{10^{12}}$ but no higher power of $\mathbf{10}$. Thus $\mathbf{52!}$ ends in exactly $\mathbf{12}$ zeroes.

*Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.*

(b) What is the rightmost nonzero digit of $\mathbf{1 \times 2 \times \cdots \times 52}$? (For example, the rightmost nonzero digit of $\mathbf{1 \times 2 \times \cdots \times 12 = 479\,001\,600}$ is $\mathbf{6}$.)

*Solution by the editors.*

In Part (a), our solver began factoring $\mathbf{52!}$ into primes. Now complete the process: Since $\mathbf{\frac{52}{3} \approx 17.3}$, $\mathbf{\frac{52}{9} \approx 5.8}$, and $\mathbf{\frac{52}{27} \approx 1.9}$, it follows that $\mathbf{52!}$ is divisible by $\mathbf{3^{17+5+1} = 3^{23}}$. Likewise, $\mathbf{52!}$ is divisible by $\mathbf{7^{7+1} = 7^8}$, $\mathbf{11^4}$, $\mathbf{13^4}$, $\mathbf{17^3}$, $\mathbf{19^2}$, $\mathbf{23^2}$, and once each by $\mathbf{29}$, $\mathbf{31}$, $\mathbf{37}$, $\mathbf{41}$, $\mathbf{43}$, and $\mathbf{47}$. Thus $\mathbf{52! = 2^{49} \cdot 3^{23} \cdot 5^{12} \cdot 7^8 \cdot 11^4 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47}$.

The rightmost non-zero digit of $\mathbf{52!}$ equals the ones digit of

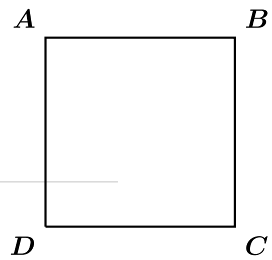
$$\mathbf{\frac{52!}{10^{12}} = 2^{37} \cdot 3^{23} \cdot 7^8 \cdot 11^4 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47}\,.$$

Let $\boldsymbol{a \equiv b}$ denote that the whole numbers $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same ones digit. Since the ones digit of a product depends only on the ones digits of the factors, $\mathbf{2^{37} = 2^2 \cdot (2^5)^7 = 2^2 \cdot (32)^7 \equiv 2^2 \cdot 2^7 = 2^9 = 512 \equiv 2}$. Likewise, $\mathbf{3^{23} = 3^3 \cdot (3^4)^5 = 3^3 \cdot (81)^5 \equiv 3^3 \cdot 1 = 27 \equiv 7}$, and $\mathbf{7^8 = (7^4)^2 = (2401)^2 \equiv 1}$, and $\mathbf{11^4 \equiv 1}$, and $\mathbf{13^4 \equiv 3^4 = 81 \equiv 1}$, and $\mathbf{17^3 \equiv 7^3 = 343 \equiv 3}$, and $\mathbf{19^2 \equiv 9^2 = 81 \equiv 1}$, and $\mathbf{23^2 \equiv 3^2 = 9}$, and, of course, $\mathbf{29 \equiv 9}$, $\mathbf{31 \equiv 1}$, $\mathbf{37 \equiv 7}$, $\mathbf{41 \equiv 1}$, $\mathbf{43 \equiv 3}$, and $\mathbf{47 \equiv 7}$. Thus

$$\mathbf{\frac{52!}{10^{12}} \equiv 2 \cdot 7 \cdot 1 \cdot 1 \cdot 1 \cdot 3 \cdot 1 \cdot 9 \cdot 9 \cdot 1 \cdot 7 \cdot 1 \cdot 3 \cdot 7 = 500094 \equiv 4}\,.$$

Therefore the rightmost non-zero digit of $\mathbf{52!}$ is $\mathbf{4}$.

**6**. Juliette and Philippe play the following game: At the beginning of the game, each corner of a square is covered with a number of chips. In turn, each player chooses one side of the square and removes as many chips as (s)he wants from the endpoints of that side provided (s)he takes at least one chip. It is not necessary to remove the same number of chips from each endpoint. The player who removes the last chip wins.
At the beginning of the game on the square $ABCD$ there are $10$ chips on corner $A$, $11$ chips on $B$, $12$ chips on $C$, and $13$ chips on $D$. If Juliette begins, how should she play?




*Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC; Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.*

Let $a$, $b$, $c$ and $d$ denote the number of chips on $A$, $B$, $C$, and $D$, respectively. Strategy: Juliette should play to ensure that $a = c$ and $b = d$.

If Philippe receives a square with $a = c$ and $b = d$, then he must remove at least one chip, and he cannot remove chips from both ends of a diagonal. Therefore he will always pass a square to Juliette with $a \neq c$ and/or $b \neq d$.

If Juliette receives a square with $a \neq c$ and/or $b \neq d$, then for each diagonal she should choose the endpoint with the larger number of chips. Then she should choose the side that connects those two endpoints and remove chips until $a = c$ and $b = d$.

Thus Juliette is always able to follow the strategy above. Eventually, $a = c = 0$ and $b = d = 0$, and Juliette wins.

**7**. Find all functions $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) + xF(-x) = 1$ for all real numbers $x$.
*Solution by Billy Suandito, Palembang, Indonesia.*

If $F(x) + xF(-x) = 1$ for *all* values of $x$, then the equation also holds for $-x$; that is, $F(-x) - xF(x) = 1$. Multiplying this last equation by $x$ yields that $xF(-x) - x^2F(x) = x$. Subtract this from the original equation to get that $F(x) + x^2F(x) = 1 - x$, so $(1 + x^2)F(x) = 1 - x$, so $F(x) = \dfrac{1-x}{1+x^2}$.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

We invite the reader to submit solutions to one or more of our problems.

# MATHEMATICAL MAYHEM



## Editorial

### Shawn Godin

Hello ***Mayhem*** readers. The CMS and the editors of ***CRUX with MAYHEM*** have been working on some changes for the journal. Over the last few months, we have set up a page on Facebook to communicate with our readers. We have surveyed the readers of ***CRUX with MAYHEM*** and have listened to their comments. One change that we are planning directly impacts the readers of ***Mayhem***.

As of volume 38, ***Mathematical Mayhem*** will no longer be part of ***Crux Mathematicorum***. It will continue to exist, but only on the web. The numbering and dates of the issues will revert back to the days before it joined ***Crux Mathematicorum***. ***Mathematical Mayhem*** will appear on line 5 times a year: September, November, January, March and May. The last volume of ***Mathematical Mayhem*** to appear as a stand alone was volume 8 [1995-1996]. The next stand alone volume starting in September 2012 will be volume 24 [2012-2013]. We will also be expanding and adding features that will be of interest to mathematics students and teachers at the high school level. The first issue will be here before you know it, so get ready!

Also, this issue marks the last installment of the Problem of the Month by Ian VanderBurgh. Ian has been writing this column since September 2004 [2004 : 264-265] (57 columns!). We will miss Ian's column, it has been a great part of ***Mayhem***. All the best Ian and thanks for sharing your passion of mathematical problem-solving with us.

Shawn Godin

# Mayhem Problems

*Please send your solutions to the problems in this edition by **1 August 2012**. Solutions received after this date will only be considered if there is time before publication of the solutions.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.*

*The editor thanks Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the problems from English into French.*

**M495**. *Proposed by the Mayhem Staff.*

All possible lines are drawn through the point $(0, 0)$ and the points $(x, y)$, where $x$ and $y$ are whole numbers with $1 \leq x, y \leq 10$. How many distinct lines are drawn?

**M496**. *Proposed by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.*

Show that if we write the numbers from $1$ to $n$ around a circle, in any order, then, for all $x = 1, 2, \ldots, n$, we are guaranteed to find a block of $x$ consecutive numbers that add up to at least $\left\lceil \frac{x(n+1)}{2} \right\rceil$. Here $\lceil y \rceil$ is the ceiling function, that is, the least integer greater than or equal to $y$. So $\lceil 6.2 \rceil = 7$, $\lceil \pi \rceil = 4$, $\lceil -8.3 \rceil = -8$ and $\lceil 10 \rceil = 10$.

**M497**. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

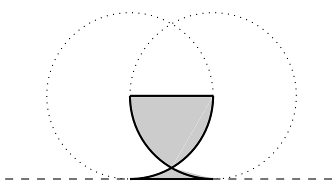
Find all integers $a$, $b$, $c$ where $c$ is a prime number such that $a^b + c$ and $a^b - c$ are both perfect squares.

**M498**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Right triangle $ABC$ has its right angle at $C$. The two sides $CB$ and $CA$ are of integer length. Determine the condition for the radius of the incircle of triangle $ABC$ to be a rational number.

**M499**. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Two circles of radius $1$ are drawn so that each circle passes through the centre of the other circle. Find the area of the goblet like region contained between the common radius, the circumferences and one of the common tangents as shown in the diagram to the right.



**M500**. *Proposed by Edward T.H. Wang and Dexter S.Y. Wei, Wilfrid Laurier University, Waterloo, ON.*

Let $\mathbb{N}$ denote the set of natural numbers.

(a) Show that if $n \in \mathbb{N}$, there do not exist $a, b \in \mathbb{N}$ such that $\dfrac{[a,b]}{a+b} = n$, where $[a,b]$ denotes the least common multiple of $a$ and $b$.

(b) Show that for any $n \in \mathbb{N}$, there exists infinitely many triples $(a,b,c)$ of natural numbers such that $\dfrac{[a,b,c]}{a+b+c} = n$, where $[a,b,c]$ denotes the least common multiple of $a$, $b$ and $c$.

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**M495**. *Proposé par l'Équipe de Mayhem.*

Toutes les droites possibles sont tracées à partir du point $(0,0)$ et des points $(x,y)$, où $x$ et $y$ sont des entiers tels que $1 \le x, y \le 10$. Combien de droites distinctes ont été tracées ?

**M496**. *Proposé par Sally Li, Institut collégial Marc Garneau, Toronto ON.*

Démontrer que si on place tous les entiers de $1$ à $n$ autour d'un cercle dans un ordre quelconque alors, pour tout $x = 1, 2, \ldots, n$, on pourra certainement trouver un bloc de $x$ nombres consécutifs dont la somme sera d'au moins $\left\lceil \dfrac{x(n+1)}{2} \right\rceil$, $\lceil y \rceil$ désignant la partie "plafond" de $y$, c'est-à-dire le plus petit entier plus grand ou égal à $y$. Ainsi $\lceil 6.2 \rceil = 7$, $\lceil \pi \rceil = 4$, $\lceil -8.3 \rceil = -8$ and $\lceil 10 \rceil = 10$.

**M497**. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

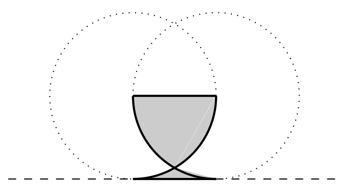
Déterminer tous les entiers $a$, $b$ et $c$ où $c$ est un nombre premier tel que $a^b + c$ et $a^b - c$ sont tous les deux des carrés d'entiers.

**M498**. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Le triangle rectangle $ABC$ a l'angle droit à $C$ ; les deux côtés $CB$ et $CA$ sont de longueurs entières. Déterminer la condition pour que le rayon du cercle inscrit du triangle $ABC$ soit un nombre rationnel.

**M499**. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Deux cercles de rayon $1$ sont tracés de façon à ce que chacun passe par le centre de l'autre. Déterminer la surface de la région en forme de gobelet qui se trouve entre le rayon commun, les circonférences et une des tangentes communes, tel qu'illustré à droite.



**M500**. *Proposé par Edward T.H. Wang et Dexter S.Y. Wei, Université Wilfrid Laurier, Waterloo, ON.*

Soit $\mathbb{N}$ l'ensemble des nombres naturels.

(a) Démontrer que si $n \in \mathbb{N}$ alors il n'existe aucun $a, b \in \mathbb{N}$ tels que $\dfrac{[a,b]}{a+b} = n$, où $[a, b]$ dénote le plus petit commun multiple de $a$ et $b$.

(b) Démontrer que si $n \in \mathbb{N}$ alors il existe un nombre infini de triplets $(a, b, c)$ d'entiers naturels tels que $\dfrac{[a,b,c]}{a+b+c} = n$, où $[a, b, c]$ dénote le plus petit commun multiple de $a$, $b$ et $c$.

# Mayhem Solutions

**M457**. *Proposed by the Mayhem Staff.*

Suppose that $A$ is a digit between $0$ and $9$, inclusive, and that the tens digit of the product of $2A7$ and $39$ is $9$. Determine the digit $A$.

*Solution by Florencio Cano Vargas, Inca, Spain.*

We write $2A7 = 2 \cdot 10^2 + A \cdot 10 + 7$ and $39 = 3 \cdot 10 + 9$. Multiplying both numbers and grouping we get:

$$2A7 \cdot 39 = 8 \cdot 10^3 + 3A \cdot 10^2 + (9A + 7) \cdot 10 + 3\,.$$

The condition stated in the problem implies that $9A + 7 \equiv 9 \pmod{10}$ which implies that $9A \equiv 2 \pmod{10}$. Hence, the solution is $A = 8$.

*Also solved by JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; LUIZ ERNESTO LEITÃO, Pará, Brazil; TRAVIS B. LITTLE, students, Angelo State University, San Angelo, TX, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia;*

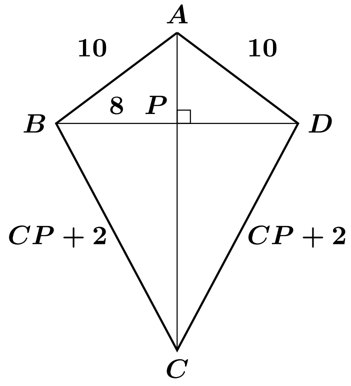
**M458**. *Proposed by the Mayhem Staff.*

Convex quadrilateral $ABCD$ has $AB = AD = 10$ and $BC = CD$. Also, $AC$ is perpendicular to $BD$, with $AC$ and $BD$ intersecting at $P$. If $BP = 8$ and $CD = CP + 2$, determine the area of quadrilateral $ABCD$.

*Solution by Ingesti Bilkis Zulpatina, student, SMPN 8, Yogyakarta, Indonesia.*

From the properties which are written above, $ABCD$ is surely a kite since $AB = AD$, $BC = CD$, and $AC \perp BD$.

Using the Pythagorean theorem:




$$AP^2 = AB^2 - BP^2$$
$$AP^2 = 36$$
$$\therefore AP = 6$$

and

$$BP^2 + CP^2 = CB^2$$
$$64 + CP^2 = CP^2 + 4CP + 4$$
$$60 = 4CP$$
$$CP = 15$$

Hence $AC = AP + PC = 6 + 15 = 21$.

Thus the area of quadrilateral $ABCD$ is

$$[ABCD] = \frac{AC \times BD}{2} = \frac{21 \times 16}{2} = 168$$

square units.

*Also solved by SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia.*

## M459. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine whether or not it is possible to create a collection of ten distinct subsets of $S = \{1, 2, 3, 4, 5, 6\}$ so that each subset contains three elements, each element of $S$ appears in five subsets, and each pair of elements from $S$ appears in two subsets.

*Solution by Jaclyn Chang, student, University of Calgary, Calgary, AB.*

It is possible to create ten distinct subsets of $S = \{1, 2, 3, 4, 5, 6\}$ such that each subset contains three elements, each element of $S$ appears in five subsets, and each pair of elements from $S$ appears in two subsets.

Each of the following distinct subsets contains three elements of $S$:

$S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2, 4\}$, $S_3 = \{1, 3, 5\}$, $S_4 = \{1, 4, 6\}$,
$S_5 = \{1, 5, 6\}$, $S_6 = \{2, 3, 6\}$, $S_7 = \{2, 4, 5\}$, $S_8 = \{2, 5, 6\}$,
$S_9 = \{3, 4, 5\}$, $S_{10} = \{3, 4, 6\}$.

Each element of $S$ appears in five subsets of $S$:

Element 1 in $S_1$, $S_2$, $S_3$, $S_4$ ,$S_5$; Element 2 in $S_1$, $S_2$, $S_6$, $S_7$, $S_8$;
Element 3 in $S_1$, $S_3$, $S_6$, $S_9$, $S_{10}$; Element 4 in $S_2$, $S_4$, $S_7$, $S_9$, $S_{10}$;
Element 5 in $S_3$, $S_5$, $S_7$, $S_8$, $S_9$; Element 6 in $S_4$, $S_5$, $S_6$, $S_8$, $S_{10}$.

Each pair of elements from $S$ appears in two subsets of $S$:

| | | | |
|---|---|---|---|
| $\{1,2\}$ in $S_1$, $S_2$; | $\{2,3\}$ in $S_1$, $S_6$; | $\{3,5\}$ in $S_3$, $S_9$; | $\{1,3\}$ in $S_1$, $S_3$; |
| $\{2,4\}$ in $S_2$, $S_7$; | $\{3,6\}$ in $S_6$, $S_{10}$; | $\{1,4\}$ in $S_2$, $S_4$; | $\{2,5\}$ in $S_7$, $S_8$; |
| $\{4,5\}$ in $S_7$, $S_9$; | $\{1,5\}$ in $S_3$, $S_5$; | $\{2,6\}$ in $S_6$, $S_8$; | $\{4,6\}$ in $S_4$, $S_{10}$; |
| $\{1,6\}$ in $S_4$, $S_5$; | $\{3,4\}$ in $S_9$, $S_{10}$; | $\{5,6\}$ in $S_5$, $S_8$. | |

*Also solved by ALEX SONG, Detroit Country Day School, Detroit, MI, USA, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. One incomplete solution was received.*

## M460. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let $a$ and $b$ be positive real numbers. Define $A = \dfrac{a+b}{2}$, $G = \sqrt{ab}$, and $K = \sqrt{\dfrac{a^2+b^2}{2}}$. Prove that (a) $G^2 + K^2 = 2A^2$, (b) $A^2 \geq KG$, (c) $G + K \leq 2A$, and (d) $G^4 + K^4 \geq 2A^4$.

*Solution by Jaclyn Chang, student, University of Calgary, Calgary, AB.*

(a) By direct computation we get

$$G^2 + K^2 = ab + \frac{a^2+b^2}{2} = \frac{a^2+2ab+b^2}{2} = \frac{(a+b)^2}{2} = 2\left(\frac{a+b}{2}\right)^2 = 2A^2\,.$$

(b) Since $(a-b)^4 \geq 0$ we get

$$\begin{aligned} & a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0 \\ \Rightarrow\ & a^4 + 4a^3b + 6a^2b^2 + 4ab^2 + b^4 \geq 8a^3b + 8ab^2 \\ \Rightarrow\ & \frac{(a+b)^4}{16} \geq \frac{a^3b+ab^3}{2} \Rightarrow \left(\frac{a+b}{2}\right)^4 \geq \frac{ab(a^2+b^2)}{2} \\ \Rightarrow\ & \left(\frac{a+b}{2}\right)^2 \geq \left(\sqrt{ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}}\right) \Rightarrow A^2 \geq KG\,. \end{aligned}$$

[*Ed.: Note that from the AM-GM inequality* $\frac{G^2+K^2}{2} \geq GK$. *Thus, using the result from (a) we get* $A^2 = \frac{G^2+K^2}{2} \geq GK$.]

(c) We have$(G+K)^2 = G^2 + 2KG + K^2$, but from (b) we know that $2KG \leq 2A^2$, thus $(G+K)^2 \leq G^2 + K^2 + 2A^2$. Using part (a) we can deduce $(G+K)^2 \leq 4A^2 = (2A)^2$ and therefore $G + K \leq 2A$.

(d) Since $(a-b)^4 \geq 0$ we have $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0$, hence

$$\begin{aligned} & a^4 + 6a^2b^2 + b^4 \geq 4a^3b + 4ab^3 \Rightarrow \frac{a^4+6a^2b^2+b^4}{8} \geq \frac{4a^3b+4ab^3}{8} \\ \Rightarrow\ & \frac{2a^4+12a^2b^2+2b^4}{8} \geq \frac{a^4+4a^3b+6a^2b^2+4ab^3+b^4}{8} \\ \Rightarrow\ & a^2b^2 + \frac{a^4+2a^2b^2+b^4}{4} \geq 2\left(\frac{a+b}{2}\right)^4 \Rightarrow G^4 + K^4 \geq 2A^4\,. \end{aligned}$$

[*Ed.: Note that from (a) and (b), we have* $G^4 + K^4 = (G^2 + K^2)^2 - 2K^2G^2 = \left(2A^2\right)^2 - 2K^2G^2 \geq 4A^4 - 2A^4 = 2A^4$.]

*Also solved by MIHÁLY BENCZE, Brasov, Romania; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA(parts a, b, c); FLORENCIO CANO VARGAS, Inca, Spain; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia(parts a, c); LUIZ ERNESTO LEITÃO, Pará, Brazil(part a); RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia; and the proposer.*

**M461**. *Proposed by Landelino Arboniés, Colegio Marcelino Champagnat, Santo Dominigo, Dominican Republic.*

A *Champagnat* number is equal to the sum of all the digits in a set of consecutive positive integers, one of which is the number itself. Thus, $42$ is a Champagnat number, since $42$ is the sum of all of the digits of $39$, $40$, $41$, $42$, $43$, $44$. Prove that there exist infinitely many Champagnat numbers.

*Solution by the proposer.*

We prove that for any $n > 6$ there is at least one Champagnat number with $n+1$ digits. Indeed, consider the number $10^n$ and suppose if is not a Champagnat number. Let $k_n$ be the greatest number such that the digital sum of the numbers $10^n, 10^n+1, 10^n+2, \ldots, 10^n+k_n$ is less than $10^n$. Consider now the number $N$ equal to the digital sum of all the integers from $10^n$ to $10^n+k_n+1$ inclusive. Now, since $k_n$ is at least $\frac{10^n}{9(n+1)}$ (since each of the numbers is less than $10^{n+1}-1$ which has a digital sum of $9(n+1)$) and $N$ is at most $10^n + 9(n+1)$ (only if $k_n + 1 = 10^{n+1} - 1$), then (if $n > 6$) $N$ is one of the numbers between $10^n$ and $10^n + k_n + 1$ inclusive, and hence it is a Champagnat number being the sum of a set of consecutive numbers, one of which is itself.

*No other solutions were received.*

**M462**. *Proposed by Alex Song, Detroit Country Day School, Detroit, MI, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding $x$ and let $\lceil x \rceil$ denote the smallest integer greater than or equal to $x$. For example, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor -1.4 \rfloor = -2$, and $\lceil -1.4 \rceil = -1$. Determine all real numbers $x$ for which $\lfloor x \rfloor \lceil x \rceil = x^2$.

*Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.*

If $x = n \in \mathbb{Z}$, then $\lfloor x \rfloor = n$ and $\lceil x \rceil = n$. Hence, $\lfloor x \rfloor \lceil x \rceil = n^2 = x^2$ for all $x \in \mathbb{Z}$. If $x \notin \mathbb{Z}$ and $x > 0$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $n < x < n+1$. We can then conclude that $\lfloor x \rfloor = n$ and $\lceil x \rceil = n+1$. Consequently, $\lfloor x \rfloor \lceil x \rceil = n(n+1) = x^2$, hence $x = \sqrt{n(n+1)}$. If $x \notin \mathbb{Z}$ and $x < 0$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $-(n+1) < x < -n$. We can then conclude that $\lfloor x \rfloor = -(n+1)$ and $\lceil x \rceil = -n$. Consequently,

$\lfloor x\rfloor\lceil x\rceil = n(n+1) = x^2$, hence $x = -\sqrt{n(n+1)}$. Thus, the set of all real numbers for which $\lfloor x\rfloor\lceil x\rceil = x^2$ is $x = \pm n$ or $x = \pm\sqrt{n(n+1)}$, $n \in \mathbb{N}\cup\{0\}$.

*Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposers. Three incomplete solutions were received.*

# Problem of the Month

Ian VanderBurgh

Many problems that appear on contests are word problems. Here is a problem that appeared last year on a Scottish competition:

**Problem** (2010-2011 Scottish Mathematical Challenge) Katie had a collection of red, green and blue beads. She noticed that the number of beads of each colour was a prime number and that the numbers were all different. She also observed that if she multiplied the number of red beads by the total number of red and green beads she obtained a number exactly $120$ greater than the number of blue beads. How many beads of each colour did she have?

Often, the first step with a word problem is to translate the words into mathematics. Since this problem is dealing with the numbers of red, green and blue beads, let's assign a variable to each of these numbers – say, $r$, $g$ and $b$, respectively. (We'll write this up nicely in a minute.) These seem to be the relevant quantities.

We are next told that each of these quantities is a prime number. Let's make a mental note to come back to this, and keep reading. The fact that the product of the number of red beads with the sum of the numbers of red and green beads is $120$ more than the number of blue beads translates into the equation $r(r+g) = 120 + b$.

Now, I seem to remember that usually when we have three variables, one equation is not enough to determine the values of the variables. (Often, we need three equations.) This is mildly concerning, but let's persevere to see what happens.

What information haven't we used? We haven't used the fact that each of $r$, $g$ and $b$ is a prime number. How can we use this information? Again, let's back up half a step. What do we know about prime numbers? It's good to check the definition first: a prime number is a positive integer larger than $1$ (remember, $1$ is not prime) that has no positive divisors other than $1$ and itself. Is there a "formula" for prime numbers? There isn't a good one that we know. However, there are lots and lots of properties of prime numbers: all prime numbers other than $2$ are odd, there are infinitely many prime numbers, every prime number greater than $\mathbf{3}$ is either one more or one less than a multiple of $\mathbf{6}$... The list goes on! Many mathematicians spend much of their professional lives investigating properties of prime numbers.

Given such a vast number to choose from, how do we know what properties to use? Therein lies the essence of problem solving! Figuring this out is not always easy.

Here's a solution to the problem.

**Solution** Suppose that $\boldsymbol{r}$, $\boldsymbol{g}$ and $\boldsymbol{b}$ are the numbers of red, green and blue beads, respectively. We are told that each of $\boldsymbol{r}$, $\boldsymbol{g}$ and $\boldsymbol{b}$ is a prime number and that $\boldsymbol{r(r+g) = 120 + b}$.

Let's focus on the fact that the only even prime number is $\mathbf{2}$ and on the parity of the two sides of the equation. (Remember, checking parity means checking to see if an integer is even or odd.) If both $\boldsymbol{r}$ and $\boldsymbol{g}$ are odd, then $\boldsymbol{r+g}$ is even, so the left side of the equation is even, which means the right side is even. If $\boldsymbol{120+b}$ is even, then $\boldsymbol{b}$ is even, which means that $\boldsymbol{b = 2}$. In this case, $\boldsymbol{r(r+g) = 122}$. Since $\mathbf{122 = 2 \times 61}$ and each of $\mathbf{2}$ and $\mathbf{61}$ is prime, then we must have $\boldsymbol{r = 2}$ or $\boldsymbol{r + g = 2}$. Neither of these is possible, since $\boldsymbol{r}$ cannot equal $\boldsymbol{b}$ (since $\boldsymbol{r}$, $\boldsymbol{b}$ and $\boldsymbol{g}$ are all different) and since $\boldsymbol{r+g}$ is at least $\mathbf{4}$.

Also, since $\mathbf{2}$ is the only even prime number, then $\boldsymbol{r}$ and $\boldsymbol{g}$ can't both be even, since then they would both be $\mathbf{2}$, which would contradict the given hypothesis that $\boldsymbol{r}$, $\boldsymbol{b}$ and $\boldsymbol{g}$ are all different.

Therefore, $\boldsymbol{r}$ and $\boldsymbol{g}$ are even and odd in some order. In other words, one of $\boldsymbol{r}$ and $\boldsymbol{g}$ equals 2 and the other is an odd prime number. Which is which? If $\boldsymbol{r = 2}$, then the equation becomes $\boldsymbol{2(2+g) = 120 + b}$. Since the left side is even, then the right side is even too, so again $\boldsymbol{b = 2}$ which is impossible since our assumption is that $\boldsymbol{r = 2}$.

Thus, it must be the case that $\boldsymbol{g = 2}$ and $\boldsymbol{r}$ is an odd prime. This gives $\boldsymbol{r(r+2) = 120 + b}$.

So we've got one value ($\boldsymbol{g = 2}$), but still have one equation and two unknowns. What to do?

Let's try solving for $\boldsymbol{b}$, which gives $\boldsymbol{b = r^2 + 2r - 120}$. At this point, it might occur to try to factor the right side to obtain $\boldsymbol{b = (r+12)(r-10)}$.

How does this help? Since $\boldsymbol{b}$ is a prime number, then it can't be factored in many ways! Aha – that is probably useful. If $\boldsymbol{b}$ is a prime number that is written as the product of two integers, then one of the factors is either $\mathbf{1}$ or $\mathbf{-1}$. This gives us four possibilities to check ($\boldsymbol{r+12}$ equals $\mathbf{1}$ or $\mathbf{-1}$ and $\boldsymbol{r-10}$ equals $\mathbf{1}$ or $\mathbf{-1}$). The only one that yields a positive value of $\boldsymbol{r}$ that is a prime number is $\boldsymbol{r - 10 = 1}$, giving $\boldsymbol{r = 11}$. In this case, $\boldsymbol{b = (11+12)(11-10) = 23}$, which is (thankfully) a prime number.

Therefore, $\boldsymbol{g = 2}$, $\boldsymbol{r = 11}$ and $\boldsymbol{g = 23}$. We can check that these satisfy the original hypotheses. □

# THE OLYMPIAD CORNER

## No. 295

R.E. Woodrow and Nicolae Strugaru

The problems from this issue come from the Italian Team Selection Test, the British Mathematical Olympiad, the Macedonian Mathematical Olympiad, the China Western Mathematical Olympiad, the Austrian Mathematical Olympiad, the Olimpiadi Italiane della Matematica and the Chinese Mathematical Olympiad. Our thanks go to Adrian Tang for sharing the material with the editor.

*The solutions to the problems are due to the editor by* **1 August 2012**.

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

*The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.*

**OC21**. A sequence of real numbers $\{a_n\}$ is defined by $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, and $a_{n+1} = 1 - a_n(1 - a_n)$ for $n = 1, 2, \cdots$. Prove that for any positive integer $n$, we have

$$a_0 a_1 \cdots a_n \left( \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) = 1\,.$$

**OC22**. Consider a standard $8 \times 8$ chessboard consisting of $64$ small squares coloured in the usual pattern, so $32$ are black and $32$ are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

**OC23**. Determine all nonnegative integers $n$ such that

$$n(n-20)(n-40)(n-60)\cdots r + 2009$$

is a perfect square where $r$ is the remainder when $n$ is divided by $20$.

**OC24**. Let $O$ be the circumcentre of the triangle $ABC$. Let $K$ and $L$ be the intersection points of the circumcircles of the triangles $BOC$ and $AOC$ with the bisectors of the angles at $A$ and $B$ respectively. Let $P$ be the midpoint of $KL$, $M$ symmetrical to $O$ relative to $P$ and $N$ symmetrical to $O$ relative to $KL$. Prove that $KLMN$ is cyclic.

**OC25**. Show that the inequality $3^{n^2} > (n!)^4$ holds for all positive integers $n$.

**OC26**. Find all functions $f$ from the real numbers to the real numbers which satisfy

$$f\left(x^3\right) + f\left(y^3\right) = (x+y)(f\left(x^2\right) + f\left(y^2\right) - f(xy))$$

for all real numbers $x$ and $y$.

**OC27**. A natural number $k$ is said to be $n$-*squared* if, for every colouring of the squares in a chessboard of size $2n \times k$ with $n$ colours, there are $4$ squares with the same colour whose centres are the vertices of a rectangle with sides parallel to the sides of the chessboard.

For any given $n$, find the smallest natural number $k$ which is $n$-squared.

**OC28**. A flea is initially at the point $(0,0)$ of the Euclidean plane. It then takes $n$ jumps. Each jump is taken in any of the four cardinal directions (north, east, south or west). The first jump has length $1$, the second jump has length $2$, the third jump has length $4$, and so on, the $n^{\text{th}}$ jump has length $2^{n-1}$.

Prove that if we know the number of jumps and the final position, we can uniquely determine the path the flea took.

**OC29**. Let $n \geq 3$ be a given integer, and $a_1, a_2, \cdots, a_n$ be real numbers satisfying $\displaystyle\min_{1\leq i<j\leq n} |a_i - a_j| = 1$. Find the minimum value of $\displaystyle\sum_{k=1}^{n} |a_k|^3$.

**OC30**. Let $P$ be an interior point of a regular $n$-gon $A_1A_2\cdots A_n$. The lines $A_iP$ meet $A_1A_2\cdots A_n$ at another point $B_i$, where $i = 1, 2, \cdots, n$. Prove that

$$\sum_{i=1}^{n} PA_i \geq \sum_{i=1}^{n} PB_i\,.$$

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**OC21**. On définit une suite de nombres réels $\{a_n\}$ par $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, et $a_{n+1} = 1 - a_n(1 - a_n)$ pour $n = 1, 2, \cdots$. Montrer que pour tout entier positif $n$, on a

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) = 1\,.$$

**OC22**. On considère un échiquier standard $8 \times 8$ formé de $64$ petits carrés de couleur et disposition usuelles, soit $32$ blancs et $32$ noirs. Un chemin en *zigzag* sur l'échiquier est une collection de huit carrés blancs, un par rangée, se touchant à leurs coins. Combien de chemins en zigzag y-a-t'il ?

**OC23**. Trouver tous les entiers non négatifs $n$ tels que

$$n(n-20)(n-40)(n-60)\cdots r+2009$$

soit un carré parfait où $r$ est le reste de la division de $n$ par $20$.

**OC24**. Soit $O$ le centre du cercle circonscrit du triangle $ABC$. Soit $K$ et $L$ les points d'intersection respectifs des cercles circonscrits $BOC$ et $AOC$ avec les bissectrices des angles en $A$ et $B$. Soit $P$ le milieu de $KL$, $M$ le symétrique de $O$ par rapport à $P$ et $N$ le symétrique de $O$ par rapport à $KL$. Montrer que les points $KLMN$ sont cocycliques.

**OC25**. Montrer que l'inégalité $3^{n^2} > (n!)^4$ est valable pour tous les entiers positifs $n$.

**OC26**. Trouver toutes les fonctions $f$ d'une variable à valeurs réelles et satisfaisant

$$f\left(x^3\right)+f\left(y^3\right)=(x+y)(f\left(x^2\right)+f\left(y^2\right)-f(xy))$$

pour tous les nombres réels $x$ et $y$.

**OC27**. Un nombre naturel $k$ est appelé $n$-*carré* si, pour tout coloriage des cases d'un échiquier de dimension $2n\times k$ avec $n$ couleurs, il y a $4$ cases de même couleur dont les centres sont les sommets d'un rectangle dont les côtés sont parallèles à ceux de l'échiquier.

Pour tout $n$ donné, trouver le plus petit nombre naturel $k$ qui soit $n$-carré.

**OC28**. On imagine une puce à l'origine $(0,0)$ du plan euclidien. La voilà qui effectue $n$ sauts. Chaque saut a lieu dans l'une quelconque des quatre directions cardinales (nord, est, sud ou ouest). Les longueurs des sauts consécutifs sont, dans l'ordre, de $1$, $2$, $4$ et ainsi de suite, le $n$-ième saut étant de $2^{n-1}$.

Montrer que si l'on connaît le nombre de sauts et la position finale, on peut en déduire univoquement le chemin suivi par la puce.

**OC29**. On donne un entier $n\geq 3$ et soit $a_1,a_2,\cdots,a_n$ des nombres réels satisfaisant $\min\limits_{1\leq i<j\leq n}|a_i-a_j|=1$. Trouver la valeur minimale de $\sum\limits_{k=1}^{n}|a_k|^3$.

**OC30**. Soit $P$ un point intérieur d'un polygone régulier $A_1A_2\cdots A_n$ à $n$ côtés. Les droites $A_iP$ coupent ce polygone en un autre point $B_i$, où $i=1,2,\cdots,n$. Montrer que

$$\sum_{i=1}^{n}PA_i\geq\sum_{i=1}^{n}PB_i\,.$$

Next we turn to our file of solutions from readers to problems of the Youth Mathematical Olympiad of the Asociación Venezolana de Competencias Matemáticas, 2006, given at [2009 : 380].

**1**. A positive integer has $223$ digits and the product of these digits is $3^{446}$. What is the sum of the digits?

*Solved by Geoffrey A. Kandall, Hamden, CT, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Manes.*

Let $n$ be the positive integer with $223$ digits. Then the maximum product of the digits of $n$ is $9^{223} = 3^{446}$ when all of the digits of $n$ are nines. Therefore, the digits of $n$ consist of $223$ nines and so, the sum of the digits is $9(223) = 2007$.

**2**. Find all solutions of the equation $m^2 - 3m + 1 = n^2 + n - 1$, where $m$ and $n$ are positive integers.

*Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Bataille.*

The solutions are the pairs $(m,n) = (a+2, a)$ where $a$ is a positive integer. The equation rewrites as $(m-\frac{3}{2})^2-\frac{5}{4} = (n+\frac{1}{2})^2-\frac{5}{4}$ or $(m-\frac{3}{2})^2-(n+\frac{1}{2})^2 = 0$ that is,

$$(m + n - 1)(m - n - 2) \;=\; 0.$$

Thus positive integers $m$, $n$ are solutions if and only if $m - n - 2 = 0$. The result follows.

**3**. Define the sequence $a_1$, $a_2$, $a_3$, $\ldots$ as follows: let $a_1 = a_2 = 1003$; $a_3 = a_2 - a_1 = 0$; $a_4 = a_3 - a_2 = -1003$; and in general $a_{n+1} = a_n - a_{n-1}$ for any $n \geq 2$. Compute the sum of the first $2006$ terms of the sequence.

*Solved by Oliver Geupel, Brühl, NRW, Germany; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Geupel.*

Let us compute the next items of the sequence:

$$\begin{aligned} a_5 = a_4 - a_3 &= -1003\,, & a_6 = a_5 - a_4 &= 0\,, \\ a_7 = a_6 - a_5 &= 1003\,, & a_8 = a_7 - a_6 &= 1003\,. \end{aligned}$$

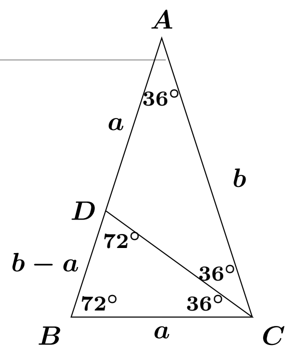
We see that the sequence is periodic with period 6 and $\sum_{n=1}^{6} a_n = 0$. Noticing that $2006 \equiv 2 \pmod{6}$, we find that the sum of the first 2006 items is

$$\sum_{n=1}^{2006} a_n = a_1 + a_2 = 2006.$$

**5**. Let $ABC$ be an isosceles triangle with $\angle B = \angle C = 72^\circ$. Find the value of $\dfrac{BC}{AB - BC}$. *Hint:* Consider the bisector $CD$ of $\angle ACB$.

*Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Kandall.*

Let $CD$ be the bisector of $\angle ACB$, and let $a = BC$, $b = AB = AC$. We have $\angle BAC = 36^\circ$ and $\angle BDC = 72^\circ$. Therefore, $AD = CD = BC = a$, so $BD = b - a$.




Triangles $ABC$ and $CDB$ are similar, hence $\frac{b}{a} = \frac{a}{b-a} \equiv \lambda$. (Note that $\lambda > 0$.)

We have $\frac{1}{\lambda} = \frac{b-a}{a} = \frac{b}{a} - 1 = \lambda - 1$, hence $\lambda^2 - \lambda - 1 = 0$. Consequently, $\frac{BC}{AB-BC} = \frac{a}{b-a} = \lambda = \frac{1+\sqrt{5}}{2}$.

To finish the material from the October 2009 files we turn to solutions from our readers to problems of the 42nd Mongolian Mathematical Olympiad, 10th Grade, given at [2009 : 380–381].

**1**. Let $a$, $b$, $c$, $d$, $e$, and $f$ be positive integers satisfying the relation $ab + ac + bc = de + df + ef$, and let $N = a + b + c + d + e + f$. Prove that if $N \mid (abc + def)$, then $N$ is a composite number.

*Solution by Jan Verster, Kwantlen University College, BC.*

Expanding the following polynomial, and using the relation $ab+ac+bc = de + df + ef$, we get

$$\begin{aligned}(x + a)(x + b)(x + c) &- (x - d)(x - e)(x - f)\\ &= x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc\\ &\qquad - x^3 + (d + e + f)x^2 - (de + df + ef)x + def\\ &= Nx^2 + abc + def\end{aligned}$$

Then, if we let $x = d$, say, this becomes

$$(d + a)(d + b)(d + c) = Nd^2 + (abc + def)$$

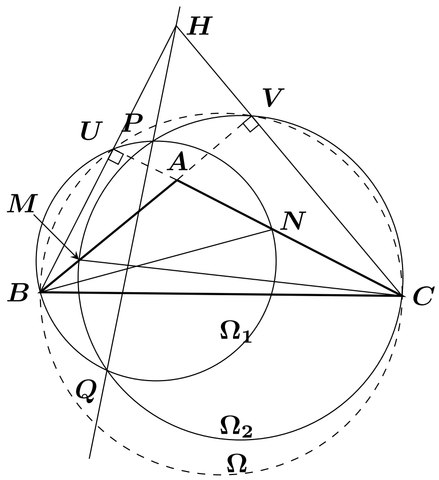
Thus, if $N|(abc + def)$, we also have $N|(d + a)(d + b)(d + c)$. Let $p$ be a prime such that $p|N$. Then $p$ must divide at least one of $d + a$, $d + b$ or $d + c$. Thus $p \le \max(d + a, d + b, d + c) < N$, so $p$ is a proper factor of $N$, and $N$ must be composite.

Now we turn to solutions from readers to problems of the Olympiade Suisse de mathématiques 2005, tour final, given at [2009 : 82–83].

**8**. Soint $ABC$ un triangle aigu. Soient $M$ et $N$ des points arbitraires sur les côtés $AB$ et $AC$ respectivement. Les cercles de diamètre $BN$ et $CM$ se coupent en $P$ et $Q$. Montrer que les points $P$, $Q$ et l'orthocentre du triangle $ABC$ se trouvent sur une droite.

*Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Michel Bataille, Rouen, France. We give the write-up of Amengual Covas.*

Let $\Omega$, $\Omega_1$, and $\Omega_2$ the circles on $BC$, $BN$, and $CM$ as diameters, respectively.




Since $\Omega_1$ and $\Omega_2$ intersect at $P$ and $Q$, the line $PQ$ is the radical axis of $\Omega_1$ and $\Omega_2$.

Let $H$ denote the orthocenter of $\triangle ABC$ and let $U$ and $V$ be the feet of the altitudes from $B$ and $C$ respectively.

Since $\angle BUN = \angle BUC = 90°$, both the circles $\Omega$ and $\Omega_1$ pass through $U$. Hence the line $BU$ is the radical axis of $\Omega$ and $\Omega_1$.

Similarly, the line $CV$ is the radical axis of $\Omega$ and $\Omega_2$.

Since $BU$ and $CV$ intersect at $H$, the orthocenter of $\triangle ABC$ is the radical center of $\Omega$, $\Omega_1$, and $\Omega_2$. Hence $H$ lies on the radical axis of $\Omega_1$ and $\Omega_2$, that is, $H$ lies on the line $PQ$. This completes the proof of the collinearity of $P$, $Q$ and the orthocenter of $\triangle ABC$.

As shown in the proof, the condition $\triangle ABC$ acute is not necessary.

Next we finish up the solutions to problems of the $55^{\text{th}}$ Czech and Slovak Mathematical Olympiad 2006 given at [2009: 81–82].

**6**. (J. Švrček, P. Calábek) Solve in real numbers the system of equations

$$\left.\begin{aligned} \tan^2 x + 2\cot^2 2y &= 1\,, \\ \tan^2 y + 2\cot^2 2z &= 1\,, \\ \tan^2 z + 2\cot^2 2x &= 1\,. \end{aligned}\right\} \qquad (1)$$

*Solved by Arkady Alt, San Jose, CA, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Zelator's write-up.*

First note that the real numbers $x = k\pi \pm \frac{\pi}{4}$, $y = m\pi \pm \frac{\pi}{4}$, $z = n\pi \pm \frac{\pi}{4}$; where $k$, $m$, $n$ are (arbitrary) integers; are solutions of the system (1).

We see, by inspection, that any solutions must satisfy $0 \le \tan^2 x$, $\tan^2 y$, $\tan^2 z \le 1$ and $0 \le \cot^2 2y$, $\cot^2 2x$, $\cot^2 2z \le \frac{1}{2}$. However, in fact none of $\tan x$, $\tan y$, $\tan z$ can be zero, since if $\tan x = 0$; then, and only then $x = t\pi$, for some $t \in \mathbb{Z}$. But then $2x = 2t\pi$; and so $\cot 2x$ would be undefined. Thus,

$$\left.\begin{array}{rcl} 0 < & \tan^2 x, \quad \tan^2 y, \quad \tan^2 z & \le 1 \\ \text{and } 0 \le & \cot^2 2y, \cot^2 2x, \cot^2 2z & < \frac{1}{2} \end{array}\right\} \quad (2)$$

Next, observe if one of $\tan^2 x$, $\tan^2 y$, $\tan^2 z$ is equal to $1$; then all three of them are and the three cotangent terms are zero. Indeed, suppose that $\tan^2 x = 1$; then $\tan x = 1$ or $-1$; which implies $x = k\pi \pm \frac{\pi}{4}$; $k \in \mathbb{Z}$. But then from the first equation in (1) we obtain $\cot 2y = 0$; $2y = \rho\pi + \frac{\pi}{2}$, $\rho \in \mathbb{Z}$; $y = \frac{\rho}{2}\pi + \frac{\pi}{4}$; when $\rho =$ even $= 2\lambda$; $y = \lambda\pi + \frac{\pi}{4}$; and when $\rho =$ odd $= 2\lambda + 1$; $y = \lambda\pi + \frac{3\pi}{4}$. These two formulas, since $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$; can be condensed into one: $y = m\pi \pm \frac{\pi}{4}$; $m \in \mathbb{Z}$.

Then $\tan^2 y = 1$; and from the second equation in (1), we obtain $\cot 2z = 0$; and from which (via a similar argument) we obtain $z = n\pi \pm \frac{\pi}{4}$; $n \in \mathbb{Z}$. We conclude that either $\tan^2 x = \tan^2 y = \tan^2 z = 1$, which produces the solutions $x = k\pi \pm \frac{\pi}{4}$, $y = m\pi \pm \frac{\pi}{4}$, $z = n\pi \pm \frac{\pi}{4}$. Or alternatively,

$$\left.\begin{array}{rcl} 0 < & \tan^2 x, \quad \tan^2 y, \quad \tan^2 z & < 1 \\ \text{and } 0 < & \cot^2 2y, \quad \cot^2 2x, \quad \cot^2 2z & < \frac{1}{2} \end{array}\right\} \quad (3)$$

Below, we shall find all the real solutions to system (1) that satisfy the conditions in (3). Suppose $(x_1, y_1, z_1)$ is a solution satisfying (3). We put $r_1 = \tan^2 x_1$, $r_2 = \tan^2 y_1$, and $r_3 = \tan^2 z_1$. So that by (3),

$$\left.\begin{array}{l} 0 < r_1 = \tan^2 x_1, \quad r_2 = \tan^2 x_1, \quad r_3 = \tan^2 z_1 < 1 \\ \text{and } 0 < \cot^2 2y_1, \quad \cot^2 2x_1, \quad \cot^2 2z_1 < \frac{1}{2} \end{array}\right\} \quad (4)$$

Since $(x_1, y_1, z_1)$ satisfies the system (1) we have,

$$\left.\begin{array}{rcl} \tan^2 x_1 + 2\cot^2 2y_1 & = & 1 \\ \tan^2 y_1 + 2\cot^2 2z_1 & = & 1 \\ \tan^2 z_1 + 2\cot^2 2x_1 & = & 1 \end{array}\right\} \quad (5)$$

Consider the first equation in (12) and multiply across by $\tan^2 2y_1$ in order to obtain, since $\cot^2 2y_1 \cdot \tan^2 2y_1 = 1$,

$$\tan^2 2y_1 \cdot \tan^2 x_1 + 2 = \tan^2 2y_1 .$$

Applying the double-angle identity $\tan 2y_1 = \frac{2\tan y_1}{1-\tan^2 y_1}$ gives $4\tan^2 y_1 \tan^2 x_1 + 2(1 - \tan^2 y_1)^2 = 4\tan^2 y_1$.

$$\Leftrightarrow \tan^4 y_1 + 2\tan^2 y_1 \tan^2 x_1 - 4\tan^2 y_1 + 1 = 0;$$

(with $r_1 = \tan^2 x_1$, $r_2 = \tan^2 y_1$) $r_2^2 + 2r_2r_1 - 4r_2 + 1 = 0$. Working similarly with the second and third equations in (12) we altogether obtain

$$\left.\begin{aligned} r_2^2 + 2r_2r_1 - 4r_2 + 1 &= 0 \\ r_3^2 + 2r_3r_2 - 4r_3 + 1 &= 0 \\ r_1^2 + 2r_1r_3 - 4r_1 + 1 &= 0 \end{aligned}\right\} \tag{6}$$

Adding the three equations in (6) yields

$$(r_1 + r_2 + r_3)^2 - 4(r_1 + r_2 + r_3) + 3 = 0;$$

or equivalently,

$$[(r_1 + r_2 + r_3) - 1][(r_1 + r_2 + r_3) - 3] = 0\,. \tag{7}$$

By (4), it follows that $0 < r_1 + r_2 + r_3 < 3$; which implies in conjunction with (7) that

$$r_1 + r_2 + r_3 = 1 \tag{8}$$

Now, we go to (6) and we multiply the first equation by $r_1r_3$, the second equation by $r_1r_2$; and the third equation by $r_2r_3$, in order to obtain

$$\left.\begin{aligned} r_1r_3r_2^2 + 2r_2r_3r_1^2 - 4r_1r_2r_3 + r_1r_3 &= 0 \\ r_1r_2r_3^2 + 2r_1r_3r_2^2 - 4r_1r_2r_3 + 4_14_2 &= 0 \\ r_2r_3r_1^2 + 2r_1r_2r_3^2 - 4r_1r_2r_3 + r_2r_3 &= 0 \end{aligned}\right\} \tag{9}$$

We add the three equations in (9) to get,

$$\begin{aligned} &r_1r_2r_3(r_1 + r_2 + r_3) + 2r_1r_2r_3(r_1 + r_2 + r_3) \\ &\quad -12r_1r_2r_3 + (r_1r_3 + r_1r_2 + r_2r_3) = 0; \end{aligned}$$

and by (8) we arrive at

$$r_1r_3 + r_1r_2 + r_2r_3 = 9r_1r_2r_3 = 9P \text{ (product)}. \tag{10}$$

By the Arithmetic-Geometric Mean inequality we also have,

$$r_1r_3 + r_1r_2 + r_2r_3 \ge 3\sqrt[3]{(r_1r_3)(r_1r_2)(r_2r_3)};$$

and by (10);

$$9P \ge 3P^{2/3} \Leftrightarrow P^{1/3} \ge \frac{1}{3};$$

or equivalently,

$$P \ge \frac{1}{27}\,. \tag{11}$$

Next, consider the cubic polynomial of degree three,

$$f(t) \; = \; (t - r_1)(t - r_2)(t - r_3); \tag{12}$$

$$f(t) = t^3 - (r_1 + r_2 + r_3)t^2 + (r_1 r_2 + r_2 r_3 + r_3 r_1)t - r_1 r_2 r_3;$$

and by (8) and (10);

$$f(t) \; = \; t^3 - t^2 + 9P \cdot t - P \,. \tag{13}$$

Next, we make use of the following lemma. The facts stated in the Lemma are well-known, standard material on cubic polynomial functions.

**Lemma 1**. Suppose that $g(t)$ is a cubic polynomial function of degree $3$ with leading coefficient $a > 0$; and let $g'(t)$ be the derivative of $g(t)$; $g'(t)$ being a quadratic trinomial.

(i) If the discriminant of $g'(t)$ is negative; then the function $g(t)$ has no critical numbers in its domain, and therefore it has no points of local maximum or local minimum. The function $g(t)$ is increasing throughout $\mathbb{R}$, and has only one inflection point. And the function $g(t)$ has *exactly one real root*, and two conjugate complex roots.

(ii) If the discriminant of $g'(t)$ is zero; then the function $g(t)$ has exactly one critical number $\rho$ in its domain $\mathbb{R}$. The point $(\rho, g(\rho))$ is both an inflection point and a critical point on the graph of $g(t)$. The function $g(t)$ increases throughout $\mathbb{R}$, it has no points of local maximum or minimum. Furthermore $g(t)$ has the form, $g(t) = a(t - \rho)^3 + \kappa$, for some $\kappa \in \mathbb{R}$. Thus, $g(t)$ has a triple real root, the real number

$$r \; = \; \sqrt[3]{-\frac{\kappa}{a} + \rho}\,.$$

Consider the derivative of $f(t)$ in (13): $f'(t) \; = \; 3t^2 - 2t + 9P$. The discriminant of $f'(t)$ is $D = 4 - 4 \cdot 3 \cdot 9 \cdot P = 4(1 - 27P)$. By (11) it follows that $D = 0$ or $D < 0$. The second possibility, $D < 0$ is eliminated by Lemma 1(i), since according to (12), $f(t)$ has three real roots. Thus we must have $D = 0$.

By Lemma 1(ii), it follows that $r_1 = r_2 = r_3$; and therefore from (8) we obtain $r_1 = r_2 = r_3 = \frac{1}{3}$. Back to (4):

$$\tan^2 x_1 = \frac{1}{3} \;\Leftrightarrow\; \tan x_1 \; = \; \pm\frac{1}{\sqrt{3}}; \qquad x_1 \; = \; k\pi \pm \frac{\pi}{6}; \qquad \text{for some } k \in \mathbb{Z}\,.$$

Likewise, we obtain

$$y_1 \; = \; m\pi \pm \frac{\pi}{6}\,, \qquad \text{for } m \in \mathbb{Z}\,,$$

and

$$z_1 \; = \; n\pi \pm \frac{\pi}{6}\,, \qquad \text{for } n \in \mathbb{Z}\,.$$

Conversely, one can verify directly that if the triple $(x_1, y_1, z_1)$ has the above form; then it satisfies the system (1).

**Conclusion**. The set of solutions $S$ of system (1) is the union of two disjoint families or solution sets $S_1, S_2$;

$$S = S_1 \cup S_2 ,$$

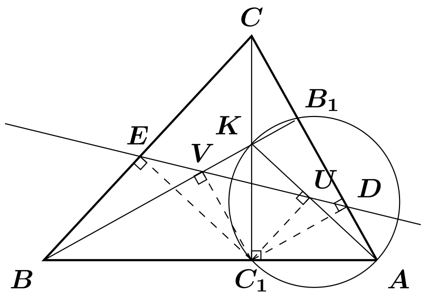
where $S_1 = \{(x, y, z) \mid x, y, z \in \mathbb{R}$ and such that $x = k\pi \pm \frac{\pi}{4}$, $y = m\pi \pm \frac{\pi}{4}$, $z = n\pi \pm \frac{\pi}{4}\}$; where $m$, $n$, $k$ can be any integers; and all eight combinations of signs are allowed. And $S_2 = \{(x, y, z) \mid x, y, z \in \mathbb{R}$ and such that $x = k\pi \pm \frac{\pi}{6}$, $y = m\pi \pm \frac{\pi}{6}$, $z = n\pi \pm \frac{\pi}{6}\}$; where $m$, $n$, $k$ can be any integers; and all eight combinations of signs are allowed.

**7**. (I. Voronovich) The point $K$ (distinct from the orthocentre) lies on the altitude $CC_1$ of the acute triangle $ABC$. Prove that the feet of the perpendiculars from $C_1$ to the segments $AC$, $BC$, $BK$, and $AK$ lie on a circle.

*Solution by Michel Bataille, Rouen, France.*

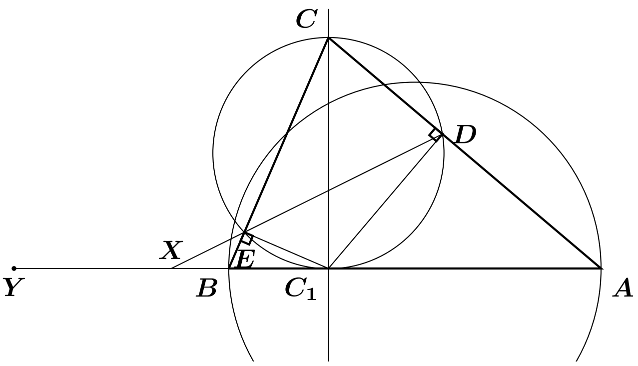
We denote by $D$, $U$, $V$, $E$ the feet of the perpendiculars from $C_1$ to $AC$, $AK$, $BK$, $BC$, respectively, and by $H$ the orthocentre of $\Delta ABC$.

First, we show that $D, U, V, E$ are not collinear. Assuming the contrary, let $BK$ meet $AC$ at $B_1$. From Simson's theorem, $C_1$ lies on the circumcircle of $\Delta AKB_1$. Since $\angle AC_1K = 90°$, it follows that $AK$ is a diameter of the circle $(AKB_1)$ and so $\angle KB_1A = 90°$. Thus $BB_1$ is the altitude from $B$ of $\Delta ABC$ and $K$ is its orthocentre, contradicting the hypothesis.




If $CA = CB$, since the line $CC_1$ is an axis of symmetry of the figure, $DUVE$ is an isosceles trapezium and $D, U, V, E$ are concyclic. From now on, we assume that $CA \neq CB$.

Let $X$ be the point of intersection of the lines $DE$ and $AB$. Since the circle with diameter $CC_1$ passes through $D$ and $E$, we have $XD \cdot XE = XC_1^2$. On the other hand, since $\angle XEB = \angle CED = \angle CC_1D = \angle DAX$, the triangle $XBE$ and $XDA$ are similar.




It follows that $XD \cdot XE = XA \cdot XB$ and so

$$XA \cdot XB \;=\; XC_1^2. \tag{1}$$

Now, if $Y$ is the pole of the line $CC_1$ with respect to the circle with diameter $AB$, points $Y, C_1$ are harmonic conjugates with respect to $A, B$ and so $(1)$ characterizes $X$ as being the midpoint of $YC_1$. Reasoning in the same way with triangle $KAB$ instead of $CAB$, we see that the line $UV$ passes through $X$ as well and that $XU \cdot XV = XC_1^2 = XD \cdot XE.$ Since $U, V, D, E$ are not collinear, the relation $XU \cdot XV = XD \cdot XE$ implies that $U, V, D, E$ are concyclic, as required.

**8**. (E. Barabanov, V. Kaskevich, S. Mazanik, I. Voronovich) An equilateral triangle of side $n$ is divided into $n^2$ unit equilateral triangles by lines parallel to its sides. Determine the smallest possible number of small triangles that must be marked so that any unmarked triangle has at least one side in common with a marked triangle.

*Solution by Oliver Geupel, Brühl, NRW, Germany.*

Let $T_n$ be the smallest number of marked triangles. We prove: $T_n = \left\lceil \frac{n^2}{4} \right\rceil$.

Since each marked triangle has at most three neighboring unmarked triangles, we have $3T_n \geq n^2 - T_n$; hence $T_n \geq \left\lceil \frac{n^2}{4} \right\rceil$. Vice versa, we show that

$$T_n \;\leq\; \left\lceil \frac{n^2}{4} \right\rceil. \tag{1}$$

The proof is by induction. The relation (1) holds for $n \in \{1, 2\}$; see Figure 1 for $n = 2$.
Assume that(1) holds for each $n \in \{1, 2, \ldots, N-1\}$.
Let $\Delta_k$ denote an equilateral triangle of side length $k$.
We make three cases.

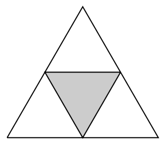


**Figure 1**

**Case 1**: $N$ is even.

The triangle $\Delta_N$ can be covered by one $\Delta_{N-2}$ and $N-1$ triangles $\Delta_2$ (Figure 2). By induction,

$$T_N \;\leq\; T_{N-2} + (N-1)T_2 \;\leq\; \left(\frac{N}{2} - 1\right)^2 + N - 1 \;=\; \left\lceil \frac{N^2}{4} \right\rceil.$$

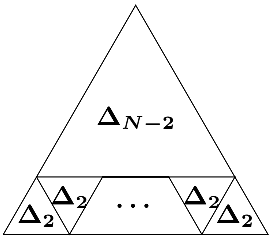



**Figure 2**

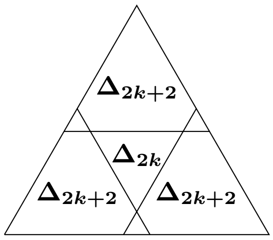



**Figure 3**

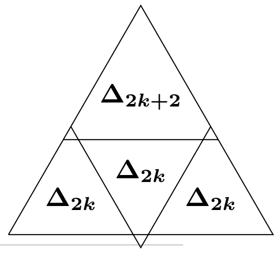



**Figure 4**

**Case 2**: $N = 4k + 3 \quad (k \geq 0)$.

The triangle $\Delta_N$ can be covered by one $\Delta_{2k}$ and three $\Delta_{2k+2}$ (Figure 3). By induction,

$$T_N \;\leq\; T_{2k} + 3T_{2k+2} \;\leq\; k^2 + 3(k+1)^2 \;=\; \left\lceil \frac{(4k+3)^2}{4} \right\rceil .$$

**Case 3**: $N = 4k + 1 \quad (k \geq 1)$. The triangle $\Delta_N$ can be covered by three $\Delta_{2k}$ and one $\Delta_{2k+2}$ (Figure 4). By induction,

$$T_N \;\leq\; 3T_{2k} + T_{2k+2} \;\leq\; 3k^2 + (k+1)^2 \;=\; \left\lceil \frac{(4k+1)^2}{4} \right\rceil .$$

This completes the proof.

Next we look at readers' solutions to problems given in the November 2009 number of the *Corner* and the 24th Iranian Mathematical Olympiad, First Round, given at [2009 : 435].

**1**. Given integers $m > 2$ and $n > 2$, prove there is a sequence of integers $a_0$, $a_1$, $\dots$, $a_k$ such that $a_0 = m$, $a_k = n$, and $(a_i + a_{i+1}) \mid (a_i a_{i+1} + 1)$ for each $i = 0, 1, \dots, k-1$.

*Solution by Titu Zvonaru, Comăneşti, Romania.*

Taking $a_0 = m$, $a_1 = 1$, $a_2 = 1, \dots, a_{k-1} = 1$, $a_k = n$ we have

$$\begin{aligned} a_0 + a_1 \;&=\; m+1; \quad a_0a_1 + 1 \;=\; m+1 \\ a_1 + a_2 \;&=\; 2; \quad a_1a_2 + 1 \;=\; 2 \\ &\;\vdots \quad \vdots \\ a_{k-2} + a_{k-1} \;&=\; 2; \quad a_{k-2}a_{k-1} + 1 \;=\; 2 \\ a_{k-1} + a_k \;&=\; n+1; \quad a_{k-1}a_k + 1 \;=\; n+1 \end{aligned}$$

hence $(a_i + a_{i+1}) \mid (a_i a_{i+1} + 1)$ for each $i = 0, 1, \dots, k-1$.

**4**. Find all two-variable polynomials $p(x, y)$ with real coefficients such that $p(x+y, x-y) = 2p(x, y)$ for all real numbers $x$ and $y$.

*Solution by Arkady Alt, San Jose, CA, USA.*

Note that $p(2x, 2y) = p((x+y)+(x-y), (x+y)-(x-y)) = 2p(x+y, x-y) = 4p(x, y)$. Excluding the trivial case $p(x, y) \equiv 0$ we assume further that $p(x, y) \neq 0$.

Since $p(0, 0) = p(2 \cdot 0, 2 \cdot 0) = 4p(0, 0)$ then $p(0, 0) = 0$ and $p(x, y)$ is not constant, moreover $p(x, 0)$ and $p(0, y)$ are not constants. Note that any such two-variable polynomial $p(x, y)$ can be represented in the form $p(x, y) = A(x) + B(y) + xyC(x, y)$, where $\deg A(x) = n > 0, \deg B(y) = m > 0,$ more precisely $A(x) = p(x, 0) = a_n x^n + a_{n-1}x^{n-2} + \ldots + a_1 x,\ B(y) = p(0, y) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x$, where $a_n \neq 0, b_m \neq 0.$

Since $p(2x, 2y) = 4p(x, y)$ then in particular for $y = 0$ and any real $x$ we have $p(2x, 0) = 4p(x, 0)$ if and only if $2^n a_n = 4a_n, 2^{n-1}a_{n-1} = 4a_{n-1}, \ldots, 2a_1 = 4a_1$ if and only if $n = 2, k_1 = 0,$ Similarly we obtain $m = 2, b_1 = 0.$

Thus, $p(x, y) = ax^2 + xyC(x, y) + by^2.$ Since $p(x, x) = x^2(a + b + C(x, x))$ and $p(2x, 2x) = 4p(x, x)$ then for $x \neq 0$ we have $4x^2(a + b + C(2x, 2x)) = 4x^2(a + b + C(x, x))$ if and only if $C(2x, 2x) = C(x, x)$ if and only if $C(x, x)$ is constant.

Indeed, since $C(x, x) = c + c_1 x^2 + \cdots + c_k x^{2k}$ then $C(2x, 2x) = C(x, x)$ if and only if $c_i = 2^{2i}c_i, i = 1, 2, \ldots, k$ if and only if $c_i = 0$, $i = 1, 2, \ldots, k.$ So, $p(x, y) = ax^2 + cxy + by^2$ and since $p(x+y, x-y) = 2p(x, y)$ if and only if $a(x+y)^2 + c(x^2 - y^2) + b(x-y)^2 = 2ax^2 + 2cxy + 2by^2$ if and only if $(b + c - a)x^2 + (a - b - c)y^2 + 2(a - c - b)xy = 0$ for any $x, y$ then $c = a - b.$

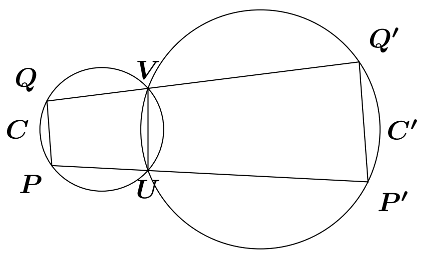
Therefore, $p(x, y) = ax^2 + (a - b)xy + by^2$ and all such two-variable polynomials $p(x, y)$ of the second degree satisfy $p(x+y, x-y) = 2p(x, y).$

Indeed, $p(x+y, x-y) = ax^2 + 2axy + ay^2 + bx^2 - 2bxy + by^2 + (b - a)(x^2 - y^2) = 2(ax^2 + (a - b)xy + by^2) = 2p(x, y).$

**5**. Let $\omega_1$ and $\omega_2$ be two circles such that the centre of $\omega_1$ is located on $\omega_2$. If the circles intersect at $M$ and $N$, $AB$ is an arbitrary diameter of $\omega_1$, and $A_1$ and $B_1$ are the second intersections of $AM$ and $BN$ with the circle $\omega_2$ (respectively), prove that $A_1B_1$ is equal to the radius of $\omega_1$.

*Solution by Michel Bataille, Rouen, France.*

The following lemma will be used twice: Let $C$ and $C'$ be two circles intersecting at $U, V$. If points $P, Q$ on $C$ and $P', Q'$ on $C'$ are such that $P, U, P'$ and $Q, V, Q'$ are collinear, then $PQ$ and $P'Q'$ are parallel.

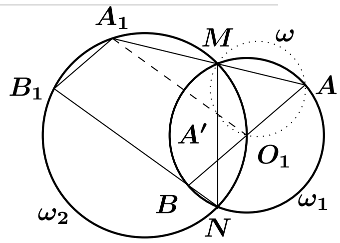


Indeed, denoting by $\angle(\cdot,\cdot)$ the directed angle of lines, we have (modulo $\pi$)

$$\begin{aligned}\angle(PQ, P'Q') &= \angle(PQ, PU) + \angle(P'U, P'Q') \\ &= \angle(VQ, VU) + \angle(VU, VQ') = \angle(VQ, VQ') = 0\end{aligned}$$

where the second equality follows from the concyclicity of $P$, $Q$, $U$, $V$ and $P'$, $Q'$ ,$U$ ,$V$.

This lemma clearly implies that $A_1B_1$ and $AB$ are parallel. Now, consider the inversion $\mathbf{I}$ in the circle $\omega_1$. Since $\mathbf{I}(M) = M$ and $\mathbf{I}(N) = N$, $\mathbf{I}$ exchanges the line $MN$ and the circle $\omega_2$. It follows that $\mathbf{I}(A_1) = A'$ is the point of intersection of $MN$ and $O_1A_1$.




We remark that $A'$ is also on the the circumcircle $\omega$ of $\Delta O_1AM$, which is the image of $AM$ under $\mathbf{I}$. Applying again the lemma, but this time to the circles $\omega$ and $\omega_1$, we obtain $BN \parallel O_1A'$. As a result, we have $A_1B_1 \parallel O_1B$ and $BB_1 \parallel O_1A_1$ so that $A_1B_1BO_1$ is a parallelogram and $A_1B_1 = O_1B$, the radius of $\omega_1$.

Next we look at solutions from the file for the 24th Iranian Mathematical Olympiad, Third Round, given at [2009 : 437–438].

**2**. Does there exist a sequence of positive integers $a_0$, $a_1$, $a_2$, $\ldots$ such that $\gcd(a_i, a_j) = 1$ whenever $i \neq j$, and for all $n$ the polynomial $\sum\limits_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?

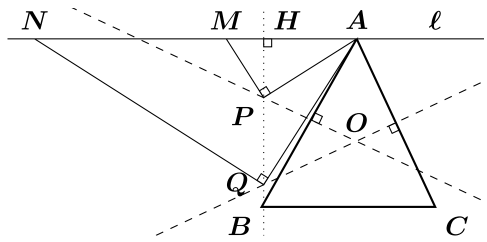
*Solved by Mohammed Aassila, Strasbourg, France.*

Consider, for example, $a_n = p_n$ where $(p_n)$ is a sequence of prime numbers such that $p_n > p_0 + p_1 + \cdots + p_{n-1}$. Then, by this condition, all the roots of $\sum_{i=0}^{n} p_i z^i$ lie inside the unit circle $|z| < 1$.

**3**. Triangle $ABC$ is isosceles with $AB = AC$. The line $\ell$ passes through $A$ and is parallel to $BC$. The points $P$ and $Q$ are on the perpendicular bisectors of $AB$ and $AC$, respectively, and such that $PQ \perp BC$. The points $M$ and $N$ are on $\ell$ and such that $\angle APM$ and $\angle AQN$ are right angles. Prove that

$$\frac{1}{AM} + \frac{1}{AN} \leq \frac{2}{AB}\,.$$

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give the solution by Bataille.*




Let $H$ be the orthogonal projection of $P$ (or $Q$) onto $\ell$. Since triangles $\Delta APM$ and $\Delta AQN$ are right-angled at $P$ and $Q$, respectively, we have $AP^2 = AH \cdot AM$ and $AQ^2 = AH \cdot AN$, hence

$$\frac{1}{AM} = \frac{AH}{AP^2} \text{ and } \frac{1}{AN} = \frac{AH}{AQ^2}.$$

Now, let $O$ denote the circumcentre of $\Delta ABC$. Then, from $\angle OPQ = \angle ABC = B$ and $\angle PQO = \angle ACB = C = B$ (acute angles with perpendicular sides) we deduce $OP = OQ$ and $\angle POA = B, \angle AOQ = 180^\circ - B$. Using the law of cosines and denoting by $R$ the circumradius of $\Delta ABC$, it follows that

$$\begin{aligned} AP^2 &= OP^2 + R^2 - 2R \cdot OP \cdot \cos B \\ &= (OP - R)^2 + 2R \cdot OP(1 - \cos B) \\ &= (OP - R)^2 + 4R \cdot OP \sin^2(B/2) \\ AQ^2 &= OP^2 + R^2 + 2R \cdot OP \cdot \cos B \\ &= (OP - R)^2 + 2R \cdot OP(1 + \cos B) \\ &= (OP - R)^2 + 4R \cdot OP \cos^2(B/2) \end{aligned}$$

and so

$$AP^2 \geq 4R \cdot OP \sin^2(B/2), \qquad AQ^2 \geq 4R \cdot OP \cos^2(B/2).$$

Observing that $\dfrac{AH}{OP} = \sin\angle(POA) = \sin B$, we obtain

$$\begin{aligned} \frac{1}{AM} + \frac{1}{AN} &= \frac{AH}{AP^2} + \frac{AH}{AQ^2} \leq \frac{1}{2R}\left(\frac{\cos(B/2)}{\sin(B/2)} + \frac{\sin(B/2)}{\cos(B/2)}\right) \\ &= \frac{1}{2R\sin(B/2)\cos(B/2)} = \frac{2}{AB}. \end{aligned}$$

(since $AB = 2R\sin B$ by the law of sines). Clearly equality holds if and only if $OP = OQ = R$.

**6**. Find all polynomials $p(x)$ of degree $3$ such that for all nonnegative real numbers $x$ and $y$

$$p(x+y) \ \geq \ p(x) + p(y)\,.$$

*Solved by Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Bataille.*

Let us say that $p(x)$ is superadditive if $p(x+y) \geq p(x) + p(y)$ for all nonnegative $x$ and $y$. We show that the superadditive polynomials of degree 3 are the polynomials

$$ax^3 + bx^2 + cx + d$$

where $c \in \mathbb{R},\ a > 0,\ d \le 0$ and $8b^3 \ge 243da^2$.

Let $p(x) = ax^3 + bx^2 + cx + d$ where $a, b, c, d$ are real numbers and $a \neq 0$ and let

$$\delta(x, y) = p(x + y) - p(x) - p(y) = 3axy(x + y) + 2bxy - d.$$

Clearly, $p(x)$ is superadditive if and only if $\delta(x, y) \ge 0$ for all $x, y \ge 0$. First, assume that it is the case. Then, $\delta(0, 0) \ge 0$, hence $d \le 0$. Also, for $x > 0$,

$$3a + \frac{b}{x} - \frac{d}{2x^3} = \frac{\delta(x, x)}{2x^3} \ge 0$$

so that $3a \ge 0$ (by letting $x$ approach infinity) and $a > 0$ is obtained. Lastly, $8b^3 \ge 243da^2$ holds if $b \ge 0$ (since $d \le 0$) and if $b < 0$ results from

$$0 \le \delta\left(\frac{-2b}{9a}, \frac{-2b}{9a}\right) = \frac{8b^3 - 243da^2}{243a^2}.$$

Conversely, assume that the conditions $a > 0,\ d \le 0$ and $8b^3 \ge 243da^2$ hold. Clearly, $\delta(x, y) \ge 0$ for all $x, y \ge 0$ if $b \ge 0$. Suppose that $b < 0$. Observing that

$$\delta(x, y) \ge 3axy \cdot 2\sqrt{xy} + 2bxy - d = \phi(\sqrt{xy})$$

where $\phi(t) = 6at^3 + 2bt^2 - d$, it suffices to show that $\phi(t) \ge 0$ for $t \ge 0$. Since the derivative of $\phi$ is given by $\phi'(t) = 18at\left(t + \frac{2b}{9a}\right)$, it is readily seen that the minimum of $\phi$ on $[0, \infty)$ is $\phi\left(\frac{-2b}{9a}\right) = \frac{8b^3 - 243da^2}{243a^2}$, which is nonnegative by assumption. Thus, $\phi(t) \ge 0$ for $t \ge 0$ and the proof is complete.

We continue with solutions from our readers to problems given in the May 2010 number of the *Corner* with the XVIII Olimpiada de Matemática de Paises del Cono Sur given at [2010: 217–218].

**1**. Find all the pairs $(x, y)$ of nonnegative integers that satisfy

$$x^3y + x + y \ = \ xy + 2xy^2 \,.$$

*Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Geoffrey A. Kandall, Hamden, CT, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's write-up.*

The solutions are the pairs $(0, 0),\ (1, 1),\ (2, 2)$.

It is readily checked that these pairs are solutions. Conversely, let $(x, y)$ be any solution. Then we have

$$y = x(y + 2y^2 - 1 - x^2y) \tag{1}$$

and

$$x = y(x - x^3 - 1 + 2xy). \tag{2}$$

From (1), $x$ divides $y$ (since $y + 2y^2 - 1 - x^2y$ is an integer) and from (2), $y$ divides $x$ (since $x - x^3 - 1 + 2xy$ is an integer). It follows that $|x| = |y|$ and since $x, y$ are nonnegative integers, $x = y$.

Now, from the equation, we must have $x^4 + 2x = x^2 + 2x^3$ that is,

$$x(x^2 - 1)(x - 2) = 0,$$

which implies $x \in \{0, 1, 2\}$. The result follows.

**2**. Given are $100$ positive whole numbers whose sum equals their product. Determine the minimum number of occurrences of the number $1$ among the $100$ numbers.

*Solved by Henry Ricardo, Tappan, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the version by Zvonaru.*

Let $x_{100} \le x_{99} \le \cdots \le x_2 \le x_1$ be the given $100$ positive integers and let $k_{\min}$ be the searched minimum number of $1$'s.

The equation $x_{100} + x_{99} + \cdots + x_2 + x_1 = x_{100}x_{99} \ldots x_2x_1$ can be rewritten as

$$\frac{x_{100}}{x_{100} \cdot x_{99} \cdots x_2x_1} + \cdots + \frac{x_2}{x_{100} \cdot x_{99} \cdots x_2x_1} + \frac{x_1}{x_{100} \cdot x_{99} \cdots x_2x_1} = 1.$$

Since $x_{100} \le x_{99} \le \cdots \le x_2 \le x_1$, it results that $\frac{x_1}{x_{100} \cdot x_{99} \cdots x_2x_1} \ge \frac{1}{100}$, that is $x_{100} \cdot x_{99} \cdots x_2 \le 100$.

We deduce that at most $6$ numbers among $x_{100}, x_{99}, \cdots, x_2$ may be greater than $1$, hence $k_{\min} \ge 93$.

Suppose that $k_{\min} = 93$. We must solve the equation

$$93 + x_7 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1 = x_1x_2x_3x_4x_5x_6x_7,$$

where $x_1 \ge x_2 \ge x_3 \ge x_4 \ge x_5 \ge x_6 \ge x_7 \ge 2$.

We have

$$93 + 7x_1 \ge 93 + x_1 + x_2 + x_3 + x_4 + x_5 + x_5 + x_7 = x_1x_2x_3x_4x_5x_6x_7 \ge 2^6x_1$$

that is $57x_1 \le 93$, a contradiction with $x_1 \ge 2$.

Suppose now that $k_{\min} = 94$. As above, we must solve the equation

$$94 + 6x_1 \ge 94 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1 = x_1x_2x_3x_4x_5x_6.$$

We have

$$94 + 6x_1 \ge x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = x_1x_2x_3x_4x_5x_6 \ge 32x_1,$$

that is $26x_1 \le 94$.

If $x_1 = 2$, then $x_2 = x_3 = x_4 = x_5 = x_6$ and we have no solution (because $94 + 6 \cdot 2 \neq 2^6$).

If $x_1 = 3$, we have to solve the equation

$$97 + x_6 + x_5 + x_4 + x_3 + x_2 = 3x_2x_3x_4x_5x_6.$$

We deduce that

$$97 + 5x_2 \geq 97 + x_2 + x_3 + x_4 + x_5 + x_6 = 3x_2x_3x_4x_5x_6 \geq 48x_2$$

and it follows that $43x_2 \leq 97$, hence $x_2 = 2$.

We obtain $x_2 = x_3 = x_4 = x_5 = x_6 = 2$ and we have no solution (because $97 + 5 \cdot 2 \neq 3 \cdot 32$).

Since the equation

$$95 + x_1 + x_2 + x_3 + x_4 + x_5 = x_1x_2x_3x_4x_5$$

has solution $x_1 = x_2 = x_3 = 3$, $x_4 = x_5 = 2$

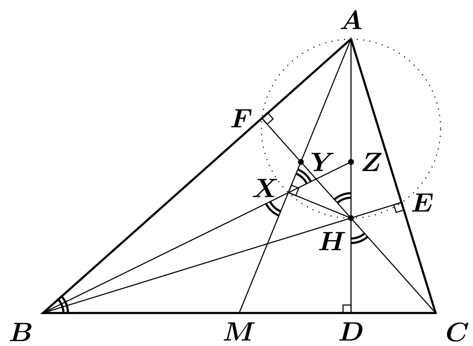
$$(95 + 3 + 3 + 3 + 2 + 2 = 108,\ 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 108),$$

it results that $k_{\min} = 95$.

**3**. Let $ABC$ be an acute triangle with altitudes $AD$, $BE$, $CF$, where $D, E, F$ lie on $BC$, $AC$ $AB$, respectively. Let $M$ be the midpoint of $BC$. The circumcircle of triangle $AEF$ cuts the line $AM$ at $A$ and $X$. The line $AM$ cuts the line $CF$ at $Y$. Let $Z$ be the point of intersection of $AD$ and $BX$. Show that the lines $YZ$ and $BC$ are parallel.

*Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Geupel's solution.*

Let $\alpha = \angle CAB$, $\beta = \angle ABC$, and let $H$ be the orthocentre of $\triangle ABC$.




Because of $\triangle ABD \sim \triangle AHF$, we have $AD : AB = AF : AH$, so that $AD \cdot AH = AB \cdot AF = AB \cdot AC \cos\alpha$. For the median line $AM$, we have

$$4AM^2 = 2AB^2 + 2AC^2 - BC^2 .$$

By the Law of Cosines it now follows that

$$BC^2 = 2AB^2 + 2AC^2 - BC^2 - 4AB \cdot AC \cos\alpha = 4AM^2 - 4AD \cdot AH;$$

hence

$$BM^2 = AM^2 - AD \cdot AH.$$

By $\angle AEH = \angle AFH = 90^\circ$, the point $H$ lies on the circumcircle of quadrilateral $AEXF$. Hence $\angle AXH = \angle AFH = 90^\circ$. Thus $\triangle AHX \sim \triangle AMD$, which implies that $AH : AX = AM : AD$ and therefore

$$\begin{aligned} AM^2 - BM^2 &= AD \cdot AH = AM \cdot AX \\ &= AM(AM - MX) = AM^2 - AM \cdot MX \,. \end{aligned}$$

We obtain $BM^2 = AM \cdot MX$ and therefore $AM : BM = BM : XM$. Noticing that $\angle AMB = \angle BMX$, we obtain $\triangle ABM \sim \triangle BXM$. Hence, $\angle YXZ = \angle BXM = \angle ABM = \beta$. Also, $\angle YHZ = \angle CHD = \angle DBA = \beta$. We obtain $\angle YXZ = \angle YHZ$, so the quadrilateral $HXYZ$ is cyclic, which establishes $\angle HZY = 180^\circ - \angle HXY = 90^\circ$ and $YZ \parallel BC$.

**4**. Some cells of a $2007 \times 2007$ table are coloured. The table is "charrua" if none of the rows and none of the columns are completely coloured.

(a) What is the maximum number $k$ of coloured cells that a charrua table can have?

(b) For such $k$, calculate the number of distinct charrua tables that exist.

*Solved by Titu Zvonaru, Comăneşti, Romania.*

(a) We will determine the minimum number $k'$ of uncoloured cells that a charrua table can have.

It is easy to see that this minimum of uncoloured cells is $2007$ (if $k' < 2007$, then there is at least a row or a column which is completely coloured).

An example of a charrua table with $k' = 2007$: the uncoloured cells are $c(1,1), c(2,2), \ldots, c(2007,2007)$.

It results that the maximum $k$ of coloured cells that a charrua table can have is $2007 \times 2007 - 2007 = 2006 \times 2007$.

(b) We can choose in $2007!$ ways the uncoloured cells, one in each row (but not two in the same columns).

**5**. Let $ABCDE$ be a convex pentagon that satisfies the following:

(i) There is a circle $\Gamma$ tangent to each of the sides.

(ii) The lengths of the sides are all whole numbers.

(iii) At least one of the sides of the pentagon has length $1$.

(iv) The side $AB$ has length $2$.

Let $P$ be the point of tangency of $\Gamma$ with $AB$.

(a) Determine the length of segments $AP$ and $BP$.

(b) Give an example of a pentagon satisfying the given conditions.

*Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We print Zvonaru's solution.*

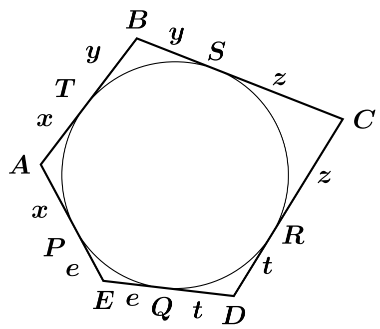
(a) Let $Q, R, S, T$ be the points of tangency of $\Gamma$ with $BC$, $CD$, $DE$, $EA$ respectively.

Let $AP = AT = x$, $BP = BQ = y$, $CQ = CR = z$, $DR = DS = t$, $ES = ET = e$.

We denote by $\mathbb{Z}$ the set of integers.

Without loss of generality, we may assume that $x \geq y$.

We have




$$\left.\begin{array}{r} x+y \in \mathbb{Z} \\ y+z \in \mathbb{Z} \end{array}\right\} \Rightarrow \quad x+y-(y+z) \in \mathbb{Z} \Rightarrow x-z \in \mathbb{Z} \qquad (1)$$

$$\left.\begin{array}{r} z+t \in \mathbb{Z} \\ t+e \in \mathbb{Z} \end{array}\right\} \Rightarrow \quad z-e \in \mathbb{Z} \qquad (2)$$

$$\left.\begin{array}{r} e+x \in \mathbb{Z} \\ x+y \in \mathbb{Z} \end{array}\right\} \Rightarrow \quad e-y \in \mathbb{Z} \qquad (3)$$

By (1), (2), and (3) we obtain $x - z + e - y \in \mathbb{Z} \Rightarrow x - y \in \mathbb{Z}$. It results that there is one integer $m$ such that

$$\begin{aligned} x - y &= m \\ x + y &= 2. \end{aligned}$$

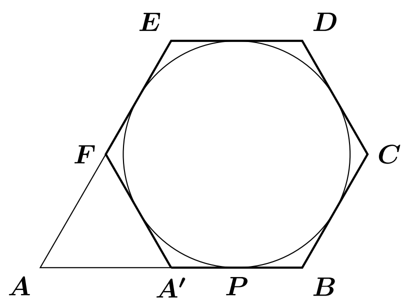
It follows that $x = 1 + \frac{m}{2}$. Since $1 \leq x < 2$, we deduce that $1 \leq 1 + \frac{m}{2} < 2 \Leftrightarrow 0 \leq m < 2$.

If $m = 0$, then $x = y = 1$ and we obtain that $z$, $e$ and $t$ are positive integers; this leads to a contradiction with condition (iii).

If $m = 1$, then $x = \frac{3}{2}$, $y = \frac{1}{2}$, hence $AP = \frac{3}{2}$, $BP = \frac{1}{2}$.

(b) Let $A'BCDEF$ be a regular hexagon with $A'B = 1$. This hexagon has an inscribed circle (tangent to each of the sides).

The line $EF$ intersects the line $A'B$ at the point $A$. Since $\triangle AA'F$ is equilateral, we have $AB = AE = 2$, $BC = CD = DE = 1$, hence the pentagon $ABCDE$ satisfies the given conditions.




**6**. Show that for each positive whole number $n$, there is a positive whole number $k$ such that the decimal representation of each of the numbers $k, 2k, \ldots, nk$ contains all the digits $0, 1, 2, \ldots, 9$.

*Solved by Oliver Geupel, Brühl, NRW, Germany.*

Let the decimal representation of the number $k$ consist of the leading digit $1$, followed by $99$ blocks of length $\lfloor \log n \rfloor + 3$ where the $m^{th}$ block is the decimal representation of the number $m+1$ with leading zeros ($m = 1, 2, \ldots, 99$):

$$1\ \underbrace{0\ldots0002}_{\text{block } 1}\ \underbrace{0\ldots0003}_{\text{block } 2}\ \ldots \underbrace{0\ldots0099}_{\text{block } 98}\ \underbrace{0\ldots0100}_{\text{block } 99}$$

Note that the length of the decimal representation of the integer $j(m+1)$ is $\lfloor \log(j(m+1)) \rfloor + 1 \le \lfloor \log n + \log 100 \rfloor + 1 \le \lfloor \log n \rfloor + 3$ $(j = 1, \ldots, n)$. The decimal representation of the number $jk$ consists of the decimal representation of the number $j$ followed by $99$ blocks of length $\lfloor \log n \rfloor + 3$ where the $m^{th}$ block is the decimal representation of the number $j(m+1)$ with leading zeros.

It therefore suffices to prove that, for each $j$ ($j = 1, \ldots, n$), every digit occurs in the decimal representation of one of the numbers $j, 2j, 3j, \ldots, 100j$. To this end, for any fixed $j$ consider the integer $\ell$ such that $10^{\ell-1} \le j < 10^{\ell}$. It holds $10^{\ell+1} \le 100j$ and each of the nine intervals

$$[10^l, 2 \cdot 10^l), [2 \cdot 10^l, 3 \cdot 10^l), \ldots, [9 \cdot 10^l, 10^{l+1})$$

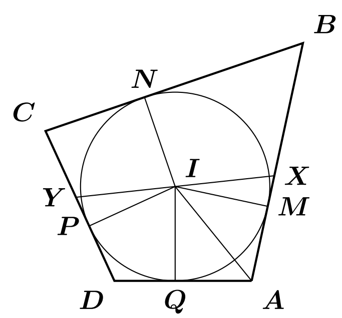
contains a number from the arithmetic sequence $j, 2j, 3j, \ldots, 100j$. The leading digit of the numbers in the $m^{th}$ interval is $m$ ($m = 1, 2, \ldots, 9$). Thus, the digits $1, 2, \ldots, 9$ occur. The last digit of $100j$ is zero. This completes the proof.

Next we move to the file of solutions from our readers to problems given in the September 2010 number of the *Corner* and the 2007 Bulgarian National Olympiad at [2010: 274].

**1**. (Emil Kolev, Alexandar Ivanov) The quadrilateral $ABCD$ is such that $\angle BAD + \angle ADC > 180^\circ$ and is circumscribed around a circle of center $I$. A line through $I$ meets $AB$ and $CD$ at points $X$ and $Y$, respectively. Prove that if $IX = IY$ then $AX \cdot DY = BX \cdot CY$.

*Solved by Titu Zvonaru, Comăneşti, Romania.*

Let $M, N, P, Q$ be the projections of point $I$ onto the sides $AB$, $BC$, $CD$, $DA$, respectively. Since $IX = IY$ and $IM = IP$, the right-angled triangles $IMX$ and $IPY$ are congruent, hence $\angle IXM = \angle IYP$.




If $Y$ lies between $D$ and $P$, and $M$ lies between $A$ and $X$, then $AB \| CD$, a contradiction with $\angle BAD + \angle AOC > 180^\circ$. We may assume that $Y$ lies between $C$ and $P$, and $X$ lies between $B$ and $M$.

We denote $a = AQ = AM$, $b = BM = BN$, $c = CN = CP$, $d = DP = DQ$, $\alpha = \angle BAD$, $\beta = \angle CBA$, $\gamma = \angle DCB$, $\delta = \angle ADC$, $\varphi = \angle IXM = \angle IYP$, $m = XM = YP$, $r = IM = IN = IP = IQ$.

Since $\tan\frac{\alpha}{2} = \tan\angle MAI = \frac{IM}{AM} = \frac{r}{a}$, we have

$$\varphi = \frac{360^\circ - \alpha - \delta}{2} = \frac{\beta+\gamma}{2},$$

hence

$$\tan\varphi = -\tan\frac{\alpha+\delta}{2} = -\frac{\frac{r}{a}+\frac{r}{d}}{1-\frac{r}{a}\cdot\frac{r}{d}} = \frac{r(a+d)}{r^2-ad}$$

and

$$\tan\varphi = \tan\frac{\beta+\gamma}{2} = \frac{r(b+c)}{bc-r^2}.$$

Since $\tan\varphi = \frac{r}{m}$, we deduce that

$$\frac{1}{m} = \frac{a+d}{r^2-ad} = \frac{b+c}{bc-r^2} = \frac{a+b+c+d}{bc-ad},$$

hence

$$m = \frac{bc-ad}{a+b+c+d}. \qquad (1)$$

It follows that $AX\cdot DY = BX\cdot CY$ is equivalent to

$$\begin{aligned}(a+m)(d+m) &= (b-m)(c-m)\\ ad+m(a+d) &= bc-m(b+c)\\ m &= \frac{bc-ad}{a+b+c+d},\end{aligned}$$

which is true by (1).

**3**. (Nikolai Nikolov, Oleg Mushkarov) Find the least natural number $n$ for which $\cos\frac{\pi}{n}$ cannot be expressed in the form $p+\sqrt{q}+\sqrt[3]{r}$, where $p$, $q$ and $r$ are rational numbers.

*Solved by Mohammed Aassila, Strasbourg, France.*

We will prove that the least positive integer is $n = 7$. Let $\omega = e^{\frac{i\pi}{7}}$. We have $0 = \omega^7+1 = (\omega+1)(1-\omega+\omega^2-\cdots+\omega^6)$, hence $0 = (\omega^3+\omega^{-3}) - (\omega^2+\omega^{-2}) + (\omega+\omega) - 1$. Let $x = \omega+\omega^{-1} = 2\cos\frac{\pi}{n}$, then $x$ verifies the equation $0 = (x^3-3x)-(x^2-2)+x-1 = x^3-x^2-2x+1$. Let $P(x) = x^3-x^2-2x+1$. Note that the roots of $P$ are $2\cos(\frac{\pi}{7})$, $2\cos(\frac{3\pi}{7})$ and $2\cos(\frac{5\pi}{7})$. Assume, by contradiction, that $x = P+\sqrt{q}+\sqrt[3]{r}$, then $x$ is irrational since if $x = \frac{a}{b}$ with $\gcd(a,b) = 1$ then $a^3-a^2b-2ab^2+b^3 = 0$, hence $b \mid a^3$, and $b = \pm1$ and similarly $a = \pm1$, in conclusion $x = \pm1$, which is false.

*Case 1*: $r = 0$.

Then $x^2 - 2px + p^2 - q = 0$. By the Euclidean division we have $P(X) = (X - a)(X^2 - 2pX + p^2 - q) + (bX + c)$. Hence $bx + c = 0$. Since $x$ is irrational, we have $b = c = 0$ and hence $P(X) = (X - a)(X^2 - 2pX + p^2 - q)$. Consequently, $P$ has a rational root $a$. Impossible.

*Case 2*: $q = 0$.

We have $(x - p)^3 = r$. Since $X^3 - 3pX^2 + 3p^2X - p^3 - r$ is not proportional to $P(X)$ (compare the coefficients of $X$ and of $X^2$), an Euclidean division yields that $x$ is a root of a polynomial (not equal to $0$) and with degree $\leq 2$, hence we are in case 1.

*Case 3*: $q \neq 0$ and $r \neq 0$.

We have $r = (x - p - \sqrt{q})^3$. Like in case 2, there exists a polynomial $A \neq 0$ of degree $\leq 2$ in $\mathbb{Q}[\sqrt{q}][X]$ such that $A(x) = 0$. If $\deg A = 1$ then we are in case 1. If $\deg A = 2$ by an Euclidean division we have $P(X) = A(X)(X - a) + B(X)$. Since $B(x) = 0$, then thanks to case 1 we have $B = 0$, hence $P$ has $a$ for a root in $\mathbb{Q} + \mathbb{Q}\sqrt{q}$. As in the first case, we have a contradiction.

**Alternative solution**: (in french and using higher algebra).

Soit $x = \cos(\pi/7)$. Soit $K = \mathbb{Q}(x)$. Alors $K$ est une extension galoisienne de degré $\varphi(14)/2 = 3$ de $\mathbb{Q}$. Si $x$ était de la forme $p + \sqrt{q} + \sqrt[3]{r}$, alors en élevant au cube $x - p - \sqrt{q}$, on voit que $\sqrt{q} \in K$. Comme $[K : \mathbb{Q}]$ est impair, il ne peut pas contenir d'extension quadratique de $\mathbb{Q}$ donc $\sqrt{q}$ est rationnel. Par conséquent, $x$ est de la forme $p + y$, où $y = \sqrt[3]{r}$. Comme $K/\mathbb{Q}$ est galoisienne, $K$ contient $jy$. On vérifie ensuite que $K = \mathbb{Q}[j, y]$ est de degré $6$ sur $\mathbb{Q}$. Contradiction.

**4**. ((Emil Kolev, Alexandar Ivanov) Let $k > 1$ be an integer. A set of positive integers $S$ is called *good* if all positive integers can be painted in $k$ colors such that no element of $S$ is a sum of two distinct numbers of the same color. Find the largest positive integer $t$ for which the set

$$S = \{a + 1, a + 2, a + 3, \ldots, a + t\}$$

is good for all positive integers $a$.

*Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

We claim that the largest such $t$, which we denote $t_{\max}(k)$, is $2k - 2$.

*Case 1.*

To show that $t_{\max}(k) \geq 2k - 2$, consider even and odd $a$ separately.

(a) If $a = 2r - 1$ and $t = 2k - 2$, then

$$S = \{2r, 2r + 1, \ldots, 2r + 2k - 3\}.$$

We assign colours as follows:

| | |
|---|---|
| Colour 1 | $1, 2, 3, \ldots, r$ |
| Colour 2 | $r + 1$ |
| Colour 3 | $r + 2$ |
| $\vdots$ | |
| Colour $k - 1$ | $r + k - 2$ |
| Colour $k$ | $r + k - 1$, $r + k$, $r + k + 1, \ldots, 2r + 2k - 4$ |

All other positive integers are assigned colour $k$. The sum of any two distinct integers of colour $1$ is smaller than any element of $S$; the sum of any two distinct integers of colour $k$ is larger than any element of $S$. Thus, no element of $S$ is a sum of two distinct integers of the same colour.

(b) If $a = 2r$ and $t = 2k - 2$, then

$$S = \{2r + 1, 2r + 2, \ldots, 2r + 2k - 2\}.$$

We assign colours as before except that colour $k$ is assigned to all integers from $n + k - 1$ through $2r + 2k - 3$. Again, no element of $S$ is the sum of two distinct integers of the same colour.

*Case 2.* To show that $t_{\max}(k) \leq 2k - 2$, suppose $t = 2k - 1$, and set $a = 2$. Then $S = \{3, 4, 5, \ldots, 2k + 1\}$. Consider the colours assigned to the integers $1, 2, 3, \ldots, k+1$. Suppose $i$ and $j$ are elements of this list with $i < j$, and assume $i$ and $j$ are assigned the same colour. Then $3 \leq i + j \leq k + (k + 1) = 2k + 1$, so $i + j \in S$. Hence the set $S$ is not good.

Thus, the largest $t$ is given by $t_{\max}(k) = 2k - 2$.

**5**. (Oleg Mushkarov, Nikolai Nikolov) Find the least number $m$ for which any five equilateral triangles with combined area $m$ can cover an equilateral triangle of area $1$.

*Solved by George Apostolopoulos, Messolonghi, Greece; and Oliver Geupel, Brühl, NRW, Germany. We give the version of Geupel.*

The answer is $m = 2$. For convenience of presentation, we rescale the given equilateral triangle $T$ such that its side length is $1$.

We firstly prove that any five equilateral triangles $T_1, \ldots, T_5$ with side lengths $a_1 \leq \cdots \leq a_5$ and $a_1^2 + \cdots + a_5^2 = 2$ can cover $T$.

Since the case $a_5 \geq 1$ is obvious, we may assume $a_5 < 1$. Then

$$(a_3 + a_4)^2 \geq 3a_3^2 + a_4^2 \geq 2 - a_5^2 > 1\,.$$

Let us place $T_3$, $T_4$, $T_5$ in one of the angles of $T$ each, as shown in figure 1. If $T_3$, $T_4$, and $T_5$ cover $T$ then we are done. Otherwise, there remains an uncovered equilateral triangle $T'$ in the interior of $T$. Each point of $T$ is covered by at most two of the triangles $T_3$, $T_4$, $T_5$, $T'$. Hence, the combined area of $T_1$ and $T_2$ is not less than twice the area of $T'$; thus $T_2$ covers $T'$. We have proved that $m \leq 2$.

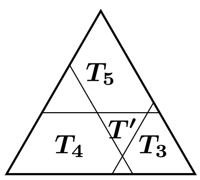


**Figure 1**

In order to show that $m \geq 2$, for arbitrary $\epsilon > 0$ we present equilateral triangles $T_1, \ldots, T_5$ with side lengths $a_1 \leq \cdots \leq a_5$ such that $a_1^2 + \cdots + a_5^2 > 2 - \epsilon$ that cannot cover $T$. Let

$$\delta = \frac{\epsilon}{20}\,, \qquad a_1 = a_2 = a_3 = \delta\,, \qquad a_4 = a_5 = 1 - 5\delta\,.$$

Indeed,

$$a_1^2 + \cdots + a_5^2 > 2 - 20\delta = 2 - \epsilon\,.$$

Let us pin congruent equilateral triangles $U$, $V$, $W$ with side length $2\delta$ to the vertices of $T$ as shown in figure 2. On applying $T_3$ and $T_4$ to $T$, each of $T_3$ and $T_4$ can meet at most one of the triangles $U$, $V$, $W$. Hence, there is a triangle among $U$, $V$, $W$ which is not met by $T_3$ and $T_4$, say $U$ has this property. But the combined area of $T_1$, $T_2$, $T_3$ is less than the area of $U$, so $T_1$, $T_2$, $T_3$ cannot cover $U$. This completes both our counterexample and the proof that $m \geq 2$.

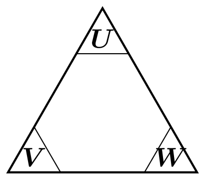


**Figure 2**

Next we look at solutions for the 48th IMO Bulgarian Team, First Selection Test, given at [2010: 275].

**1**. The sequence $\{a_i\}_{i=1}^{\infty}$ is such that $a_1 > 0$ and $a_{n+1} = \frac{a_n}{1+a_n^2}$ for $n \geq 1$.

(a) Prove that $a_n \leq \frac{1}{\sqrt{2n}}$ for $n \geq 2$;

(b) Prove that there exists $n$ such that $a_n > \frac{7}{10\sqrt{n}}$.

*Solved by Arkady Alt, San Jose, CA, USA; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give the solution by Alt.*

Since $a_n > 0, n \geq 1$ then

$$a_{n+1} = \frac{a_n}{1+a_n^2} \iff \frac{1}{a_{n+1}^2} = \left(\frac{1}{a_n} + a_n\right)^2$$
$$\iff \frac{1}{a_{n+1}^2} = \frac{1}{a_n^2} + a_n^2 + 2 \iff b_{n+1} = b_n + 2 + \frac{1}{b_n}\,,$$

where $b_n := \frac{1}{a_n^2}$, $n \geq 1$ and we will prove:

a) $b_n \geq 2n$ for $n > 2$;

b) There is $n$ such that $b_n < \frac{100n}{49}$.

a) Since $b_{n+1} = b_n + 2 + \frac{1}{b_n} > b_n + 2$ and $b_2 \geq 4$, by induction $b_n \geq 2n$.

b) Since $b_n \geq 2n$ for $n \geq 2$ then $b_{n+1} = b_n + 2 + \dfrac{1}{b_n} \leq b_n + 2 + \dfrac{1}{2n}$, $n \geq 2$ and, therefore,

$$b_{n+1} - b_2 = \sum_{k=2}^{n} (b_{k+1} - b_k) \leq \sum_{k=2}^{n} \left(2 + \frac{1}{2n}\right) = 2(n-1) + \frac{1}{2}(h_n - 1) ,$$

where $h_n = \sum\limits_{k=1}^{n} \dfrac{1}{n}$. Thus,

$$b_{n+1} \leq 2n - \frac{5}{2} + \frac{1}{2}h_n + b_2 < 2(n+1) + \frac{1}{2}h_{n+1} + b_2, \; n \geq 2$$
$$\implies b_n < 2n + \frac{1}{2}h_n + b_2, \; n \geq 3 .$$

Note that $h_n < \sqrt{2n}$, $n \in \mathbb{N}$. Indeed, by the Cauchy Inequality we have

$$h_n^2 \leq n \cdot \sum_{k=1}^{n} \frac{1}{k^2}$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{(k-1)k} = 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 + 1 - \frac{1}{n} < 2 .$$

Since $\frac{100n}{49} = 2n + \frac{2n}{49}$ and $h_n < \sqrt{2n}$ then it suffices to prove that there is $n$ such that $\frac{1}{2}\sqrt{2n} + b_2 < \frac{2n}{49} \iff 49b_2 < \sqrt{2n}\left(\sqrt{2n} - \frac{49}{2}\right)$.

It is easy to see that the latter inequality holds for any

$$n \geq n_0 = \max\left\{\frac{49^2 b_2^2}{2}, \frac{51^2}{8}\right\} .$$

Another variant of ending solution (b):

Since $2n < b_n < 2n + \dfrac{1}{2}\sqrt{2n} + b_2$ and $\lim\limits_{n\to\infty} \dfrac{2n + \frac{1}{2}\sqrt{2n} + b_2}{n} = 2$ then $\lim\limits_{n\to\infty} \dfrac{b_n}{n} = 2$ and, therefore, for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\dfrac{b_n}{n} < 2 + \varepsilon \iff b_n < (2+\varepsilon)\,n$ for all $n > n_0(\varepsilon)$. In particular for $\varepsilon = \dfrac{2}{49}$ we have $b_n < \left(2 + \dfrac{2}{49}\right) n = \dfrac{100n}{49}$ for all $n > n_0\left(\dfrac{2}{49}\right)$.

That completes the *Corner* for this issue.

# BOOK REVIEWS

Amar Sodhi

*Lobachevski Revisited* by Seth Braver
The Mathematical Association of America, 2011
ISBN: 978-0-88385-979-7 (e-book)
237 + x pages, US$35.00 (print on demand, US$95.00)
*Reviewed by* ***J. Chris Fisher****, University of Regina, Regina, SK*

During the 1820s Lobachevski (1792-1856) in Russia and Bolyai (1802-1860) in Hungary independently discovered non-Euclidean Geometry—the geometry in which there are two lines parallel to a given line through a given point not on that line. Because of the overwhelming importance of their ideas, it might be hard for us today to understand how their work could have been so thoroughly ignored by their contemporaries—it was only after their deaths that the mathematical world paid much attention to the subject, and several decades elapsed before the full implication of their achievements became appreciated. Whereas Bolyai was so discouraged that he gave up publishing mathematics, Lobachevski optimistically produced further Russian accounts of non-Euclidean geometry; when his fellow Russians failed to recognize the significance of his work, he then published treatments of his theory in French and in German. His little German book of 1840, *Geometrische Untersuchungen zur Theorie der Parallellinien*, forms the core of the book under review here. Seth Braver's translation, with the title shortened to *The Theory of Parallels*, appears twice: at the end as a 23-page appendix, and spread out over the first 200 pages, printed in red type and supplemented by Braver's introduction and notes, printed in black. The resulting "illuminated" Lobachevski is intended for "student, professional, [and] layman."

Braver's book would certainly make a superb textbook for an undergraduate course in non-Euclidean geometry. His commentary provides the historic and philosophical background that explains the mathematical environment in which Lobachevski worked, as well as the significance of his work. His mathematical ideas are clearly motivated, and the relevant achievements of predecessors and contemporaries are briefly outlined. Explanations are provided to fill in details that many of today's students might otherwise find difficult; the commentary is informative and entertaining, which most students would appreciate. For example, when I taught such a course a few years back I was surprised when several students complained about the diagrams: in *imaginary geometry* (to use Lobachevski's terminology), straight lines in diagrams are often represented by curves so as to avoid unwanted intersections. Braver points out that although space in imaginary geometry looks the same at every point, "it looks very different at different *scales*. On a tiny scale, it resembles Euclidean geometry, and serious deviations become noticeable only on a large, possibly astronomical, scale. Since similar figures do not exist in imaginary geometry, accurate scaled down drawings are impossible." I wish I had thought of this explanation to give my students. In general, the author provides the student with good explanations of what is the same and what is different in this new geometry. He supplements Lobachevski's proofs with further details and alternative arguments, together with related results and elegant arguments from Saccheri, Lambert, Legendre, Gauss, Bolyai, and others. A teacher using this book as a text, however, would have to provide details of Euclidean theorems and proofs, and perhaps examples of modern arguments involving betweenness axioms. Also, the book comes with no exercises. As usual, I disagree with the MAA's pricing policy: the $90 US price tag on the printed edition seems designed to encourage the student to purchase the $35 e-book. According to the book representative who sent me the printed version, the electronic version is equally hard to navigate—the references are not linked so that there is no quick way to locate an item that has been cross-referenced.

The book is less successful in its effort to reach the professional. Lobachevski's intended audience was the professional mathematician of the mid-nineteenth century. He wrote quite well; his arguments provide pretty much the same level of detail that would be expected by readers of the geometry problems in **Crux**, so not much help would be needed for any reader familiar with Euclid. The supplementary comments on philosophy and history have been taken almost entirely from standard sources that are readily available and are not entirely reliable. Whereas the light, whimsical tone of the commentary might be suitable for the undergraduate seeing the material for the first time, I found many explanations lacking in depth. This shortcoming was most evident in the notes accompanying the preface where Lobachevski lists his preliminary theorems and claims that their "proofs present no difficulties." Braver is content to denigrate the first five propositions ("A Rough Start", "[they] should be demoted to the status of descriptions (or axioms)", "... if either he or Euclid had tried to prove it rigorously, they would have found the task impossible", and on and on. Far from thinking Lobachevski's arguments were faulty, I was struck by how much thought he put into the foundations, and how far his thoughts had advanced beyond Euclid. Compare Euclid's second postulate, "A finite straight line may be extended continuously in a straight line." with Lobachevski's Paragraph 3, "By extending both sides of a straight line sufficiently far, it will break out of any bounded region. In particular, it will separate a bounded plane region into two parts." Surely, he wanted to make it clear that lines stretch to infinity in both directions (which is not clear in Euclid, who demands only that his segments be extendable). We can also see here a primitive notion of separation decades before Pasch made the concept rigorous (in 1882). How more valuable it would have been had Braver discussed where the initial five "theorems" originated, why Lobachevski thought they were easily established and, most importantly, where and how he used them in subsequent proofs. At any rate, the reader does not have to be reminded every time Proposition 3 is invoked, that Lobachevski lacked $20^{\text{th}}$ century tools.

The author fails to mention why he felt the need for a new translation from the original German, which seems to be only superficially different from Bruce Halsted's 1881 translation. But whichever translation you can get your hands on, any person who likes geometry should read *The Theory of Parallels*—the final half-dozen propositions constitute some of the most clever and exciting mathematics ever conceived. That is where Lobachevski develops the geometry of the horosphere and then establishes that the geometry of the sphere is independent of the parallel postulate (that is, the geometry and trigonometry of a sphere is the same in imaginary geometry as in Euclidean geometry); these results lead him directly to the key formulas of imaginary geometry.

## Unsolved Crux Problems

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux*[2010 : 545, 547]. Below is a sample of two of these unsolved problems:

**342★**. [1978 : 133, 297; 1980 : 319-22] *Proposed by James Gary Propp, Great Neek, NY, USA.*

For fixed $n \geq 2$, the set of all positive integers is partitioned into the (disjoint) subsets $S_1$, $S_2$, $\ldots$, $S_n$ as follows: for each positive integer $m$, we have $m \in S_k$ if and only if $k$ is the largest integer such that $m$ can be written as the sum of $k$ distinct elements from one of the $n$ subsets.

Prove that $m \in S_n$ for all sufficiently large $m$. (If $n = 2$, this is essentially equivalent to Problem 226 [1977 : 205]).

**1754★**. [1992 : 175; 1993 : 151-2; 1994 : 196-9; 1995 : 236-8] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n$ and $k$ be positive integers such that $2 \leq k < n$, and let $x_1$, $x_2$, $\ldots$, $x_n$ be non-negative real numbers satisfying $\sum_{i=1}^{n} x_i = 1$. Prove or disprove that

$$\sum x_1 x_2 \ldots x_k \;\leq\; \max\left\{\frac{1}{k^k}, \frac{1}{n^{k-1}}\right\},$$

where the sum is cyclic over $x_1$, $x_2$, $\ldots$, $x_n$. [The case $k = 2$ is known — see inequality (1) in the solution of ***CRUX* 1662** [1992 : 188].]

Good luck solving these problems. We would love to receive your solutions so that we could cross them off our list.

# RECURRING CRUX CONFIGURATIONS

J. Chris Fisher

## Triangles for which $2b^2 = c^2 + a^2$

Many configurations have popped up again and again in **Crux** geometry problems over the years. In this occasionally appearing column we will recall some of them—they might not be as common as a right triangle, but their properties are generally useful and often surprising. This month we investigate triangles with sides $a, b, c$ that satisfy $2b^2 = c^2 + a^2$. Following the suggestion of Leon Bankoff we will call them **root-mean-square triangles** because $b = \sqrt{\frac{c^2+a^2}{2}}$ is the root mean square of $a$ and $c$. These triangles have also been called *automedian* and *self-median* (see property 1 below), as well as *quasi-isosceles* (see problem 727 below).

Leon Bankoff reported [1978 : 13-16] that in a series of *Mathesis* articles [4], 69 properties of these triangles are listed, generally with an abundance of earlier references instead of proofs. He listed his favorite 10 properties and supplied the short proofs. Here we shall leave the proofs as exercises with hints where appropriate. For these ten properties we assume that $ABC$ is a triangle whose sides satisfy

$$2b^2 = c^2 + a^2. \qquad (1)$$

Let $m_a, m_b, m_c$ denote the medians to sides $a, b, c$, respectively, and let $H, O$, and $G$ denote the orthocentre, circumcentre, and centroid. The symmedians $k_a, k_b$, and $k_c$ are the reflections of the medians in the respective angle bisectors; they meet in the symmedian point $K$. As usual, let $s = \frac{a+b+c}{2}$ be the semiperimeter, and $R$ denote the circumradius.

**Property 1**. $m_a = \frac{\sqrt{3}}{2}c,\ m_b = \frac{\sqrt{3}}{2}b,\ m_c = \frac{\sqrt{3}}{2}a$.

This property follows immediately by plugging (1) into the formula for the length of a median in terms of the sides. Note that the medians can be translated to form a triangle that is oppositely similar to triangle $ABC$; this property is the source of the "automedian" terminology that was used throughout [4]. Perhaps in French the word sounds less asinine, but to me it does not convey the image of a triangle that is oppositely similar to the triangle formed by its three medians. K.R.S. Sastry favored the terminology self-median. He proved a converse in "The Triangle: A Parametric Description" [2004 : 497-501]: *If $b$ is the length of the middle side, then $\Delta ABC$ is (oppositely) similar to its medial triangle if and only if $b = \frac{\sqrt{3}}{2}m_b$, if and only if (1) holds.* Note that the ratio of similitude of corresponding sides tells us that the area of the triangle formed by the medians is $3/4$ that of the original triangle. It is also seen that $m_a + m_b + m_c = \sqrt{3}s$, a familiar property of the equilateral triangle, which is, of course, a special case of a root-mean-square triangle.

**Property 2**. $b^2 = 2ca\cos B$. (This is the cosine law applied to (1).)

**Property 3.** $2\cot B = \cot C + \cot A$. (You can get started by applying the sine law to both sides of (1).)

**Property 4.** $2\cos 2B = \cos 2C + \cos 2A$.

**Property 5.** $b^2 = AG^2 + BG^2 + CG^2$.

**Property 6.** $4\cdot\text{Area}(ABC) = b^2 \tan B$.

**Property 7.** $2m_b^2 = m_c^2 + m_a^2$.

**Property 8.** $2BH^2 = CH^2 + AH^2$.

**Property 9.** *If $B'$ and $B''$ are the third vertices of equilateral triangles constructed externally and internally on side $AC$, then $\angle B'BB''$ is a right angle.*

*Hint.* A 1796 theorem of Nicholas Fuss implies that $B'B^2 + B''B^2 = a^2 + b^2 + c^2$ [3, p. 220, par. 354c]; more easily than tracking down a reference, one can prove Fuss's theorem by applying the cosine law to triangles $BCB'$ and $BCB''$.

**Property 10.** $\frac{m_b}{k_b} = 2\cos B$.

Also in [1978 : 13], W.J. Blundon showed how to construct a triangle satisfying (1) given the segment $AC$: Construct the equilateral triangle $ACD$ and let $O$ be the midpoint of $AC$; Property 1 tells us that a triangle $ABC$ satisfies (1) if and only if $OB = \frac{\sqrt{3}}{2}b$, whence the locus of $B$ is the circle with centre $O$ and radius $OD$. The editor Léo Sauvé added (on page 16) that if you want to generate specific examples of root-mean-square triangles with integer sides, you could use the theorem that all primitive solutions of $2b^2 = c^2 + a^2$ are given by

$$a = u^2 + 2uv - v^2, \quad b = u^2 + v^2, \quad c = |u^2 - 2uv - v^2|,$$

where $u > v$, with $u$ and $v$ relatively prime positive integers of different parity [1], [2, pp. 435 ff.]. Of course, your values of $a, b, c$ must satisfy the triangle inequality. Another approach to this result was taken by K.R.S. Sastry in "Pythagoras Strikes Again!" [1998 : 276-280]; specifically, he proved that

> If $a_0, b_0, c_0$ are the sides of a right triangle in which $c_0^2 + a_0^2 = b_0^2$, $a_0 > c_0$, and $b_0 > 2c_0$, then $a = a_0 + c_0$, $c = a_0 - c_0$, and $b = b_0$ are the sides of a triangle that satisfies $c^2 + a^2 = 2b^2$; conversely, if the sides of a triangle satisfy $a > b > c$ and $c^2 + a^2 = 2b^2$, then $a_0 = \frac{1}{2}(a + c)$ and $c_0 = \frac{1}{2}(a - c)$ are the legs of a right triangle whose hypotenuse is $b$, $a_0 > c_0$, and $b > 2c_0$.

For a list of all $36$ primitive root-mean-square triangles with perimeters less than 1000, see [1978 : 194]. The smallest (that is not equilateral) has side lengths $17, 13, 7$, which comes from a 12-13-5 right triangle by Sastry's theorem.

The 1978 discussion of these triangles came out of the solution to problem 210 [1977 : 10, 160-164, 196-197; 1978 : 13-16, 193-194] (proposed by Murray S. Klamkin): $P, Q, R$ denote points on the sides $BC, CA$, and $AB$, respectively, of a given triangle $ABC$; determine all triangles $ABC$ such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \qquad \left(\neq 0, \frac{1}{2}, 1\right),$$

then $\boldsymbol{PQR}$ (in some order) is similar to $\boldsymbol{ABC}$. The solution revealed that if the triangles are directly similar, then they must be equilateral. When $\boldsymbol{a > b > c}$ the value of $\boldsymbol{k}$ in one of the three resulting solutions has $\boldsymbol{2b^2 - c^2 - a^2}$ in its denominator, whence *root-mean-square triangles* $\boldsymbol{ABC}$ *would have only two oppositely similar triangles* $\boldsymbol{PQR}$ *instead of three, and those two triangles would be congruent.*

**Problem 309** [1978 : 12, 200-202] (Proposed by Peter Shor). Let $\boldsymbol{ABC}$ be a triangle with $\boldsymbol{a \geq b \geq c}$ or $\boldsymbol{a \leq b \leq c}$. Let the bisectors of $\boldsymbol{\angle c m_a}$ and $\boldsymbol{\angle a m_c}$ meet at $\boldsymbol{R}$. Prove that

(a) $\boldsymbol{AR \perp CR}$ if and only if $\boldsymbol{2b^2 = c^2 + a^2}$;

(b) if $\boldsymbol{2b^2 = c^2 + a^2}$, then $\boldsymbol{R}$ lies on $\boldsymbol{m_b}$.

Is the converse of (b) true?

*Yes*, the converse turns out to be true (when $\boldsymbol{b}$ is assumed to be the middle side). The featured solution of Daniel Sokolowsky made use of yet another interesting property:

**Theorem**. If $\boldsymbol{D}$ and $\boldsymbol{E}$ are the midpoints of sides $\boldsymbol{AB}$ and $\boldsymbol{BC}$, and $\boldsymbol{G}$ is the centroid, then $\boldsymbol{2b^2 = c^2 + a^2}$ if and only if $\boldsymbol{BDGE}$ is a cyclic quadrilateral.

**Problem 313** (revised) [1978 : 35, 207-209] (Proposed by Leon Bankoff). The sides of a nonequilateral triangle satisfy $\boldsymbol{2b^2 = c^2 + a^2}$ if and only if $\boldsymbol{GK}$ (the join of the centroid and the symmedian point) is parallel to $\boldsymbol{AC}$.

Bankoff seems not to have worried much about converses! Although his version of Problem 313 called for a proof of sufficiency only, both featured solutions make clear that (1) is both necessary and sufficient for $\boldsymbol{GK}$ to be parallel to $\boldsymbol{AC}$. He found the theorem (without a converse) in *Mathesis*, t. IX (1889) p. 208, where it was attributed to Lemoine (*Mathesis*, t. V (1885) p. 104). As for Bankoff's list of ten properties (reproduced above), although I did not carefully write down the proofs, I believe that all ten converses hold for triangles with $\boldsymbol{a > b > c}$ or $\boldsymbol{c > b > a}$.

**Problem 383** [1978 : 250, 174-176] (Proposed by Daniel Sokolowsky). Let $\boldsymbol{m_a, m_b, m_c}$ be respectively the medians $\boldsymbol{AD, BE, CF}$ of a triangle $\boldsymbol{ABC}$ with centroid $\boldsymbol{G}$. Prove that

(a) if $\boldsymbol{m_a : m_b : m_c = a : b : c}$, then $\boldsymbol{\Delta ABC}$ is equilateral;

(b) if $\boldsymbol{\frac{m_b}{m_c} = \frac{c}{b}}$, then either (i) $\boldsymbol{b = c}$ or (ii) quadrilateral $\boldsymbol{AEGF}$ is cyclic;

(c) if both (i) and (ii) hold in (b), then $\boldsymbol{\Delta ABC}$ is equilateral.

By the theorem from the proof of Problem 309 above, quadrilateral $\boldsymbol{AEGF}$ is cyclic in part (b) if and only if $\boldsymbol{2a^2 = b^2 + c^2}$. Part (b) should be compared with the quasi-isosceles property of Bottema and Groenman discussed in the next problem, number 727.

**Problem 727** [1982 : 78; 1983 : 115, 180-181] (Proposed by J.T. Groenman). Let $\boldsymbol{t_b}$ and $\boldsymbol{t_c}$ be the symmedians issued from vertices $\boldsymbol{B}$ and $\boldsymbol{C}$ of triangle $\boldsymbol{ABC}$ and terminating in the opposite sides $\boldsymbol{b}$ and $\boldsymbol{c}$, respectively. Prove that $\boldsymbol{t_b = t_c}$ if and only if $\boldsymbol{b = c}$.

It turned out that this problem had already appeared several times elsewhere; instead of a solution, three references were provided: *Amer. Math.*

*Monthly*, **51** (Dec 1944) 590-591; *Scripta Mathematica*, **22** (1956) 102; and *Mathematics Magazine*, **40** (May 1967) 165, and **41** (Jan 1968) 48-49. There was also reference to a problem in the *Pi Mu Epsilon Journal* that claimed to prove that the analogous result holds also for exsymmedians. Not so, according to a paper by Groenman with O. Bottema in *Nieuw Tijdschrift voor Wiskunde*, **70** (1983) 143-151. For the details, recall that an exmedian is a line through a vertex parallel to the opposite side of the triangle, while an exsymmedian is the reflection of the exmedian in the external angle bisector. The Bottema-Groenman result says that if the exsymmedians from $B$ and $C$ are equal in length, then either $b = c$ (and $\Delta ABC$ is isosceles), or $2a^2 = b^2 + c^2$ (and the triangle is a root-mean-square triangle). In the latter case they call the triangle *quasi-isosceles*. Warning: their article is written in Dutch.

After Problem 727 (from 28 years ago!) it seems as if Sastry has single-handedly kept the subject of root-mean-square triangles alive. Besides the **Crux** articles from 1998 and 2004 mentioned earlier, he published the article [5] and posed Problem 473 in [6]: *Prove that in a scalene triangle $ABC$, the bisectors of $\angle ABC$ and $\angle AGC$ intersect on the side $AC$ if and only if* $c^2 + a^2 = 2b^2$. He is also responsible for

**Problem 2252** [1997 : 300; 1998 : 375-376] (Proposed by K.R.S. Sastry). Prove that the nine-point circle of a triangle trisects a median if and only if the side lengths of the triangle are proportional to the lengths of its medians in some order.

This completes the list of results on root-mean-square triangles that I found in the problems pages of **CRUX with MAYHEM**. Triangles for which $2b = c + a$ and triangles for which $2B = C + A$ have likewise been enthusiastically investigated in this journal. We will discuss them in future columns.

## References

[1] Leon Bankoff, Solution to Problem 91 (A Long Probate), *Mathematics Magazine*, **41**:5 (Nov-Dec 1968) 291-293.

[2] Leonard Eugene Dickson, *History of the Theory of Numbers*, vol. II, Chelsea, 1952.

[3] Roger A. Johnson, *Modern Geometry*, Houghton Mifflin, 1929; reissued as *Advanced Euclidean Geometry*, Dover, 1960.

[4] *Mathesis*: (1902) 205-208; (1903) 196-200, 226-230, 245-248; (1926) 68-69.

[5] K.R.S. Sastry, Self-Median Triangles, *Mathematical Spectrum*, **22** (1989/90) 58-60.

[6] K.R.S. Sastry, Problem 473, *College Mathematics Journal*, **23**:2 (March 1992) 162; **24**:2 (March 1993) 186-188.

# Summations according to Gauss

Gerhard J. Woeginger

A well-known anecdote relates that when Carl Friedrich Gauss (1777–1855) was only ten years old, his school teacher wanted to keep the pupils busy and asked them to add up all the integers from 1 to 100. Almost immediately Gauss placed his slate on the table and said "There it is." The slate just contained the number 5050 without further calculations. When the teacher finally checked the results, Gauss's slate was the only one with the correct answer.

If there is any truth in this anecdote, then the young Gauss must have paired up the integers in the following (or some closely related) way. In his mind, he wrote down the summation twice: once in the standard fashion from left to right, and once he flipped it around and wrote it from right to left.

$$\begin{array}{ccccccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & \cdots\cdots & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & 97 & + & \cdots\cdots & + & 2 & + & 1 \end{array}$$

This yielded 100 vertically aligned pairs, where the numbers in each pair added up to 101. Hence the sum of all listed numbers was $100 \cdot 101$, and as each number was listed twice the answer desired by Gauss's teacher was $\frac{1}{2} \cdot 100 \cdot 101 = 5050$.

In this article we will discuss several related problems that all can be settled by this *"write it once down left-to-right and once right-to-left"* approach of Gauss. For warming up, the reader may want to generalize the above calculation to an arbitrary number of terms.

**Problem 1** *Determine a closed form expression for* $1 + 2 + 3 + 4 + \cdots + n$.

Our next problem is a standard textbook exercise in the manipulation of sums of binomial coefficients.

**Problem 2** *Determine a closed form expression for the following sum* $S$*:*

$$S \;=\; 1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 3 \cdot \binom{n}{2} + \;\cdots\; + (n+1) \cdot \binom{n}{n} \qquad (2)$$

We present a solution that is based on the trick of Gauss. We write down the sum once again, but in reversed order with its terms taken from right to left:

$$S \;=\; (n+1) \cdot \binom{n}{n} + n \cdot \binom{n}{n-1} + (n-1) \cdot \binom{n}{n-2} + \;\cdots\; + 1 \cdot \binom{n}{0} \quad (3)$$

This yields $n + 1$ vertically aligned pairs, where the $k$th pairs consists of the $k$th term in (2) and the $k$th term in (3). By applying the well-known relation

$\binom{n}{\ell} = \binom{n}{n-\ell}$ with $\ell = k-1$, we derive that the terms in the $k$th pair add up to

$$k \cdot \binom{n}{k-1} + (n-k+2) \cdot \binom{n}{n-k+1} \;=\; (n+2) \cdot \binom{n}{k-1}. \qquad (4)$$

Although the resulting value in the right hand side of (4) still depends on the parameter $k$, we have made substantial progress. By using (4), we see that the sum of all terms listed in (2) and (3) is

$$2S \;=\; (n+2) \cdot \left\{ \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \right\} \;=\; (n+2) \cdot 2^n.$$

Here we used the fact that the binomial coefficients in the $n$th row of Pascal's triangle add up to $2^n$. Hence the answer to this problem is $S = (n+2) \cdot 2^{n-1}$.

The following problem can be settled by a very similar argument. The reader is encouraged to verify that the answer is $T_n = n$.

**Problem 3** *For* $n \geq 1$ *evaluate* $T_n = \sum_{k=0}^{2n} k \cdot \cos\left(\frac{k\pi}{2n}\right)$.

Now let us turn to a summation problem from the 2000 Asian Pacific Mathematical Olympiad (APMO'2000).

**Problem 4** *Compute the sum* $\sum_{k=0}^{101} \frac{x_k^3}{3x_k^2 - 3x_k + 1}$ *with* $x_k = k/101$ *for* $k = 0, \ldots, 101$.

Every single term in this summation is bulky, and there are lots of bulky terms that must be added up. Computing this sum by hand is certainly not a good idea. Let us follow Gauss and let us pair up the terms in the given sum with the terms in reversed order: term 0 is paired with term 101, term 1 with term 100, term 2 with term 99, and so on. Observe that in every resulting pair we have $x_k + x_{101-k} = k/101 + (101-k)/101 = 1$, which implies

$$\begin{aligned} 3x_k^2 - 3x_k + 1 &= (1-x_k)^3 + x_k^3 \\ &= x_{101-k}^3 + (1-x_{101-k})^3 \;=\; 3x_{101-k}^2 - 3x_{101-k} + 1. \end{aligned}$$

Adding up the term for $x_k$ and the term for $x_{101-k}$ then yields

$$\begin{aligned} &\frac{x_k^3}{3x_k^2 - 3x_k + 1} + \frac{x_{101-k}^3}{3x_{101-k}^2 - 3x_{101-k} + 1} \;= \\ &\quad = \frac{x_k^3}{(1-x_k)^3 + x_k^3} + \frac{x_{101-k}^3}{(1-x_k)^3 + x_k^3} \;=\; \frac{x_k^3 + (1-x_k)^3}{(1-x_k)^3 + x_k^3} \;=\; 1. \end{aligned}$$

Since altogether there are 102 pairs, we see that twice the value of the sum equals 102, and that hence the answer to the APMO problem is 51. The bulky summation can actually be performed in a routine fashion!

The following problem was posed as problem A3 on the 1980 William Lowell Putnam Mathematics Competition.

**Problem 5** *Evaluate* $I = \int\limits_{x=0}^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$ .

The integrand in this problem looks horrible. A little bit of playing around soon confirms our impression that it is hopeless to search for an antiderivative in closed form. But remember that a definite integral is really just some kind of fancy summation, as it is obtained by adding up lots of very small numbers. Hence let us try to apply the trick of young Gauss. We write the integral down once again, but we flip it over so that this time we integrate from right to left. This flipping over operation corresponds to the substitution $y = \pi/2 - x$ with $dy = -dx$. We derive:

$$I = \int\limits_{y=\pi/2}^{0} \frac{-dy}{1+(\tan(\pi/2-y))^{\sqrt{2}}} = \int\limits_{y=0}^{\pi/2} \frac{dy}{1+(\tan(\pi/2-y))^{\sqrt{2}}}. \quad (5)$$

We recall the trigonometric identity $\tan(\pi/2-\alpha) = \cot(\alpha)$, and with its help we rewrite the integrand in (5) as

$$\frac{1}{1+(\tan(\pi/2-y))^{\sqrt{2}}} = \frac{1}{1+(\cot y)^{\sqrt{2}}} = \frac{(\tan y)^{\sqrt{2}}}{(\tan y)^{\sqrt{2}}+1}. \quad (6)$$

Now (5) and (6) imply

$$\begin{aligned} 2I &= \int\limits_{x=0}^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}} + \int\limits_{y=0}^{\pi/2} \frac{(\tan y)^{\sqrt{2}}}{(\tan y)^{\sqrt{2}}+1}\, dy \\ &= \int\limits_{x=0}^{\pi/2} \frac{1+(\tan x)^{\sqrt{2}}}{1+(\tan x)^{\sqrt{2}}}\, dx = \int\limits_{x=0}^{\pi/2} dx = \pi/2. \end{aligned}$$

Therefore the answer to this Putnam problem is $I = \pi/4$. Note that the exponent $\sqrt{2}$ does not play any special role in our calculations. If we replace it by an arbitrary positive real number, the answer will still remain $\pi/4$.

The same integration theme resurfaced seven years later as problem B1 on the 1987 Putnam exam. The reader should have little difficulty in finding the right substitution for the following problem, which leads to the answer $J = 1$.

**Problem 6** *Evaluate* $J = \int_{x=2}^{4} \frac{\sqrt{\ln(9-x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$ .

The occurrence of the function $\sqrt{\ln x}$ in this problem is purely artificial. The solution virtually remains the same, if $\sqrt{\ln x}$ is replaced by any integrable function $f(x)$ for which $f(9-x) + f(x+3) \neq 0$ for $2 \leq x \leq 4$.

Finally we want to discuss problem A4 from the 1999 Putnam exam.

**Problem 7** *Sum the series* $S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m\,(n3^m + m3^n)}$.

Let us flip over the summation so that $m$ becomes $n$ and simultaneously $n$ becomes $m$. Of course this does not change the value of the sum, and (similarly as before) we get

$$2S \;=\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m\,(n3^m + m3^n)} \;+\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2 m}{3^n\,(m3^n + n3^m)}. \tag{7}$$

We pair up terms (similarly as before), and by some simple algebra derive

$$\frac{m^2 n}{3^m\,(n3^m + m3^n)} + \frac{n^2 m}{3^n\,(m3^n + n3^m)} \;=\; \frac{m\,n}{3^m\,3^n}. \tag{8}$$

The right hand side of (8) indicates that it might be useful to investigate the auxiliary sum $T = \sum_{m=1}^{\infty} m/3^m$. Since

$$3T \;=\; \sum_{m=1}^{\infty} \frac{m}{3^{m-1}} \;=\; \sum_{m=0}^{\infty} \frac{m+1}{3^m} \;=\; T + \sum_{m=0}^{\infty} \frac{1}{3^m} \;=\; T + \frac{3}{2},$$

we conclude $T = 3/4$. By combining this with (7) and (8) we derive (similarly as before)

$$2S \;=\; \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m\,n}{3^m\,3^n} \;=\; \sum_{m=1}^{\infty} \frac{m}{3^m} \cdot \sum_{n=1}^{\infty} \frac{n}{3^n} \;=\; T^2 \;=\; \frac{9}{16}.$$

Thus the final answer to our final problem is $S = 9/32$.

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# A nest of Euler Inequalities

Luo Qi

**Abstract**

For any given $\triangle ABC$, we define the *antipodal triangle*. Repeating this construction gives a sequence of triangles with circumradii $R_n$ and inradii $r_n$ obeying a generalized form of Euler's inequality

$$2^n R_n \geq \cdots \geq 2^2 R_2 \geq 2R_1 \geq R_0 \geq 2r_0 \geq 2^2 r_1 \geq \cdots \geq 2^{n+1} r_n,$$

$(n = 1, 2, \cdots)$, with equalities iff $\triangle ABC$ is equilateral.

**Key words:** Euler inequality; antipodal triangle

Let $R, r$ be the radius of circumcircle and inscribed circle of a triangle; then $R \geq 2r$, with equalities iff the triangle is equilateral ([1], p.50). This is the famous Euler inequality. In this note, we are going to build a nest of Euler inequalities for a certain family of related triangle.

**Definition 1** *If a vertex $A$ of a triangle $ABC$ and another point $A'$ on the perimeter divide the perimeter into two equal parts (that is, $|AB| + |BA'| = |AC| + |CA'|$) we call $|A'|$ the* antipode *of $|A|$, and the triangle $\triangle A'B'C'$ of which three vertices are antipodes of $A, B, C$ respectively the* antipodal triangle *of $\triangle ABC$.*

Note that $A'$ is necessarily on the (non-extended) edge $BC$ and in fact it is the point where that edge touches the appropriate escribed circle.[2] Thus, we can easily find a way to draw an antipodal triangle $\triangle A'B'C'$ of a given triangle $\triangle ABC$.

**Lemma 1** *Denote by $a, b, c, a_1, b_1, c_1, s, s_1, \mathrm{A}, \mathrm{A}_1$ the sides, semiperimeters, and areas of $\triangle ABC$ and its antipodal triangle $\triangle A_1B_1C_1$, and let $R$ and $r$ be the circumradius and inradius of $\triangle ABC$. Then*

1. $|AB_1| = |BA_1| = s - c, |AC_1| = |CA_1| = s - b, |BC_1| = |CB_1| = s - a$;
2. $\dfrac{\mathrm{A}_1}{\mathrm{A}} = \dfrac{r}{2R}$;
3. $\dfrac{a_1 b_1 c_1}{abc} \geq \dfrac{r}{4R}$ *(with equality iff $\triangle ABC$ is equilateral );*
4. $2s_1 \geq s$*( with equality iff $\triangle ABC$ is equilateral ).*

**Proof** (See the figure 1)
**(1)** We have

$$|AB_1| = \frac{1}{2}(|AB| + |BC| + |CA|) - |AB|,$$
$$|BA_1| = \frac{1}{2}(|AB| + |BC| + |CA|) - |AB|$$

and so $|AB_1| = |BA_1| = s - c.$ In the same way, we have

$$|AC_1| = |CA_1| = s - b,$$
$$|BC_1| = |CB_1| = s - a.$$

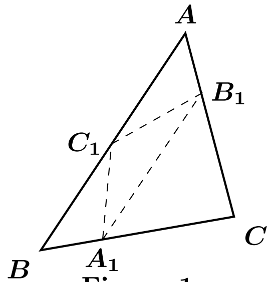


**Figure 1**

**(2)** Denote by $\mathbf{A}_{AB_1C_1}$, $\mathbf{A}_{BA_1C_1}$, $\mathbf{A}_{CA_1B_1}$, the areas of $\triangle AB_1C_1$, $\triangle BA_1C_1$, $\triangle CA_1B_1$.

Because

$$\frac{\mathbf{A}_{AB_1C_1}}{\mathbf{A}} = \frac{|AB_1| \cdot |AC_1|}{|AB| \cdot |AC|} = \frac{(s-c)(s-b)}{c \cdot b},$$
$$\frac{\mathbf{A}_{BA_1C_1}}{\mathbf{A}} = \frac{|BA_1| \cdot |BC_1|}{|BA| \cdot |BC|} = \frac{(s-c)(s-a)}{c \cdot a},$$
$$\frac{\mathbf{A}_{CA_1B_1}}{\mathbf{A}} = \frac{|CA_1| \cdot |CB_1|}{|CA| \cdot |CB|} = \frac{(s-b)(s-a)}{b \cdot a}$$

we have

$$\begin{aligned}\frac{\mathbf{A}_1}{\mathbf{A}} &= \frac{\mathbf{A} - \mathbf{A}_{AB_1C_1} - \mathbf{A}_{BA_1C_1} - \mathbf{A}_{CA_1B_1}}{\mathbf{A}} \\ &= 1 - \frac{\mathbf{A}_{AB_1C_1}}{\mathbf{A}} - \frac{\mathbf{A}_{BA_1C_1}}{\mathbf{A}} - \frac{\mathbf{A}_{CA_1B_1}}{\mathbf{A}} \\ &= 1 - \frac{(s-c)(s-b)}{c \cdot b} - \frac{(s-c)(s-a)}{c \cdot a} - \frac{(s-b)(s-a)}{b \cdot a} \\ &= \frac{2(s-a)(s-b)(s-c)}{a \cdot b \cdot c}\end{aligned}$$

By Heron's and other well-known formulas,

$$\mathrm{A} = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = sr$$

Thus

$$\frac{\mathrm{A}_1}{\mathrm{A}} = \frac{2(s-a)(s-b)(s-c)}{a \cdot b \cdot c} = \frac{2\mathrm{A}^2}{s \cdot 4\mathrm{A} \cdot R} = \frac{\mathrm{A}}{2sR} = \frac{r}{2R}$$

**(3)** Using the law of sines on $\triangle AB_1C_1$, we have

$$\begin{aligned}\frac{a_1}{\sin A} &= \frac{s-b}{\sin\angle AB_1C_1} = \frac{s-c}{\sin\angle AC_1B_1} = \frac{s-b+s-c}{\sin\angle AB_1C_1 + \sin\angle AC_1B_1} \\ &= \frac{a}{2\sin\frac{\angle AB_1C_1+\angle AC_1B_1}{2}\cos\frac{\angle AB_1C_1-\angle AC_1B_1}{2}} \\ &\geq \frac{a}{2\sin\frac{\pi-A}{2}} = \frac{a}{2\cos\frac{A}{2}}\end{aligned}$$

(with equality iff $\angle AB_1C_1 = \angle AC_1B_1$).

Therefore we have

$$\frac{a_1}{a} \geq \frac{\sin A}{2\cos\frac{A}{2}} = \sin\frac{A}{2}$$

(with equality iff $\angle AB_1C_1 = \angle AC_1B_1$).

In the same way, we have

$$\frac{b_1}{b} \geq \sin\frac{B}{2}$$

(with equality iff $\angle BA_1C_1 = \angle BC_1A_1$), and

$$\frac{c_1}{c} \geq \sin\frac{C}{2}$$

(with equality iff $\angle CB_1A_1 = \angle CA_1B_1$).

Multiplying these three inequalities, we obtain

$$\frac{a_1b_1c_1}{abc} \geq \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(with equality iff $\triangle ABC$ is equilateral).

Also, because

$$\mathrm{A} = \frac{1}{2}ab\sin C = \frac{abc}{4R} = \frac{r(a+b+c)}{2}$$

we have

$$\frac{(2R)^3\sin A\sin B\sin C}{4R} = \frac{2Rr(\sin A+\sin B+\sin C)}{2}.$$

So

$$\begin{aligned}\frac{r}{R} &= \frac{2\sin A\sin B\sin C}{\sin A+\sin B+\sin C}\\ &= \frac{16\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}}\\ &= \frac{16\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\cos\frac{C}{2}(\cos\frac{A-B}{2}+\sin\frac{C}{2})}\\ &= \frac{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{A+B}{2}}\\ &= \frac{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}\\ &= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\end{aligned}$$

Thus

$$\frac{a_1b_1c_1}{abc} \geq \frac{r}{4R}$$

(with equality iff $\triangle ABC$ is equilateral).

**(4)** Construct perpendiculars $B_1E$ and $C_1D$ to $BC$ at $E$ and $D$, respectively (See Figure 2). Then $a_1 \geq |DE|$ (with equality iff $BC \parallel B_1C_1$.) But

$$|DE| = a - |BD| - |CE| = a - (s-a)\cos B - (s-a)\cos C\ ,$$

so

$$a_1 \geq a - (s-a)(\cos B + \cos C) \text{ with equality iff } BC \parallel B_1C_1\ .$$

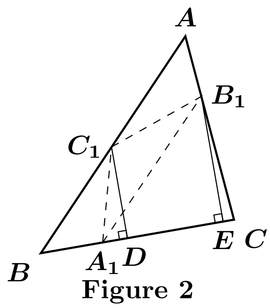

**Figure 2**

Similarly, we have:

$$b_1 \geq b - (s-b)(\cos C + \cos A) \text{ with equality iff } CA \parallel C_1A_1\ ,$$

$$c_1 \geq c - (s-c)(\cos A + \cos B) \text{ with equality iff } AB \parallel A_1B_1\ .$$

Adding up these three inequalities yields

$$\begin{aligned} 2s_1 &\geq 2s - (a\cos A + b\cos B + c\cos C) \\ &\geq 2s - \frac{1}{2}(a\cos B + a\cos C + b\cos A + b\cos C + c\cos A + c\cos B) \\ &= 2s - \frac{1}{2}(a+b+c) = s \end{aligned}$$

Therefore $2s_1 \geq s$ with equality iff $\triangle ABC$ is equilateral.

Studying these two triangles we can also find other interesting properties. The reader may verify that the antipodal triangle is "less equilateral" in that the ratio between longest and shortest side is always greater than in the original triangle.

**Theorem 1** *Denote by $R, R_1, r, r_1$, the circumradii and inradii of $\triangle ABC$ and its antipodal triangle $\triangle A_1B_1C_1$. Then $2R_1 \geq R \geq 2r \geq 4r_1$, with equalities iff $\triangle ABC$ is equilateral.*

**Proof** By (2) and (3) of lemma 1, we have

$$\frac{r}{2R} = \frac{\mathrm{A}_1}{\mathrm{A}} = \frac{a_1b_1c_1/4R_1}{abc/4R} = \frac{Ra_1b_1c_1}{R_1abc} \geq \frac{R}{R_1}\cdot\frac{r}{4R} = \frac{r}{4R_1}$$

So $2R_1 \geq R$, with equality iff $\triangle ABC$ is equilateral.

By (2) and (4) of lemma 1, we have

$$\frac{1}{4} \geq \frac{r}{2R} = \frac{\mathrm{A}_1}{\mathrm{A}} = \frac{r_1s_1}{rs} \geq \frac{r_1s_1}{r\cdot 2s_1} = \frac{r_1}{2r}$$

and so $2r \geq 4r_1$ with equality iff $\triangle ABC$ is equilateral. Hence, we get $2R_1 \geq R \geq 2r \geq 4r_1$, again with equalities iff $\triangle ABC$ is equilateral.

Using mathematical induction and the theorem we immediately get:

**Corollary 1** *(See Figure 3) Let $\triangle A_0B_0C_0$ be given, and let $R_0, R_1, \cdots, R_n$; $r_0, r_1, \cdots, r_n$ denote the circumradii and inradii of $\triangle A_0B_0C_0$, $\triangle A_1B_1C_1, \cdots$, $\triangle A_nB_nC_n$ respectively, and $\triangle A_iB_iC_i$ is the antipodal triangle of $\triangle A_{i-1}B_{i-1}C_{i-1}$, $(i = 1, 2, \cdots)$. Then*

$$2^nR_n \geq \cdots \geq 2^2R_2 \geq 2R_1 \geq R_0 \geq 2r_0 \geq 2^2r_1 \geq \cdots \geq 2^{n+1}r_n,$$

*with equalities iff $\triangle A_0B_0C_0$ is equilateral.*

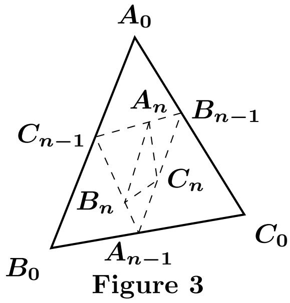

**Figure 3**

So, we build a nest of Euler inequalities.

**Open question:** In [3] Yang derives Euler-type inequalities for tetrahedra in 3-dimensional space. Can we define antipodal points for a tetrahedron in such a way that the results of this paper generalize?

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[1] O. Bottema et al. *Geometric Inequalities*, Wolter-Noordhoff, Groningen, 1969.

[2] Weisstein, Eric W. "Semiperimeter." From*MathWord–A Wolfram Web Resource*. http://mathworld.wolfram.com/Semiperimeter.html. Accessed 10 January 2012.

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# PROBLEMS

*Solutions to problems in this issue should arrive no later than* ***1 September 2012****.*
*An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

*The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.*

**3650**. *Replacement. Proposed by Michel Bataille, Rouen, France.*

Let $ABC$ be a triangle and $R$, $O$, $G$ and $K$ its circumradius, circumcentre, centroid and Lemoine point, respectively. Prove that

$$BC \cdot \frac{KA}{GA} = CA \cdot \frac{KB}{GB} = AB \cdot \frac{KC}{GC} = \sqrt{3(R^2 - OK^2)}.$$

Recall that a symmedian of a triangle is the reflection of the median from a vertex in the angle bisector of the same vertex. The Lemoine point of a triangle is the point of intersection of the three symmedians.

**3651**. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let $a$, $b$, and $c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$a^2b + b^2c + c^2a + abc + 4abc(3 - ab - bc - ca) \leq 4\,.$$

**3652**. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $\alpha$ and $\beta$ be positive real numbers. Find the value of

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^\alpha}{n^\beta}\right)\,.$$

**3653**. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $O$ be the centre of a sphere $S$ circumscribing a tetrahedron $ABCD$. Prove that:

(i) there exists tetrahedra whose four faces are obtuse triangles; and

(ii) if $O$ is inside or on $ABCD$, then at least two faces of $ABCD$ are acute triangles.

**3654**. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let $a$, $b$, $c$, and $d$ be nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 1$. Prove that

$$a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab \;\leq\; 1\,.$$

**3655**. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the integral

$$\int_0^1 \int_0^1 x \left\{ \frac{1}{1-xy} \right\} dxdy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of $a$.

**3656**. *Proposed by Michel Bataille, Rouen, France.*

Let $AB$ be a fixed chord of an ellipse that is not a diameter and let $MN$ be a variable diameter. Show that the locus of the intersection of $MA$ and $NB$ is an ellipse with the same eccentricity as that of the original ellipse, and find a geometrical description of its centre.

**3657**. *Proposed by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*

Prove that for the angles of any triangle the following inequality holds

$$\frac{\cos^2 A}{1+\cos^2 A} + \frac{\cos^2 B}{1+\cos^2 B} + \frac{\cos^2 C}{1+\cos^2 C} \;\geq\; \frac{1}{2}\,.$$

**3658**. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $-\pi < \theta_0 < \theta_1 < \cdots < \theta_k < \pi$ and let $a_j$, $j = 0, 1, \cdots, k$, be complex numbers. Prove that if

$$\lim_{n\to\infty} \sum_{j=0}^{k} a_j \cos(\theta_j n) = 0\,,$$

then $a_j = 0$ for all $j$.

**3659**. *Proposed by Michel Bataille, Rouen, France.*

Let $P$ be a point on a circle $\Gamma$ with diameter $AB$. The tangent to $\Gamma$ at $P$ intersects the tangents at $A$ and $B$ in $D$ and $C$, respectively. Let $M$ be any point of the line $BC$ and $V$ the point of intersection of $MD$ and $BP$. If the parallel to $BC$ through $V$ meets $CD$ in $U$, show that the line $MU$ is tangent to $\Gamma$.

**3660**. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Triangle $ABC$ has inradius $r$, circumradius $R$, and side lengths $a, b, c$. Prove that

$$\frac{y+z}{x}\cdot\frac{1}{a^2} + \frac{z+x}{y}\cdot\frac{1}{b^2} + \frac{x+y}{z}\cdot\frac{1}{c^2} \geq \frac{1}{Rr}\,,$$

for all positive real numbers $x$, $y$ and $z$.

**3661**. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Consider a triangle $ABC$ with the midpoints $D$, $E$, $F$ of its sides $BC$, $CA$, $AB$. For an arbitrary point $P$, let $X$, $Y$, $Z$ be the reflections of $P$ in $D$, $E$, $F$ respectively. Show that the lines $AX$, $BY$, $CZ$ are concurrent.

**3662**. *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{R}$ denote the set of positive integers whose base ten expression is a single repeated digit (e.g. $5 \in \mathcal{R}$, $222 \in \mathcal{R}$, $88888 \in \mathcal{R}$). Let $T(n) = (n-2)^2 + n^2 + (n+2)^2$ where $n$ is a non-negative integer.

(a) Find all even integers $n$ such that $T(n) \in \mathcal{R}$.

(b) Find one odd integer $n > 1$ such that $T(n) \in \mathcal{R}$. Extra credit will be given to anyone who finds more than one odd integer $n > 1$ such that $T(n) \in \mathcal{R}$.

**3663**. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Let $a$, $b$, $c$ be positive real numbers. Prove that

$$\sqrt[3]{\frac{2a}{4a+4b+c}} + \sqrt[3]{\frac{2b}{4b+4c+a}} + \sqrt[3]{\frac{2c}{4c+4a+b}} < 2\,.$$

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**3650**. *Remplacement. Proposé par Michel Bataille, Rouen, France.*

Soit $ABC$ un triangle et soit respectivement $R$, $O$, $G$ et $K$ le rayon et le centre de son cercle circonscrit, son centre de gravité et son point de Lemoine. Montrer que

$$BC \cdot \frac{KA}{GA} = CA \cdot \frac{KB}{GB} = AB \cdot \frac{KC}{GC} = \sqrt{3(R^2 - OK^2)}.$$

Rappelons qu'une symédiane d'un triangle est la réflexion d'une médiane par rapport à la bissectrice issue d'un même sommet. Le point Lemoine d'un triangle est le point d'intersection des trois symédianes.

**3651**. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit $a$, $b$ et $c$ trois nombres réels non négatifs tels que $a + b + c = 3$. Montrer que

$$a^2b + b^2c + c^2a + abc + 4abc(3 - ab - bc - ca) \;\le\; 4\,.$$

**3652**. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $\alpha$ et $\beta$ deux nombres réels positifs. Trouver la valeur de la limite

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^\alpha}{n^\beta}\right) .$$

**3653**. *Proposé par Peter Y. Woo, Université Biola, La Mirada, CA, É-U.*

Soit $O$ le centre d'une sphère $S$ circonscrite à un tétrahèdre $ABCD$. Montrer que

(i) il existe des tétrahèdres dont les quatre faces sont des triangles obtusangles ; et

(ii) si $O$ est à l'intérieur ou sur $ABCD$, alors au moins deux faces de $ABCD$ sont des triangles acutangles.

**3654**. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Soit $a$, $b$, $c$ et $d$ quatre nombres réels non négatifs tels que $a^2+b^2+c^2+d^2=1$. Montrer que

$$a^3+b^3+c^3+d^3+abc+bcd+cda+dab \ \leq \ 1 .$$

**3655**. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Calculer l'intégrale

$$\int_0^1 \int_0^1 x \left\{\frac{1}{1-xy}\right\} dxdy,$$

où $\{a\} = a - \lfloor a \rfloor$ dénote la partie fractionnaire de $a$.

**3656**. *Proposé par Michel Bataille, Rouen, France.*

Dans une ellipse, on fixe une corde $AB$ qui ne soit pas un diamètre et soit $MN$ un diamètre variable. Montrer que le lieu du point d'intersection de $MA$ et $NB$ est une ellipse de même excentricité que l'ellipse originale, et trouver une description géométrique de son centre.

**3657**. *Proposé par Thanos Magkos, 3ième -Collège de Kozanie, Kozani, Grèce.*

Montrer qu'on a l'inégalité suivante

$$\frac{\cos^2 A}{1+\cos^2 A} + \frac{\cos^2 B}{1+\cos^2 B} + \frac{\cos^2 C}{1+\cos^2 C} \ \geq \ \frac{1}{2} .$$

pour les angles de n'importe quel triangle.

**3658**. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $-\pi < \theta_0 < \theta_1 < \cdots < \theta_k < \pi$ et soit $a_j$, $j = 0, 1, \cdots, k$, $k$ nombres complexes. Montrer que si

$$\lim_{n\to\infty} \sum_{j=0}^{k} a_j \cos(\theta_j n) = 0\,,$$

alors $a_j = 0$ pour tout les $j$.

**3659**. *Proposé par Michel Bataille, Rouen, France.*

Soit $P$ un point sur un cercle $\Gamma$ de diamètre $AB$. La tangente à $\Gamma$ en $P$ coupe respectivement les tangentes en $A$ et $B$ aux points $D$ et $C$. Soit $M$ un point quelconque sur la droite $BC$ et $V$ le point d'intersection de $MD$ et $BP$. Si la parallèle à $BC$ passant par $V$ coupe $CD$ en $U$, montrer que la droite $MU$ est tangente à $\Gamma$.

**3660**. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit $a, b, c$ les longueurs des côtés d'un triangle $ABC$, $r$ le rayon de son cercle inscrit et $R$ le rayon de son cercle circonscrit. Montrer que

$$\frac{y+z}{x}\cdot\frac{1}{a^2} + \frac{z+x}{y}\cdot\frac{1}{b^2} + \frac{x+y}{z}\cdot\frac{1}{c^2} \geq \frac{1}{Rr}\,,$$

pour tous les nombres réels positifs $x$, $y$ et $z$.

**3661**. *Proposé par Paul Yiu, Florida Atlantic University, Boca Raton, FL, É-U.*

On considère un triangle $ABC$ et $D$, $E$, $F$ les points milieu de ses côtés $BC$, $CA$, $AB$. Pour un point arbitraire $P$, soit $X$, $Y$, $Z$ les réflexions respectives de $P$ par rapport à $D$, $E$, $F$. Montrer que les droites $AX$, $BY$, $CZ$ sont concourantes.

**3662**. *Proposé par Michel Bataille, Rouen, France.*

Soit $\mathcal{R}$ l'ensemble des entiers positifs dont l'écriture en base dix est la répétition d'un seul chiffre (p.ex. $5 \in \mathcal{R}$, $222 \in \mathcal{R}$, $88888 \in \mathcal{R}$). Soit $T(n) = (n-2)^2 + n^2 + (n+2)^2$ où $n$ est un entier non négatif.

(a) Trouver tous les entiers pairs $n$ tels que $T(n) \in \mathcal{R}$.

(b) Trouver un entier impair $n > 1$ tel que $T(n) \in \mathcal{R}$. On donnera un crédit supplémentaire à quiconque trouve plus d'un tel entier.

**3663**. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

Soit $a$, $b$, $c$ trois nombres réels positifs. Montrer que

$$\sqrt[3]{\frac{2a}{4a+4b+c}} + \sqrt[3]{\frac{2b}{4b+4c+a}} + \sqrt[3]{\frac{2c}{4c+4a+b}} < 2\,.$$

# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

**3224**. [2007 : 112, 115; 2008 : 121- 124] *Proposed by J. Chris Fisher and Harley Weston, University of Regina, Regina, SK.*

Let $A_0B_0C_0$ be an isosceles triangle whose apex angle $A_0$ is not $120^\circ$. We define a sequence of triangles $A_nB_nC_n$ in which $\triangle A_{i+1}B_{i+1}C_{i+1}$ is obtained from $\triangle A_iB_iC_i$ by reflecting each vertex in the opposite side (that is, $B_iC_i$ is the perpendicular bisector of $A_iA_{i+1}$, and so forth). Prove that all three angles approach $60^\circ$ as $n \to \infty$.

[*Ed:* This problem is a special case of an open problem described by Judah Schwartz in "Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible?", *International Journal of Computers for Mathematical Learning*, **4** (1999), pp. 99–119.]

*II. Solution by Grégoire Nicollier, University of Applied Sciences of Western Switzerland, Sion, Switzerland.*

In his paper [1] Nicollier provides another solution to our problem (which is restricted to isosceles triangles). In addition he resolves Schwartz's open problem by describing those triangles for which iterating the reflection map produces a sequence of triangles whose limit is equilateral, whose limit is degenerate, and whose limit is neither equilateral nor degenerate.

**References**

[1] Grégoire Nicollier, Reflection triangles and their iterates, **Forum Geometricorum**, 12 (2012) 83–128.

**3551**. [2010 : 314, 316] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $p \geq 2$ be an integer. Find the product

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{\lfloor \sqrt[p]{n} \rfloor}\right)^{(-1)^{n-1}},$$

where $\lfloor a \rfloor$ is the greatest integer not exceeding $a$.

*Solution by Joel Schlosberg, Bayside, NY, USA.*

There are an equal number of odd and even integers in the interval $[m^p + 1, (m + 1)^p - 1]$, so $\sum\limits_{n=m^p}^{(m+1)^p-1} (-1)^{n-1} = (-1)^{m^p-1} = (-1)^{m-1}$.

Therefore,

$$\prod_{n=1}^{N}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}}$$

$$= \prod_{m=1}^{\lfloor\sqrt[p]{N}\rfloor-1}\prod_{n=m^p}^{(m+1)^p-1}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}} \cdot \prod_{n=\lfloor\sqrt[p]{N}\rfloor^p}^{N}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}}$$

$$= \prod_{m=1}^{\lfloor\sqrt[p]{N}\rfloor-1}\left(1+\frac{1}{m}\right)^{\sum_{n=m^p}^{(m+1)^p-1}(-1)^{n-1}} \cdot \left(1+\frac{1}{\lfloor\sqrt[p]{N}\rfloor}\right)^{\sum_{n=\lfloor\sqrt[p]{N}\rfloor^p}^{N}(-1)^{n-1}}$$

$$= \prod_{m=1}^{\lfloor\sqrt[p]{N}\rfloor-1}\left(\frac{m+1}{m}\right)^{(-1)^{m-1}} \cdot \left(1+O\left(\frac{1}{\lfloor\sqrt[p]{N}\rfloor}\right)\right).$$

Letting $N \to \infty$ and using Wallis' product,

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}} = \prod_{m=1}^{\infty}\left(\frac{m+1}{m}\right)^{(-1)^{m-1}}$$

$$= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots = \frac{\pi}{2}.$$

*Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; NEVEN JURIČ, Zagreb, Croatia; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incorrect solution was submitted.*

**3552**. [2010 : 314, 316] *Proposed by N. Javier Buitrago Aza, Universidad Nacional de Colombia, Bogota, Colombia.*

Let $\theta$ be a real number. Prove that

$$\sum_{k=0}^{n-1}\frac{\sin\left(\frac{2k\pi}{n}-\theta\right)}{3+2\cos\left(\frac{2k\pi}{n}-\theta\right)} = \frac{(-1)^n n\sin(n\theta)}{5F_n^2+4(-1)^n\sin^2\left(\frac{n\theta}{2}\right)},$$

where $F_n$ denotes the $n^{\text{th}}$ Fibonacci number.

*Solution by Michel Bataille, Rouen, France, modified and expanded by the editor.*

First, we have, by the well known Binet's formula that $\sqrt{5}F_n = \alpha^n - \beta^n$ where $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. Note that $\alpha\beta = -1$ so $5F_n^2 = \alpha^{2n}+\beta^{2n}-2(-1)^n$. Also, $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 3$.

Now, for real $x$, let $z = e^{-ix}$. Then it is readily checked that

$$\frac{\sin x}{3+2\cos x} = \frac{1}{2i}\cdot\frac{e^{ix}-e^{-ix}}{3+e^{ix}+e^{-ix}} = \frac{1}{2i}\cdot\frac{z^{-1}-z}{3+z^{-1}+z} = \frac{1}{2i}\cdot\frac{1-z^2}{z^2+3z+1}$$
$$= \frac{1}{2i}\left(\frac{3z+2}{z^2+3z+1}-1\right) = \frac{1}{2i}\left(\frac{2+(\alpha^2+\beta^2)z}{z^2+(\alpha^2+\beta^2)z+1}-1\right)$$
$$= \frac{1}{2i}\left(\frac{1}{1+\alpha^2 z}+\frac{1}{1+\beta^2 z}-1\right).$$

Letting $w = e^{-\frac{2\pi i}{n}}$ and replacing $x$ with $\frac{2k\pi}{n}-\theta$, $k = 0, 1, 2, \ldots, n-1$, we have $z = e^{-i\left(\frac{2k\pi}{n}-\theta\right)} = e^{i\theta}\cdot e^{-\frac{2k\pi i}{n}} = e^{i\theta}\cdot w^k$. Hence,

$$\frac{\sin x}{3+2\cos x} = \frac{1}{2i}\left(\frac{1}{1+\alpha^2 e^{i\theta}w^k}+\frac{1}{1+\beta^2 e^{i\theta}w^k}-1\right). \tag{1}$$

The identity below is known and can be verified easily by the method of partial fractions decomposition:

$$\frac{1}{t^n-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{w^k t-1}. \tag{2}$$

From (1) and (2) we have

$$\sum_{k=0}^{n-1}\frac{\sin\left(\frac{2k\pi}{n}-\theta\right)}{3+2\cos\left(\frac{2k\pi}{n}-\theta\right)}$$
$$= \frac{-1}{2i}\sum_{k=0}^{n-1}\left(\frac{1}{(-\alpha^2 e^{i\theta})w^k-1}+\frac{1}{(-\beta^2 e^{i\theta})w^k-1}+1\right)$$
$$= \frac{n}{2i}\left(\frac{1}{1-(-1)^n\alpha^{2n}e^{in\theta}}+\frac{1}{1-(-1)^n\beta^{2n}e^{in\theta}}-1\right)$$
$$= \frac{n}{2i}\left(\frac{1-e^{2in\theta}}{1-(-1)^n(\alpha^{2n}+\beta^{2n})e^{in\theta}+e^{2in\theta}}\right)$$
$$= \frac{n}{2i}\cdot\frac{e^{-in\theta}-e^{in\theta}}{e^{-in\theta}-(-1)^n(\alpha^{2n}+\beta^{2n})+e^{in\theta}}$$
$$= \frac{(-1)^n n\sin(n\theta)}{\alpha^{2n}+\beta^{2n}-2(-1)^n\cos(n\theta)}.$$

The result now follows by substituting $\alpha^{2n}+\beta^{2n} = 2(-1)^n+5F_n^2$ and $\cos(n\theta) = 1-2\sin^2\left(\frac{n\theta}{2}\right)$.

*Also solved by PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.*

*Both Bataille and Geupel pointed out that this problem (proposed by the same person) has appeared as problem U173 in Mathematical Reflections, 2010, issue 5. The solution featured above is different from the one that was published in issue 6.*

**3553**. [2010 : 314, 317] *Proposed by Michel Bataille, Rouen, France.*

Let $A$, $B$, and $C$ be the angles of a triangle. Prove that

$$\sum_{\text{cyclic}} \left(\sin A \cos\frac{B}{2}\cos\frac{C}{2}\right)^2 \ \le\ \sum_{\text{cyclic}} \cos^6\frac{A}{2}\,.$$

*Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

Using basic trigonometric identities and the fact that $A+B+C=\pi$, we obtain

$$\begin{aligned}
&\sin A\cos\frac{B}{2}\cos\frac{C}{2}\\
&= 2\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \cos\frac{A}{2}\cos\frac{C}{2}\left[\sin\frac{A+B}{2}+\sin\frac{A-B}{2}\right]\\
&= \frac{1}{2}\cos\frac{A}{2}\left[\sin\frac{A+B+C}{2}+\sin\frac{A+B-C}{2}+\sin\frac{A-B+C}{2}\right.\\
&\qquad\qquad\left.+\sin\frac{A-B-C}{2}\right]\\
&= \frac{1}{2}\cos\frac{A}{2}\left[\sin\frac{\pi}{2}+\sin\left(\frac{\pi}{2}-C\right)+\sin\left(\frac{\pi}{2}-B\right)-\sin\left(\frac{\pi}{2}-A\right)\right]\\
&= \frac{1}{2}\cos\frac{A}{2}[1+\cos C+\cos B-\cos A]\\
&= \frac{1}{2}\cos\frac{A}{2}\left[2\cos^2\frac{C}{2}+2\cos^2\frac{B}{2}-2\cos^2\frac{A}{2}\right]\\
&= \cos\frac{A}{2}\left[\cos^2\frac{C}{2}+\cos^2\frac{B}{2}-\cos^2\frac{A}{2}\right]\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(\sin A\cos\frac{B}{2}\cos\frac{C}{2}\right)^2 = \cos^6\frac{A}{2}+\cos^2\frac{A}{2}\Bigg[&\cos^4\frac{B}{2}+\cos^4\frac{C}{2}\\
&+2\cos^2\frac{B}{2}\cos^2\frac{C}{2}-2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\\
&-2\cos^2\frac{A}{2}\cos^2\frac{C}{2}\Bigg]\,,
\end{aligned}$$

with similar expressions for $\left(\sin B\cos\frac{C}{2}\cos\frac{A}{2}\right)^2$ and $\left(\sin C\cos\frac{A}{2}\cos\frac{B}{2}\right)^2$.

Therefore,

$$\begin{aligned}
&\sum_{\text{cyclic}} \left(\sin A \cos\frac{B}{2}\cos\frac{C}{2}\right)^2 \\
&= \sum_{\text{cyclic}} \cos^6\frac{A}{2} \\
&- \left[\cos^2\frac{A}{2}\cos^4\frac{B}{2} - 2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{A}{2}\cos^4\frac{C}{2}\right] \\
&- \left[\cos^2\frac{B}{2}\cos^4\frac{C}{2} - 2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{B}{2}\cos^4\frac{A}{2}\right] \\
&- \left[\cos^2\frac{C}{2}\cos^4\frac{A}{2} - 2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{C}{2}\cos^4\frac{B}{2}\right] \\
&= \sum_{\text{cyclic}} \cos^6\frac{A}{2} - \sum_{\text{cyclic}} \cos^2\frac{A}{2}\left[\cos^2\frac{B}{2} - \cos^2\frac{C}{2}\right]^2 \\
&\le \sum_{\text{cyclic}} \cos^6\frac{A}{2} .
\end{aligned}$$

Furthermore, since $0 < \frac{A}{2}, \frac{B}{2}, \frac{C}{2} < \frac{\pi}{2}$, equality is attained if and only if $\cos^2\frac{A}{2} = \cos^2\frac{B}{2} = \cos^2\frac{C}{2}$; that is, if and only if $A = B = C = \frac{\pi}{3}$, so that the given triangle is equilateral.

*Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

**3554**. [2010 : 314, 317] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let $a$, $b$, and $c$ be positive real numbers. Prove that

$$\frac{\sqrt{a^2+bc}}{b+c} + \frac{\sqrt{b^2+ca}}{c+a} + \frac{\sqrt{c^2+ab}}{a+b} \ \ge\ \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} .$$

*Solution by Joe Howard, Portales, NM, USA.*

By symmetry we can assume that $a \ge b \ge c > 0$. Then $(a-c)(b-c) \ge 0$ and thus

$$c^2 + ab \ge c(a+b) .$$

Hence

$$\frac{\sqrt{c^2+ab}}{a+b} \ge \frac{\sqrt{c(a+b)}}{a+b} = \sqrt{\frac{c}{a+b}} .$$

To complete the inequality, we will prove that

$$\left(\frac{\sqrt{a^2+bc}}{b+c}+\frac{\sqrt{b^2+ac}}{a+c}\right)^2 \geq \left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{a+b}}\right)^2 .$$

First we will show that

$$\frac{\sqrt{a^2+bc}}{b+c}\frac{\sqrt{b^2+ac}}{a+c} \geq \sqrt{\frac{a}{b+c}}\sqrt{\frac{b}{a+b}} \tag{1}$$

This simplifies to $c(a^3+b^3) \geq abc(a+b)$. Dividing by $c(a+b)$ this inequality is equivalent to

$$a^2-ab+b^2 \geq ab ,$$

or

$$(a-b)^2 \geq 0 .$$

Thus, (1) holds.
Now we also show that

$$\frac{a^2+bc}{(b+c)^2}+\frac{b^2+ac}{(a+c)^2} \geq \frac{a}{b+c}+\frac{b}{a+b} \tag{2}$$

This simplifies to

$$a^4+b^4+a^3c+b^3c+2abc^2 \geq a^3b+b^3a+a^2c^2+b^2c^2+a^2bc+ab^2c ,$$

or

$$(a-b)^2(a^2+ab+b^2)+c(a-b)[a(a-c)-b(b-c)] \geq 0 .$$

This last inequality is an immediate consequence of $a \geq b \geq c > 0$.
This completes the proof.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer .*

**3555**. [2010 : 315, 317] *Proposed by Vahagn Aslanyan, Yerevan, Armenia.*

Let $a$ and $b$ be positive integers, $1 < a < b$, such that $a$ does not divide $b$. Prove that there exists an integer $x$ such that $1 < x \leq a$ and both $a$ and $b$ divide $x^{\phi(b)+1} - x$, where $\phi$ is Euler's totient function.

*Similar solutions by Oliver Geupel, Brühl, NRW, Germany and John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA. We give Geupel the write up.*

We write $a = a_1 a_2$ ; $b = b_1 b_2$ where

$$a_1 = \prod_{i=1}^{m} p_i^{\alpha_i} \; ; \; a_2 = \prod_{i=m+1}^{n} p_i^{\alpha_i} \; ; \; b_1 = \prod_{i=1}^{m} p_i^{\beta_i} \; ; \; b_2 = \prod_{i=m+1}^{n} p_i^{\beta_i} \; ,$$

with distinct primes $p_i$ and nonnegative integers $\alpha_i, \beta_i$ satisfying $\alpha_i \geq \beta_i$ for all $1 \leq i \leq m$ and $\alpha_i < \beta_i$ for all $i > m$. That is $a_1$ is the product of prime powers that occur with at least the same exponent in $a$ as in $b$, and $b_1$ is the product of the corresponding powers in $b$.

We prove that $x = a_1$ works.

The hypothesis $a \nmid b$ yields $x \neq 1$.

We know that gcd$(b_1, b_2) = 1$, thus $\phi(b) = \phi(b_1)\phi(b_2)$. Since gcd$(x, b_2)$=1, by Euler Theorem we have

$$x^{\phi(b)} \equiv \left(x^{\phi(b_2)}\right)^{\phi(b_1)} \equiv (1)^{\phi(b_1)} \equiv 1 \pmod{b_2} \; .$$

Hence

$$x^{\phi(b)+1} \equiv x \pmod{a_1 b_2} \; .$$

From the definition of $a_1, a_2, b_1, b_2$, it follows that $b_1 | a_1$ and $a_2 | b_2$ and thus, both $a, b$ divide $a_1 b_2$.

Thus, both $a, b$ divide $x^{\phi(b)+1} - x$ which completes the proof.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer . There was also an incorrect solution.*

*Three solvers mentioned that this problem appeared simultaneously as Problem O170 in Mathematical Reflections 5 (2010). Geupel pointed that the solution featured in that journal is wrong, since it uses the incorrect fact that* $\frac{a}{gcd(a,b)}$ *and* $b$ *are relatively prime [ED:* $a = 4, b = 6$ *is a counterexample].*

**3557**. [2010 : 315, 317] *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers with $\sum\limits_{k=1}^{\infty} a_k = 1$ and $a_{k+1} \le \dfrac{a_k}{1-a_k}$. Let $S_n^{(p)} = \left(\sum\limits_{k=1}^{n} a_k^p\right)^{1/p}$, and for $p \ge 1$ prove that

$$\lim_{n\to\infty} \sum_{k=n+1}^{2n} \frac{k}{2}\left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j} a_j}{S_{k+j}^{(p)}}\right)^{1/n} \;=\; 0\,.$$

[*Ed.: The upper limit of summation is now correctly stated as* $2n$*; our apologies for this error.*]

*Solution by Albert Stadler, Herrliberg, Switzerland.*

We will only use the condition that $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive numbers such that $\sum\limits_{k\ge 1} a_k$ converges to a positive number $c$. The conditions that $c = 1$ and $a_{k+1} \le \dfrac{a_k}{1-a_k}$ are superfluous.

We first note that by the AM–GM Inequality

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n} \le \frac{c}{n}.$$

It follows that

$$\sum_{k=n+1}^{2n} \frac{k}{n}\left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j} a_j}{S_{k+j}^{(p)}}\right)^{1/n} \le 2n\cdot\frac{c}{n} \max_{n<k\le 2n}\left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j}}{S_{k+j}^{(p)}}\right)^{1/n}. \quad (1)$$

Put

$$T_n^{(p)} = \sum_{k=1}^{n} a_k^p\,,$$

$$M_n^{(p)} = \max_{n<k\le 2n}\left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j}}{S_{k+j}^{(p)}}\right)^{p/n} = \max_{n<k\le 2n}\left(\prod_{j=1}^{n} \frac{j a_{k+j}^p}{T_{k+j}^{(p)}}\right)^{1/n}.$$

We also have

$$\left(\prod_{j=1}^{n} j\right)^{1/n} \le \left(\prod_{j=1}^{n} n\right)^{1/n} = n\,, \quad (2)$$

$$\left(\prod_{j=1}^{n} a_{k+j}^p\right)^{1/n} \le \frac{1}{n}\sum_{j=1}^{n} a_{k+j}^p\,, \quad (3)$$

$$T_{k+n}^{(p)} \ge T_{k+n-1}^{(p)} \ge \cdots \ge T_{k+1}^{(p)} \ge T_k^{(p)}\,. \quad (4)$$

where (3) holds by the AM–GM Inequality.

Using (2), (3), and (4) we obtain

$$M_n^{(p)} \le \max_{n<k\le 2n} \left( \frac{\sum\limits_{j=1}^{n} a_{k+j}^p}{T_k^{(p)}} \right) .$$

By assumption, $\sum\limits_{k\ge 1} a_k$ converges, so there are only finitely many $a_k$ for which $a_k \ge 1$ and the other summands satisfy $0 \le a_k^p \le a_k < 1$. Thus, $\sum\limits_{k\ge 1} a_k^p$ converges, say to $\sigma$. Put $\epsilon_k = \sigma - \sum\limits_{j=1}^{k} a_j^p = \sum\limits_{j=k+1}^{\infty} a_j^p$. It then follows that

$$M_n^{(p)} \le \max_{n<k\le 2n} \left( \frac{\sum\limits_{j=1}^{n} a_{k+j}^p}{T_k^{(p)}} \right) \le \max_{n<k\le 2n} \left( \frac{\epsilon_k}{\sigma - \epsilon_k} \right) \to 0 \text{ as } n \to \infty .$$

Finally, we conclude from (1) that

$$\sum_{k=n+1}^{2n} \frac{k}{n} \left( \prod_{j=1}^{n} \frac{j^{1/p} a_{k+j} a_j}{S_{k+j}^{(p)}} \right)^{1/n} \to 0 \text{ as } n \to \infty ,$$

since the sum is nonnegative and bounded above by $2c\sqrt[p]{M_n^{(p)}}$, which vanishes as $n$ tends to infinity.

*Also solved by Oliver Geupel, Brühl, NRW, Germany, and the proposer.*
*Geupel also corrected the problem and removed the hypothesis $a_{k+1} \le a_k/(1-a_k)$.*

**3558**. [2010 : 315, 317] *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Given two distinct positive integers $a$ and $b$, prove that there exists a positive integer $n$ such that $an$ and $bn$ have different numbers of digits.

*Solution by Skidmore College Problem Solving Group, Skidmore College, Saratoga Springs, NY, USA(abbreviated by editor).*

Without loss of generality we assume $0 < a < b$. For each positive integer $j$ let $I_j$ denote the interval $(ja, jb)$. It will be sufficient to show that there exist positive integers $n$ and $N$ such that $10^N \in I_n$, for then $na$ will require at most $N$ digits and $nb$ more than $N$ digits in their decimal representations.

It is observed that $I_m$ and $I_{m+1}$ will overlap for all sufficiently large $m$. Indeed, for any $m > M = \lceil \frac{a}{b-a} \rceil$ it is easy to see that $m(b-a) > a$, therefore

$(m+1)a < mb$ and thus $I_m \cap I_{m+1} = ((m+1)a, mb) \neq \emptyset$. Consequently

$$\bigcup_{m>M} I_m = \Big((M+1)a, \lim_{m\to\infty} mb\Big) = ((M+1)a, \infty)$$

.

Now take any integer $N$ such that $10^N \in ((M+1)a, \infty)$ and we will have that $10^N \in I_n$ for some $n > M$ as claimed.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon(2 solutions); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer who provided several solutions.*

**3559★**. [2010 : 315, 318] *Proposed by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*

Let $ABC$ be a triangle with side lengths $a$, $b$, $c$, inradius $r$, circumradius $R$, and semiperimeter $s$. Prove that

$$\frac{(b+c)^2}{4bc} \leq \frac{s^2}{3r(4R+r)}\,.$$

*Solution I by Albert Stadler, Herrliberg, Switzerland, expanded slightly by the editor.*

Let $F$ be the area of the triangle. We have the formulae

$$F = \frac{abc}{4R} = rs = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

So

$$\begin{aligned}\frac{s^2}{3r(4R+r)} &= \frac{s^4}{3F(4Rs+F)} = \frac{s^3}{3abcs + 3(s-a)(s-b)(s-c)}\\ &= \frac{1}{3}\cdot\frac{s^2}{s^2-(a+b+c)s+ab+bc+ca}\\ &= \frac{1}{3}\cdot\frac{(a+b+c)^2}{(a+b+c)^2-2(a+b+c)^2+4(ab+bc+ca)}\\ &= \frac{1}{3}\cdot\frac{a^2+b^2+c^2+2ab+2bc+2ca}{-a^2-b^2-c^2+2ab+2bc+2ca}\,.\end{aligned}$$

Hence, the given inequality is equivalent in succession to

$$\begin{aligned} 3(b+c)^2(-a^2-b^2-c^2+2ab+2bc+2ca) & \\ \leq 4bc(a^2+b^2+c^2+2ab+2bc+2ca) & \\ 0 \leq (3b^2+10bc+3c^2)a^2-2(b+c)(3b^2+2bc+3c^2)a & \\ +(b+c)^2(3b^2-2bc+3c^2)\,. & \quad (1) \end{aligned}$$

Let $f(a)$ be the quadratic polynomial in $a$ on the right hand side of (1) and let $\Delta$ denote its discriminant.

Since the leading coefficient of $f(a)$ is clearly positive, to conclude that $f(a) \geq 0$ it suffices to show that $\Delta \leq 0$. Letting $d = 3b^2+3c^2$ we have

$$\begin{aligned} \Delta &= 4(b+c)^2[(d+2bc)^2-(d-2bc)(d+10bc)] \\ &= 4(b+c)^2(4bcd+4b^2c^2-8bcd+20b^2c^2) \\ &= 4(b+c)^2(-4bcd+24b^2c^2) \\ &= -48bc(b+c)^2(b^2+c^2-2bc) = -48bc(b+c)^2(b-c)^2 \leq 0 \end{aligned}$$

so the proof is now complete.

*Solution II by Joe Howard, Portales, NM, USA.*

We prove the following extension:

$$\frac{(b+c)^2}{4bc} \leq \frac{(a+b+c)^3}{27abc} \leq \frac{s^2}{3r(4R+r)}\,.$$

Let $A_n$ and $G_n$ denote the arithmetic and geometric means of $n$ positive numbers, respectively. Then it is known [1] that $\left(\frac{A_n}{G_n}\right)$ is monotonically increasing. The left inequality follows from $\frac{A_2}{G_2} \leq \frac{A_3}{G_3}$.

To establish the right inequality we use the known formulae: $a+b+c=2s$ and $abc = 4Rrs$. Then $\frac{(a+b+c)^3}{27abc} \leq \frac{s^2}{3r(4R+r)}$ is equivalent in succession to

$$\begin{aligned} \frac{8s^3}{27(4Rrs)} &\leq \frac{s^2}{12Rs+3r^2} \\ 24Rr+6r^2 &\leq 27Rr \\ 2r &\leq R \end{aligned}$$

which is the famous Euler's Inequality.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and PETER Y. WOO, Biola University, La Mirada, CA, USA.*

*Arslanagić, Geupel and Malikić all pointed out that equality holds if and only if the triangle is equilateral.*

*The proposer remarked that the proposed inequality sharpens the result* $3r(4R+r) \leq s^2$ *by G. Colombier and T. Doucet (see item 5.5 on p.49 of [2]).*

**References**

[1] B. Arbed, *From Tricks to Strategies for Problem Solving*, Int. J. Math., Edu. Sci. Tech. 21(3), 1990; pp. 429 – 438

[2] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

**3560**. [2010 : 315, 318] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let $x$ and $y$ be real numbers such that $x^2 + y^2 = 1$. Find the maximum value of

$$f(x,y) = |x-y| + \left|x^3 - y^3\right| .$$

*Solution by Joel Schlosberg, Bayside, NY, USA.*

Since $|xy| \leq \frac{1}{2}(x^2+y^2) = \frac{1}{2}$,

$$x^2 + xy + y^2 \geq \frac{1}{2} \geq 0, \quad 1 - 2xy \geq 0, \quad \text{and} \quad 2 + xy \geq 0 .$$

Noting that $|x^3 - y^3| = |x-y||x^2+xy+y^2|$ and using the AM-GM inequality we see that

$$\begin{aligned} (f(x,y))^2 &= |x-y|^2\left(1 + |x^2+xy+y^2|\right)^2 \\ &= (x^2 - 2xy + y^2)(1 + x^2 + xy + y^2)^2 \\ &= (1-2xy)(2+xy)^2 \\ &\leq \left(\frac{(1-2xy) + 2(2+xy)}{3}\right)^3 = \left(\frac{5}{3}\right)^3 ; \end{aligned}$$

that is, $f(x,y) \leq \left(\frac{5}{3}\right)^{3/2}$. Since $x^2+y^2 = 1$, and $f(x,y) = \left(\frac{5}{3}\right)^{3/2}$ for

$$\{x,y\} = \left\{\frac{1+\sqrt{5}}{2\sqrt{3}}, \frac{1-\sqrt{5}}{2\sqrt{3}}\right\} \quad \text{or} \quad \{x,y\} = \left\{-\frac{1+\sqrt{5}}{2\sqrt{3}}, -\frac{1-\sqrt{5}}{2\sqrt{3}}\right\} ,$$

the maximum value of $f(x,y)$ is $\left(\frac{5}{3}\right)^{3/2}$.

*Also solved by ARKADY ALT, San Jose, CA, USA(2 solutions); GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; NEVEN JURIČ, Zagreb, Croatia; KEE-WAI LAU, Hong Kong, China; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; CRISTINEL MORTICI, Valahia University of Târgovişte,* *Romania; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA(3 solutions); DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.*

**3561**. [2010 : 316, 318] *Proposed by Mihály Bencze, Brasov, Romania.*

An $n$-sided polygon has perimeter $k$ with $k^2 < 2n^2$. Prove that some three consecutive vertices along the polygon form a triangle with area less than $1$ unit.

*Solution by George Apostolopoulos, Messolonghi, Greece.*

If $s_1, \ldots, s_n, s_{n+1}$ are the side lengths in cyclic order (with $s_{n+1} = s_1$), then $s_i > 0$ and $k = \sum_{i=1}^n s_i$. A triangle formed by three consecutive vertices that determine an angle $\theta_i$ has an area $F_i$ that satisfies

$$F_i = \frac{s_i s_{i+1} \sin\theta_i}{2} \le \frac{s_i s_{i+1}}{2}.$$

From this inequality together with the AM-GM inequality and $k^2 < 2n^2$ (given) we get

$$\begin{aligned}\prod_{i=1}^n F_i &\le \prod_{i=1}^n \frac{s_i s_{i+1}}{2} = \frac{1}{2^n}\left(\prod_{i=1}^n s_i\right)^2 \\ &\le \frac{1}{2^n}\left(\left(\frac{\sum_{i=1}^n s_i}{n}\right)^n\right)^2 \\ &= \frac{1}{2^n}\left(\frac{k}{n}\right)^{2n} = \left(\frac{k^2}{2n^2}\right)^n < 1.\end{aligned}$$

We deduce that at least one $F_i$ is less than $1$, as desired.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.*

*All the submitted solutions were quite similar, although most used an indirect argument.*

**3562**. [2010 : 316, 318] *Proposed by Vahagn Aslanyan, Yerevan, Armenia.*

Let $p$ be a prime number. Prove that there exists a prime number $q$ such that $p \mid (q-1)$ and with the property that $q^k \mid \left(a^p - b^p\right)$ whenever $q^k \mid \left(a^{p^m} - b^{p^m}\right)$ for positive integers $a$, $b$, $m$, $k$ with $a$ and $b$ not divisible by $q$.

*Solution by Joel Schlosberg, Bayside, NY, USA, modified slightly by the editor.*

Since $(p^2, p+1) = 1$, by Dirichlet's Theorem, there exists infinitely many primes in the arithmetic progression $p + 1 + np^2$, $n = 0, 1, 2, \ldots$. Thus, there exists a prime $q$ such that $q = p + 1 + np^2$ or $q - 1 = p + np^2$ for some $n$. Hence $p \mid q - 1$ but $p^2 \nmid q - 1$.

Suppose that $a$, $b$, $m$, $k$ are positive integers such that $q \nmid a$, $q \nmid b$ and $q^k \mid a^{p^m} - b^{p^m}$. Then

$$(ab^{-1})^{p^m} \equiv 1 \pmod{q^k} \qquad (1)$$

where $b^{-1}$ denotes the multiplicitive inverse of $b$ modulo $q^k$. Since $bb^{-1} \equiv 1 \pmod{q^k}$, $q^k \nmid b^{-1}$ so $(q^k, ab^{-1}) = 1$. Hence, by Euler's theorem, we have

$$(ab^{-1})^{(q-1)q^{k-1}} = (ab^{-1})^{\phi(q^k)} \equiv 1 \pmod{q^k} \qquad (2)$$

where $\phi$ denotes Euler's totient function.

By (1) and (2), the multiplicative order of $ab^{-1}$ modulo $q^k$ divides $(p^m, (q-1)q^{k-1})$ which equals $p$ since $p \nmid q$, $p \mid q - 1$ and $p^2 \nmid q - 1$. Therefore, $(ab^{-1})^p \equiv 1 \pmod{q^k}$, from which $a^p \equiv b^p \pmod{q^k}$ follows.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.*

*Geupel remarked that this problem is closely related to problem 6 of the IMO 2003. [Ed: The IMO problems asked to show that for each prime $p$ there exists a prime $q$ such that $n^p - p$ is not divisible by $q$ for any positive integer $n$.]*