THE OLYMPIAD CORNER
No. 293

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In this issue we begin a transition in the Corner. Problems editor Nicolae Strugaru, from Grant MacEwan University in Edmonton, has agreed to take over from Robert Woodrow who has been the editor of the Corner since 1987. Robert’s dedication to CRUX with MAYHEM over the years is greatly appreciated and he will be sorely missed. Material from Robert will continue to appear in CRUX with MAYHEM as we wrap up the solutions to the last sets of problems he published.

The format of the Corner is changing slightly. It will still consist of problems from Olympiads from around the world, but, rather than printing the contests in their entirety, each column will consist of 10 questions, in both English and French, selected from different contests. The origin of the question will be revealed when the solutions are published.

We will have the same time lines as we do with the CRUX problems. Solutions will be due six months from the issue date and will appear in the same issue number of the next volume, one year later. The first set of new Olympiad Corner problems is below, please send your solutions to Nicolae by email (preferred) at:

crux-olympiad@cms.math.ca

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Enjoy the new Corner!
The solutions to the problems are due to the editor by 1 January 2012.

**OC1.** Find all positive integers $w$, $x$, $y$ and $z$ which satisfy $w! = x! + y! + z!$.

**OC2.** Suppose that $f$ is a real-valued function for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real numbers $x$ and $y$.

(a) Give a nonconstant polynomial that satisfies the condition.

(b) Prove that $f(x) \geq 0$ for all real $x$.

**OC3.** Let $ABCD$ be a convex quadrilateral with

$$\angle CBD = 2\angle ADB,$$

$$\angle ABD = 2\angle CDB$$

and $AB = CB$.

Prove that $AD = CD$.

**OC4.** Consider 70-digit numbers $n$, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of $n$ ten times (and 8, 9 and 0 do not appear). Show that no number of this form can divide another number of this form.

**OC5.** Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0,$$

$$a_1 + a_2 \leq 100$$

and $a_3 + a_4 + \cdots + a_{100} \leq 100$.

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

**OC6.** In the diagram, $ABCD$ is a square, with $U$ and $V$ interior points of the sides $AB$ and $CD$ respectively. Determine all the possible ways of selecting $U$ and $V$ so as to maximize the area of the quadrilateral $PUQV$. 

![Diagram of square with interior points U and V]
OC7. Let $n$ be a natural number such that $n \geq 2$. Show that 
\[ \frac{1}{n+1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right) > \frac{1}{n} \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \right). \]

OC8. For each real number $r$ let $T_r$ be the transformation of the plane that takes the point $(x, y)$ into the point $(2^r x, r^2 x + 2^r y)$. Let $F$ be the family of all such transformations i.e. $F = \{ T_r : r \in \mathbb{R} \}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in $F$.

OC9. A deck of $2n + 1$ cards consists of a joker and, for each number between 1 and $n$ inclusive, two cards marked with that number. The $2n + 1$ cards are placed in a row, with the joker in the middle. For each $k$ with $1 \leq k \leq n$, the two cards numbered $k$ have exactly $k - 1$ cards between them. Determine all the values of $n$ not exceeding 10 for which this arrangement is possible. For which values of $n$ is it impossible?

OC10. The number 1987 can be written as a three digit number $xyz$ in some base $b$. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of $x, y, z, b$.

OC1. Trouver tous les entiers positifs $w, x, y$ et $z$ qui satisfont $w! = x! + y! + z!$.

OC2. Supposer que $f$ est une fonction à valeurs réelles qui satisfait 
\[ f(xy) + f(y - x) \geq f(y + x) \]

pour tous nombres réels $x$ et $y$.

(a) Donner un polynôme non constant qui satisfait cette condition.

(b) Montrer que $f(x) \geq 0$ pour tout nombre réel $x$.

OC3. On considère un quadrilatère convexe $ABCD$ dans lequel
\[ \angle CBD = 2\angle ADB, \]
\[ \angle ABD = 2\angle CDB \]
et $AB = CB$.

Démontrer que $AD = CD$.

OC4. Considérer les nombres $n$ à 70 chiffres avec la propriété que chacun des chiffres 1, 2, 3, ..., 7 apparaît dix fois dans l’expansion décimale de $n$ (et que 8, 9 et 0 n’y apparaissent pas). Montrer qu’aucun nombre de cette forme ne peut être divisé par un autre nombre de la même forme.
OC5. Supposons que les nombres réels $a_1, a_2, \ldots, a_{100}$ satisfont aux conditions suivantes

\[ a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0, \]
\[ a_1 + a_2 \leq 100 \]
\[ a_3 + a_4 + \cdots + a_{100} \leq 100. \]

Déterminer la valeur maximale possible de $a_1^2 + a_2^2 + \cdots + a_{100}^2$, et trouver toutes les suites possibles $a_1, a_2, \ldots, a_{100}$ pour lesquelles ce maximum est atteint.

OC6. Sur le diagramme ci-dessous, $ABCD$ est un carré sur lequel on choisit des points $U$ et $V$ intérieurs aux côtés $AB$ et $CD$ respectivement. Déterminer toutes les façons possibles de choisir $U$ et $V$ de telle sorte que la surface du quadrilatère $P U Q V$ soit maximale.

OC7. Soit $n$ un nombre naturel tel que $n \geq 2$. Montrer que

\[ \frac{1}{n+1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right) > \frac{1}{n} \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \right). \]

OC8. Soit $F$ la famille des transformations $F = \{ T_r : r \in \mathbb{R} \}$ où $T_r$ transforme le point $(x, y)$ en le point $(2^r x, r^2 x + 2^r y)$. Trouver toutes les courbes $y = f(x)$ dont le graphe est invariant pour chacune des transformations de $F$.

OC9. Un jeu de $2n+1$ cartes contient un joker et, pour chaque nombre entier de 1 à $n$ inclusivement, 2 cartes marquées de ce numéro. Les $2n+1$ cartes sont alors alignées avec le joker au milieu. De plus, pour chaque nombre entier $k$ avec $1 \leq k \leq n$, les deux cartes numérotées $k$ ont exactement $k-1$ autres cartes entre elles. Trouver toutes les valeurs de $n$ ne dépassant pas 10 pour lesquelles cet arrangement est possible. Maintenant, pour quelles valeurs de $n$ est-ce impossible ?

OC10. Le nombre 1987 s’écrit à trois chiffres, $xyz$, dans une certaine base $b$. Si $x + y + z = 1 + 9 + 8 + 7$, déterminer toutes les valeurs possible de $x, y, z$ et $b$. 

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**Diagramme:**

[Diagramme avec des points U et V dans un carré.]
First we look at solutions from the files to the 20th Korean Mathematical Olympiad, given at [2010: 152–153].

1. Triangle $ABC$ is acute with circumcircle $\Gamma$ and circumcentre $O$. The circle $\Gamma'$ has centre $O'$, is tangent to $O$ at $A$ and to the side $BC$ at $D$, and intersects the lines $AB$ and $AC$ again at $E$ and $F$, respectively. The lines $OO'$ and $EO'$ intersect $\Gamma'$ again at $A'$ and $G$, respectively. The lines $BO$ and $A'G$ intersect at $H$. Prove that $DF^2 = AF \cdot GH$.

*Solved by Titu Zvonaru, Comănești, Romania.*

Since $O'E = O'G$ and $O'A = O'A'$, the quadrilateral $AEA'G$ is a parallelogram, hence

$$A'G \parallel AE \quad (1)$$

The triangles $O'A'E$ and $OAB$ are isosceles. It follows that

$$\angle O'E A = \angle EAO' = \angle BAO = \angle ABO,$$

hence

$$EO' \parallel BO. \quad (2)$$

By (1) and (2) we deduce that $BHGE$ is a parallelogram; thus $HG = BE$ and we have to prove that

$$DF^2 = AF \cdot BE \quad (3)$$

Let $a = BC$, $b = CA$, $c = AB$. If $b = c$, then $D$ is the midpoint of $BC$, $AD$ is a diameter of $\Gamma'$, $BE = CF$ and $DF \perp AC$. It is easy to see that, in $\triangle ADC$ with $DF \perp AC$, the equation (3) is true.

We may assume that $b > c$, and we denote by $T$ the intersection of the line $BC$ with the tangent to $\Gamma$ at $A$.

Using the power of point $T$ with respect to $\Gamma$, we obtain

$$TA^2 = TB \cdot TC \iff TA^2 = TB(TB + a),$$
and applying the Law of Cosines in \( \triangle ABT \) (with \( \angle ABT = 180^\circ - B \)), we have

\[
TA^2 = TB^2 + AB^2 - 2TB \cdot AB \cdot \cos \angle ABT
\]

\[
\Leftrightarrow TB^2 + aTB = TB^2 + c^2 + 2c \cdot TB \cdot \cos B,
\]

hence

\[
TB = \frac{c^2}{a - 2c \cos B}.
\]

Since \( O'A = O'D \), then \( TD = TA \) and, using again the Law of Cosines, we deduce that

\[
TB + a = \frac{c^2}{a - 2c \cos B} + a = \frac{c^2 + a^2 - 2ac \cos B}{a - 2c \cos B} = \frac{b^2}{a - 2c \cos B},
\]

\[
BD = TD - TB = \frac{bc}{1 - 2c \cos B} - \frac{c^2}{a - 2c \cos B} = \frac{c(b - c)}{a - 2c \cos B}.
\]

Denoting \( \alpha = \frac{b - c}{a - 2c \cos B} \), we have \( \alpha = \frac{a(b - c)}{a^2 - 2ac \cos B} = \frac{a(b - c)}{b^2 - c^2} = \frac{a}{b + c} \);

it results that

\[
BD = c\alpha, \; DC = b\alpha.
\]

Using the power of points \( B \) and \( C \) with respect to circle \( \Gamma' \), we get:

\[
BE \cdot BA = BD^2, \; CF \cdot CA = CD^2,
\]

hence \( BE = c\alpha^2, \; CF = b\alpha^2, \; AF = b(1 - \alpha^2) \).

The equality (3) is equivalent to:

\[
DF^2 = AF \cdot BE \Leftrightarrow DC^2 + CF^2 - 2DC \cdot CF \cdot \cos C = AF \cdot BE
\]

\[
\Leftrightarrow b^2 \alpha^2 + b^2 \alpha^4 - 2b^2 \alpha^3 \cos C = bca^2(1 - \alpha^2)
\]

\[
\Leftrightarrow b + b\alpha^2 - 2b\alpha \cos C = c(1 - \alpha^2)
\]

\[
\Leftrightarrow b - c + (b + c) \cdot \frac{a^2}{(b + c)^2} - 2b \cdot \frac{a}{b + c} \cos c = 0
\]

\[
\Leftrightarrow b^2 - c^2 + a^2 - 2ab \cos c = 0.
\]

which is true (by the Law of Cosines in \( \triangle ABC \)).

3. Find all triplets \((x, y, z)\) of positive integers satisfying \( 1 + 4^x + 4^y = z^2 \).

Solved by Michel Bataille, Rouen, France; Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of De.

Observe that \( z \) is odd and is at least 3. Let \( z = 2m + 1 \) where \( m \) is a positive integer. Then the equation reduces to

\[
4^{x-1} + 4^{y-1} = m(m + 1)
\]

(1)

Assume that \( x \geq y \) and rewrite (1) as

\[
4^{y-1}(4^{x-y} + 1) = m(m + 1)
\]

(2)

Observe that \( \gcd(m, m + 1) = 1 \). Therefore either
(a) \( m = 4^y - 1, \ m + 1 = 4^x - y + 1 \); or
(b) \( m + 1 = 4^y - 1, \ m = 4^x - y + 1 \).

If (a) holds then \( x = 2y - 1 \) and \( z = 2^{2y-1} + 1 \). If (b) holds then we obtain
\[
2^{2y-3} - 2^{2x-2y-1} = 1
\tag{3}
\]
The solution of (3) is \( (x, y) = (\frac{5}{2}, 2) \) which is inadmissible because \( x \) is not an integer.

Hence the solution set in positive integers of this equation is
\[
\{ (2k - 1, k, 2^{2k-1} + 1) : k \in \mathbb{Z}^+ \} \cup \{ (k, 2k - 1, 2^{2k-1} + 1) : k \in \mathbb{Z}^+ \}
\]
where \( \mathbb{Z}^+ \) is the set of positive integers.

4. Find all pairs \((p, q)\) of primes such that \( p^p + q^q + 1 \) is divisible by \( pq \).

*Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.*

Suppose that \((p, q)\) is such a pair of primes. Then,
\[
\{ \begin{align*}
p^p + q^q + 1 &= kpq, \\
\text{for some positive integer } k
\end{align*}\}
\tag{1}
\]

Since equation (1) is symmetric with respect to \( p \) and \( q \); and since \( p \neq q \) (by inspection, \( p = q \) would imply \( p \mid 1 \)). There is no loss of generality in assuming \( p < q \).

We distinguish between two cases: Case 1, in which \( p \) and \( q \) are both odd primes; and Case 2 wherein, \( p = 2 \) and \( q \) is an odd prime.

**Case 1.** \( p \) and \( q \) are both odd primes.

Thus, by (2) we must have
\[
3 \leq p < p + 2 \leq q, \quad 5 \leq q.
\tag{3}
\]

We will prove that no primes satisfying (1) and (3) exist.

We make use of the concept of the order of a positive integer \( a \) modulo an odd prime \( r \). If \( a \) and \( r \) are relatively prime, then the order of \( a \) modulo \( r \) is the smallest positive integer \( n \) such that \( a^n \equiv 1 \pmod{r} \). When \( a \equiv 1 \pmod{r} \), the order of \( a \) is equal to \( 1 \). Otherwise, it is some positive integer. The order of \( a \) exists, since by Fermat's Little Theorem, we know that \( a^{r-1} \equiv 1 \pmod{r} \). (Thus the set of all natural numbers \( n \) such that \( a^n \equiv 1 \pmod{r} \) is nonempty).

The following lemma is well-known in elementary number theory, we state it without proof.

**Lemma 1.** Let \( r \) be an odd prime, and \( a \) a positive integer not divisible by \( r \), and let \( n \) be the order of \( a \) modulo \( r \). Then, if \( m \) is a positive integer such that \( a^m \equiv 1 \pmod{r} \), \( n \) is a divisor of \( m \).
From (1) it follows that,

\[ q^q \equiv -1 \pmod{p} \]

\[ \Rightarrow q^{2q} \equiv (-1)^2 \equiv 1 \pmod{p}. \]  

(4)

Let \( n \) be the order of \( q \) modulo \( p \). By (4) and Lemma 1, it follows that \( n \) is a divisor of \( 2q \), which means that \( n = 1, 2, q, \) or \( 2q \).

If \( n = 1 \): then \( q \equiv 1 \pmod{p}; q = 1 + pl \), for some positive integer \( l \), and going back to (1) we have

\[ p^p + (1 + p \cdot l)^q + 1 = k \cdot p \cdot q \]  

(5)

It is evident from the binomial expansion of \((1 + p \cdot l)^2\), that (5) implies \( 2 + \lambda p = k p q \), for some positive integer \( \lambda \), which is impossible since this last equation implies \( p \mid 2 \); we know that \( p \ge 3 \).

Next, consider the case in which the order \( n \) (of \( q \) modulo \( p \)) is \( q \) or \( 2q \). We know from Fermat’s Little Theorem that

\[ q^{p-1} \equiv 1 \pmod{p}. \]

By Lemma 1, the order \( n \) (\( = q \) or \( 2q \)) must divide \( p - 1 \). Since \( p - 1 \) is even and is \( q \) odd; we see that in either case \( 2q \) must divide \( p \): therefore

\[ p - 1 = 2q \cdot t, \text{ for some positive integer } t. \]

\[ p = 2qt + 1 > q. \]

which contradicts (3).

There remains only one possibility to consider: the order \( n \) (of \( q \) modulo \( p \)) is equal to \( 2 \).

\[ q^2 \equiv 1 \pmod{p} \Leftrightarrow (q - 1)(q + 1) \equiv 0 \pmod{p} \]

\[ \Leftrightarrow q \equiv \pm 1 \pmod{p} \] (since \( p \) is prime.)  

(6)

The case \( q \equiv 1 \pmod{p} \) has already been examined above (this was done in the case order \( n = 1 \)). So, then suppose that \( q \equiv -1 \pmod{p} \),

\[ q = p \cdot v - 1, \ v \in \mathbb{Z}, v \ge 2 \]  

(7)

We go back to (1) and this time we work modulo \( q \):

\[ p^p \equiv -1 \pmod{q} \Rightarrow p^{2p} \equiv 1 \pmod{q} \]

which implies by Lemma 1 that the order \( f \) of \( p \) modulo \( q \) must be a divisor of \( 2p \). Thus, \( f = 1, 2, p, \) or \( 2p \). Once again, by Fermat’s Little Theorem, we know that the order \( f \) must divide \( q - 1 \) by virtue of \( p^{q-1} \equiv 1 \pmod{q} \). Hence, \( q - 1 = f \cdot u \)

\[ q = f \cdot u + 1, \text{ where } f = 2, p, \text{ or } 2p. \]  

(8)
Note that the possibility \( f = 1 \) is ruled out: if \( f = 1 \) then \( p \equiv 1 \pmod{q} \) which implies (since both \( p \) and \( q \) are positive and \( \geq 3 \)) that \( p > q \); contrary to (3).

If \( f = p \) or \( 2p \), then combining (7) with (8) yields \( p \cdot v - f \cdot u = 2 \), which implies (since \( f = p \) or \( 2p \)) that \( p \) divides \( 2 \); an impossibility since \( p \geq 3 \).

Finally suppose that \( f = 2 \). Then,

\[
p^2 \equiv 1 \pmod{q} \iff (p-1)(p+1) \equiv 0 \pmod{q};
\]

and since \( q \) is a prime, we must have either \( p = 1 + q \cdot w \) or \( p = -1 + q \cdot w \) for some positive integer \( w \) which again contradicts the conditions in (3); for either possibility implies \( p > q \) (note that in either case, \( w \geq 2 \)). It is now clear that there are no odd primes \( p \) and \( q \) which satisfy (1).

**Case 2.** \( p = 2 \) and \( q \) is an odd prime.

From (1) we have, \( 2^2 + q^2 + 1 = 2kq \):

\[
5 = q \cdot (2k - q^{q-1}).
\]

Equation (9) clearly shows that \( q \mid 5 \); and since \( q \) is a prime; we must have \( q = 5 \) and \( 2k - q^{q-1} = 1 \) so \( 2k = 5^4 + 1 \), thus \( k = \frac{626}{2} = 313 \).

**Conclusion:** Taking into account symmetry, there exist exactly two pairs with the problem’s property: \((p, q) = (2, 5), (5, 2)\).

**5.** For the vertex \( A \) of \( \triangle ABC \), let \( A' \) be the point of intersection of the angle bisector at \( A \) with side \( BC \), and let \( \ell_A \) be the distance between the feet of the perpendiculars from \( A' \) to the lines \( AB \) and \( C \), respectively. Define \( \ell_B \) and \( \ell_C \) similarly, and let \( \ell \) be the perimeter of \( \triangle ABC \). Prove that

\[
\frac{\ell_A \ell_B \ell_C}{\ell^3} \leq \frac{1}{64}.
\]

*Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comănești, Romania. We give Bataille’s solution.*

We adopt the standard notations for the elements of \( \triangle ABC \) and denote the orthogonal projections of \( A' \) onto \( AB, AC \) by \( H, K \), respectively. Since the line segment \( HK \) is a chord subtending \( \angle BAC \) of the circle with diameter \( AA' \), we have \( \ell_A = HK = AA' \sin A \). As it is well-known, the length of the bisector is given by \( AA' = \frac{2bc \cos(A/2)}{b + c} \) so we obtain

\[
\ell_A = \frac{2bc \sin(A/2)}{b + c} \cdot 2 \cos^2(A/2) = \frac{2bc \sin(A/2)}{b + c} \cdot \left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right).
\]

It quickly follows that

\[
\frac{\ell_A}{\ell} = \frac{2(s - a) \sin(A/2)}{b + c}.
\]
With similar results for $\ell_B$ and $\ell_C$, we finally have
\[ \frac{\ell_A\ell_B\ell_C}{\ell^3} = \frac{\sin(A/2)\sin(B/2)\sin(C/2) \cdot 8(s-a)(s-b)(s-c)}{(b+c)(c+a)(a+b)}. \]

From the following known formulas:
\[ rs = \sqrt{s(s-a)(s-b)(s-c)}, \quad \sin(A/2)\sin(B/2)\sin(C/2) = \frac{r}{4R}, \]
\[ abc = 4rRs, \quad ab + bc + ca = s^2 + r^2 + 4rR \]
we first deduce
\[ (b+c)(c+a)(a+b) = (a+b+c)(ab+bc+ca) - abc = 2s(s^2 + r^2 + 4rR) - 4rRs = 2s(s^2 + r^2 + 2rR) \]
and then
\[ \frac{\ell_A\ell_B\ell_C}{\ell^3} = \frac{r^3}{Rs^2 + Rr^2 + 2rR^2} \tag{1} \]

By AM-GM, we have
\[ \frac{s}{3} = \frac{(s-a) + (s-b) + (s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)} = \sqrt[3]{r^2s} \]
so that $s^2 \geq 27r^2$. Recalling Euler’s inequality $R \geq 2r$, we obtain
\[ Rs^2 + Rr^2 + 2rR^2 \geq 54r^3 + 2r^3 + 8r^3 = 64r^3 \]
and from (1), \[ \frac{\ell_A\ell_B\ell_C}{\ell^3} \leq \frac{1}{64}. \]

Next we turn to the 2006/2007 British Mathematical Olympiad, Round 1, given at [2010: 153].

1. Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

Solved by Arkady Alt, San Jose, CA, USA; Geoffrey A. Kandall, Hamden, CT, USA; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Ricardo’s write-up.

Using the familiar ‘difference of squares’ identity repeatedly, we can write
\[ 3^{32} - 2^{32} = 5 \prod_{k=1}^{4} (3^{2^k} + 2^{2^k}). \]
For \( k \leq 2 \), the factors \( 3^{2^k} + 2^{2^k} \) are less than 100 and it is easy to pick out 5, 13 (when \( k = 1 \)), and 97 (when \( k = 2 \)) as prime factors. Now we observe that 
\[
3^{32} = 9^{16} \equiv 1 \pmod{17}
\]
and 
\[
2^{32} = 4^{16} \equiv 1 \pmod{17}
\]
by Fermat’s Little Theorem. Thus \( 3^{32} - 2^{32} \equiv 0 \pmod{17} \) and 17 is the fourth prime factor we seek.

2. In the convex quadrilateral \( ABCD \), points \( M, N \) lie on the side \( AB \) such that \( AM = MN = NB \), and points \( P, Q \) lie on the side \( CD \) such that \( CP = PQ = QD \). Prove that

\[
\text{Area of } AMCP = \text{Area of } MNPQ = \frac{1}{3} \text{Area of } ABCD.
\]

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Geoffrey A. Kandall, Hamden, CT, USA; by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and by Titu Zvonaru, Comănești, Romania. We give the solution of Amengual Covas.

Since \( CP = PQ \), we have (figure 1)

\[
\text{Area of } \triangle CPM = \text{Area of } \triangle PQM
\]

Since \( AM = MN \), we have

\[
\text{Area of } \triangle AMP = \text{Area of } \triangle MNP
\]

Hence,

\[
\text{Area of } \triangle CPM + \text{Area of } \triangle AMP = \text{Area of } \triangle PQM + \text{Area of } \triangle MNP
\]

that is,

\[
\text{Area of } AMCP = \text{Area of } MNPQ
\]

Now, since the areas of triangles with equal altitudes are proportional to the bases of the triangles, we have (figure 2)

\[
\text{Area of } \triangle AMC = \frac{1}{3} (\text{Area of } \triangle ABC)
\]
and 
\[ \text{Area of } \triangle CPA = \frac{1}{3} (\text{Area of } \triangle CDA) \]
Hence,
\[ \text{Area of } \triangle AMC + \text{Area of } \triangle CPA = \frac{1}{3} (\text{Area of } \triangle ABC + \text{Area of } \triangle CDA) \]
that is,
\[ \text{Area of } AMCP = \frac{1}{3} (\text{Area of } ABCD) \]
and we are done.

3. The number 916238457 is an example of a nine-digit number which contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. How many such numbers are there?

*Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.*

Let \( S \) be the set of all nine-digit numbers with distinct nonzero digits and such that the digits 1 to 5 occur in their natural order, and \( m = n(S) = \text{cardinality of the set } S \).

Let \( S_1 \) be the set of all nine-digit numbers with distinct nonzero digits and such that the digits 1 to 6 occur in their natural order, and \( m_1 = n(S_1) = \text{cardinality of the set } S_1 \).

Let \( S_2 \) be the set of all nine-digit numbers with distinct nonzero digits and such that the digits 1 to 5 occur in their natural order; but the (numbers) digits 1 to 6 do not occur in their natural order; and \( m_2 = n(S_2) = \text{cardinality of the set } S_2 \).

Then \( S = S_1 \cup S_2 \) and \( S_1 \cap S_2 = \emptyset \). Therefore,
\[
\begin{align*}
    n(S) &= n(S_1) + n(S_2); \\
    m &= m_1 + m_2; \\
    m_2 &= m - m_1.
\end{align*}
\] (1)
To calculate \( m \), observe that any five of the positions 1 through 9 may be chosen; for any such choice, the numbers 1 to 5 are placed in their natural order on those positions. Moreover, for each such choice of five positions; there are \( 4! \) ways to place the remaining numbers 6 to 9, on the remaining four positions. Hence,

\[
m = (4!) \cdot \left( \begin{array}{c} 9 \\ 5 \end{array} \right) = 4! \frac{9!}{4!5!} = \frac{9!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9.
\]

Similarly,

\[
m_1 = (3!) \left( \begin{array}{c} 9 \\ 6 \end{array} \right) = 3! \frac{9!}{3!6!} = \frac{9!}{6!} = 7 \cdot 8 \cdot 9.
\]

Hence by (1)

\[
m_2 = m - m_1 = 6 \cdot 7 \cdot 8 \cdot 9 - 7 \cdot 8 \cdot 9
\]

\[
= (7 \cdot 8 \cdot 9)(6 - 1)
\]

\[
= 7 \cdot 8 \cdot 9 \cdot 5 = 2520
\]

**Conclusion:** There are exactly 2520 such numbers.

4. Two touching circles \( S \) and \( T \) share a common tangent which meets \( S \) at \( A \) and \( T \) at \( B \). Let \( AP \) be a diameter of \( S \) and let the tangent from \( P \) to \( T \) touch it at \( Q \). Show that \( AP = PQ \).

*Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the two solutions of Amengual Covas.*

**Solution 1.**

Let \( O \) and \( O' \) be the centers of \( S \) and \( T \), respectively.

Then the line \( OO' \), joining the centers of the touching circles, goes through the point of contact \( D \).

Now, \( PA \) is perpendicular to \( AB \), as is the radius \( O'B \) to the point of contact with \( AB \). Thus \( PA \) and \( O'B \) are parallel and the alternate angles \( POD \) and \( BO'D \) are equal. But triangles \( POD \) and \( BO'D \) are isosceles, and since their vertical angles are equal, so are their base angles. Therefore \( \angle ODP = \angle O'DB \), and \( D \) lies on \( PB \).

Next, diameter \( PA \) subtends a right angle at \( D \), making \( AD \) the altitude to the hypotenuse in right-triangle \( ADP \). By a standard mean proportion, then, we have

\[
PA^2 = PD \cdot PB
\]
On the other hand, the power of $P$ with respect to $T$ is $PQ^2$ and also $PD \cdot PB$; hence,

$$PQ^2 = PD \cdot PB$$

By (1) and (2), $PQ^2 = AP^2$. It follows that $PQ = AP$, as desired.

**Solution 2.**

Let $R$ and $r$ be the radii of circles $S$ and $T$, respectively. Let $O'$ be the center of $T$ and denote by $C$ the foot of the perpendicular from $O'$ to $AP$.

By the Pythagorean theorem, applied to right triangles $PQO'$ and $PCO'$,

$$PQ^2 + QO'^2 = PC^2 + CO'^2$$

that is,

$$PQ^2 + QO'^2 = PC^2 + AB^2$$

and since $AB = 2\sqrt{Rr}$ (for a proof, see e.g. *Japanese Temple Geometry Problems*, by H. Fukagawa and D. Pedoe, Canada, 1989, Example 1.1 on p. 3), $PC = 2R - r$, $QO' = r$, we have

$$PQ^2 + r^2 = (2R - r)^2 + 4Rr.$$ 

Hence

$$PQ^2 = 4R^2 = (2R)^2 = AP^2$$

It follows that $PQ = AP$, as desired.

5. For positive real numbers $a$, $b$, $c$ prove that

$$(a^2 + b^2)^2 \geq (a + b + c)(a + b - c)(b + c - a)(c + a - b).$$

*Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Giulio Loddi, High School student, Cagliari, Italy; Henry Ricardo, Tappan, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Loddi’s solution.*

We will treat $a$ and $b$ like constants and the right-hand side as a function of $c$:

$$f(c) = (a + b + c)(a + b - c)(b + c - a)(c + a - b)$$

$$= [(a + b)^2 - c^2] \cdot [c^2 - (a - b)^2]$$

$$= -c^4 + c^2[(a - b)^2 + (a + b)^2] - (a - b)^2(a + b)^2$$

$$= -c^4 + 2c^2(a^2 + b^2) - (a^2 - b^2)^2$$
Let us find the maximum of \( f(c) \). Differentiating with respect to \( c \) yields:

\[
f'(c) = -4c^3 + 4c(a^2 + b^2) = 4c[-c^2 + a^2 + b^2].
\]

The derivative of \( f(c) \) is zero when \( c = 0 \) or when \( c^2 = a^2 + b^2 \).\( f(0) = -(a^2 - b^2)^2 \) and \( f(\sqrt{a^2 + b^2}) = (a^2 + b^2)^2 - (a^2 - b^2)^2 = 4a^2b^2 \), so \( f(0) < f(\sqrt{a^2 + b^2}) \). We can guess that there is a maximum when \( c^2 = a^2 + b^2 \).

But

\[
f(\sqrt{a^2 + b^2}) = 4a^2b^2 \geq -c^4 + 2c^2(a^2 + b^2) - (a^2 - b^2)^2 = f(c)
\]

when \( c^4 - 2c^2(a^2 + b^2) + (a^2 - b^2)^2 + 4a^2b^2 \geq 0 \). By computing the discriminant of this quadratic in \( c^2 \):

\[
\Delta = 4(a^2 + b^2)^2 - 4(a^2 - b^2)^2 - 16a^2b^2 = 16a^2b^2 - 16a^2b^2 = 0,
\]

so \( f(\sqrt{a^2 + b^2}) - f(c) \geq 0 \) for all \( c \). Finally, \( (a^2 + b^2)^2 \geq 4a^2b^2 \) follows from \( (a^2 - b^2)^2 \geq 0 \) and thus

\[
LHS \geq 4a^2b^2 = f(\sqrt{a^2 + b^2}) \geq f(c) = RHS \quad \forall c > 0.
\]

Ed. – Note that \( f(c) = -(c^2 - (a^2 + b^2))^2 + 4a^2b^2 \) which yields the same result.

6. Let \( n \) be an integer. Show that, if \( 2 + 2\sqrt{1 + 12n^2} \) is an integer, then it is a perfect square.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Henry Ricardo, Tappan, NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Bataille’s write-up.

Suppose that \( 2 + 2\sqrt{1 + 12n^2} \) is a positive integer \( m \). Then \( (m - 2)^2 = 4(1 + 12n^2) \) so that \( 1 + 12n^2 \) must be a perfect square, say \( 1 + 12n^2 = a^2 \) where \( a \) is a positive integer. It follows that

\[
a^2 - 3(2n)^2 = 1
\]

(1)

and the pair \( (a, 2n) \) is a solution to the Fermat equation \( x^2 - 3y^2 = 1 \) with \( x \geq 1 \) and \( y \) even. It is well-known that the solutions to this equation in nonnegative integers are the pairs \( (x_k, y_k) \) such that \( x_k + y_k \sqrt{3} = (2 + \sqrt{3})^k, k = 0, 1, 2, \ldots \). Since \( x_{k+1} + y_{k+1} \sqrt{3} = (x_k + y_k \sqrt{3})(2 + \sqrt{3}) \) the sequences \( (x_k), (y_k) \) are given by the recursion \( x_{k+1} = 2x_k + 3y_k, y_{k+1} = x_k + 2y_k \) and \( x_0 = 1, y_0 = 0. \) Using induction, it is easy to see that \( y_k \) is even if and only if \( k \) is even. Note also that \( 2x_k = (2 + \sqrt{3})^k + (2 - \sqrt{3})^k. \)

Returning to (1) and assuming that \( n \geq 0 \) without lost of generality, we must have \( a = x_k \) and \( 2n = y_k \) for some even \( k. \) Setting \( k = 2\ell, \) we first deduce \( 2a = (2 + \sqrt{3})^{2\ell} + (2 - \sqrt{3})^{2\ell} \) and then

\[
m = 2 + 2a = (2 + \sqrt{3})^{2\ell} + (2 - \sqrt{3})^{2\ell} + 2(2 + \sqrt{3})^\ell(2 - \sqrt{3})^\ell \]
\[
= ((2 + \sqrt{3})^\ell + (2 - \sqrt{3})^\ell)^2 = x_\ell^2,
\]

a perfect square.
Next up are solutions to problems of the 2006/2007 British Mathematical Olympiad, Round 2, given at [2010: 154].

1. Triangle \(ABC\) has integer-length sides, and \(AC = 2007\). The internal bisector of \(\angle BAC\) meets \(BC\) at \(D\). Given that \(AB = CD\), determine \(AB\) and \(BC\).

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandel, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We use Kandall’s version.

Let \(BC = a, AB = c\), so that \(BD = a - c\). Note that \(2007 = 3^2 \cdot 223\) (prime factorization).

Since \(AD\) is an angle bisector, we have

\[
\frac{a - c}{c} = \frac{c}{2007}
\]  

(1)

Thus, \(c^2 = 3^2 \cdot 223(a - c)\). Both 3 and 223 divide \(c\), so \(c = 3 \cdot 223k\) (\(k\) a positive integer). From (1),

\[
a = \frac{c^2}{2007} + c = 223(k^2 + 3k).
\]

Since \(a < c + 2007\) (triangle inequality), we have

\[
223(k^2 + 3k) < 3 \cdot 223k + 9 \cdot 223,
\]

which reduces easily to \(k^2 < 9\). Thus, \(k = 1\) or \(k = 2\).

If \(k = 1\), then \(c = 3 \cdot 223, a = 4 \cdot 223\), so \(c + a = 7 \cdot 223 < 2007\), which violates the triangle inequality.

Therefore, \(k = 2\), which means that \(c = 6 \cdot 223 = 1338\) and \(a = 10 \cdot 223 = 2230\).
2. Show that there are infinitely many pairs of positive integers \((m, n)\) such that
\[
\frac{m + 1}{n} + \frac{n + 1}{m}
\]
is a positive integer.

_Solved by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India._

Let \(f(m, n) = \frac{m + 1}{n} + \frac{n + 1}{m}\). Observe that \(f(1, 1) = 4\).

We claim that there are infinitely many pairs of positive integers \((m, n)\) such that \(f(m, n) = 4\).

\[
f(m, n) = 4
\]
implies
\[
m = \frac{(4n - 1) \pm \sqrt{12n^2 - 12n + 1}}{2}. \tag{1}
\]

Observe that \(t = 12n^2 - 12n + 1\) is odd and \(m\) is an integer if and only if \(t\) is a perfect square. So let \(t = p^2\), for some positive integer \(p\), then
\[
p^2 = 3q^2 - 2 \tag{2}
\]
where \(q = 2n - 1\). If \((p, q)\) satisfies \((2)\) then both \(p\) and \(q\) must be odd.

Equation \((2)\) is satisfied by \((p_1, q_1) = (1, 1)\) and if the positive integral pair \((p_k, q_k)\) satisfies \((2)\) then so does \((p_{k+1}, q_{k+1})\) where
\[
p_{k+1} = 2p_k + 3q_k \quad \text{and} \quad q_{k+1} = p_k + 2q_k.
\]

Observe that \(\{p_k\}\) and \(\{q_k\}\) are increasing sequences and \(p_k > q_k\) for \(k > 1\).

Now define
\[
n_k = \frac{q_k + 1}{2},
\]
\[
m_k = \frac{(2n_k - 1) + \sqrt{12n_k^2 - 12n_k + 1}}{2} = \frac{2q_k + 1 + p_k}{2} = \frac{q_{k+1} + 1}{2}
\]
for \(k \geq 1\). Observe that both \(m_k\) and \(n_k\) are positive integers as \(q_k\) and \(q_{k+1}\) are odd positive integers.

The set \(S = \{(m_k, n_k) : k \geq 1\}\) is an infinite set (because \(\{q_k\}\) is an increasing sequence) and consists of pairs of positive integers satisfying
\[
f(m, n) = 4.
\]

Thus we have produced infinitely many pairs of positive integers \((m, n)\) for which
\[
\frac{m + 1}{n} + \frac{n + 1}{m}
\]
is a positive integer.
3. Let \( \triangle ABC \) be an acute-angled triangle with \( AB > AC \) and \( \angle BAC = 60^\circ \). Denote the circumcentre by \( O \) and the orthocentre by \( H \) and let \( OH \) meet \( AB \) at \( P \) and \( AC \) at \( Q \). Prove that \( PO = HQ \).

Note: The circumcentre of triangle \( \triangle ABC \) is the centre of the circle which passes through the vertices \( A, B \) and \( C \). The orthocentre is the point of intersection of the perpendiculars from each vertex to the opposite side.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give Bataille’s solution.

Note that from \( AB > AC \), we have \( C = \angle ACB > B = \angle ABC \), hence \( 2C > B + C = 180^\circ - A = 120^\circ \) and so \( C > 60^\circ \). Also, since \( \triangle ABC \) is acute-angled, \( AH = 2OA' \) where \( A' \) is the midpoint of \( BC \), that is \( AH = 2R \cos A = R \) (denoting the circumcentre by \( H \)). It follows that \( \triangle OAH \) is isosceles with \( AO = AH \). Moreover, we have \( \angle HAC = 90^\circ - C \) and \( \angle PAO = \frac{1}{2}(180^\circ - \angle BOA) = \frac{1}{2}(180^\circ - 2C) = 90^\circ - C \) as well. It follows that the angle bisectors of \( \angle BAC \) and \( \angle OAH \) are the same line \( \ell \). Now, if \( \rho_\ell \) denotes the reflection in \( \ell \), the image \( \rho_\ell(OH) \) of the line \( OH \) is \( OH \) itself (since \( OH \perp \ell \)) and the image \( \rho_\ell(AB) \) is \( AC \). As a result, the image of \( P \), the intersection of \( OH \) and \( AB \) is \( Q \), the intersection of \( OH \) and \( AC \). Finally, \( \rho_\ell(P) = Q, \rho_\ell(O) = H \) and so \( PO = QH \).
Next we move to the May 2010 number of the Corner and solutions from our readers to problems of the XV Olympíada Matemática Rioplatense, Nivel 2, given at [2010; 214].

1. Let $\triangle ABC$ be a right triangle with right angle at $A$. Consider all the isosceles triangles $\triangle XYZ$ with right angle at $X$, where $X$ lies on the segment $BC$, $Y$ lies on $AB$, and $Z$ is on the segment $AC$. Determine the locus of the medians of the hypotenuses $YZ$ of such triangles $XYZ$.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Geupel.

Let $\triangle AEDF$ be the square where $D$, $E$, and $F$ are on the segments $BC$, $AB$, and $AC$, respectively. We prove that $D = X$ and that the locus of the midpoints $M$ of segments $YZ$ is the line segment $EF$. The end point $E$ and accordingly $F$ is included if and only if $AB \geq AC$ and $AB \leq AC$, respectively.

Consider $\triangle XYZ$ as described in the problem. By $\angle YAZ = \angle YXZ = 90^\circ$, the quadrilateral $\triangle AYZ$ is cyclic. From the condition $XY = XZ$, we see that $\angle XAY = \angle XAZ = 45^\circ$; hence $X = D$. By $\angle DMY = \angle DEY = 90^\circ$, the quadrilateral $\triangle DMEY$ is cyclic. Thus, $\angle DEM = \angle DYM = 45^\circ$. Consequently, $M$ is on $EF$.

Vice versa, let $M$ lie on $EF$. The cases $M = E$ and $M = F$ are possible if and only if $AB \geq AC$ and $AB \leq AC$, respectively. Let us suppose that $M \neq E, F$. The perpendicular to $DM$ through $M$ cuts $AB$ and $AC$ at $Y$ and $Z$, respectively. By $\angle DMY = \angle DEY = 90^\circ$, the quadrilateral $\triangle DMEY$ is cyclic; hence $\angle DYM = \angle DEM = 45^\circ$. Similarly $\angle DZM = 45^\circ$. Consequently, $\triangle XYZ$ is an isosceles right triangle, which completes the proof.
A finite number of (possibly overlapping) intervals on a line are given. If the rightmost \( \frac{1}{3} \) of each interval is deleted, an interval of length 31 remains. If the leftmost \( \frac{1}{3} \) of each interval is deleted, an interval of length 23 remains. Let \( M \) and \( m \) be the maximum and minimum of the lengths of an interval in the collection, respectively. How small can \( M - m \) be?

Solved by Oliver Geupel, Brühl, NRW, Germany.

The solution is 24.

Consider the intervals \([0, 33], [19, 28], [25, 34]\). If the rightmost \( \frac{1}{3} \) of each interval is deleted, then the union of the resulting intervals is the interval \([0, 31]\) with length 31. If the leftmost \( \frac{1}{3} \) of each interval is deleted, then the union of the resulting intervals is \([11, 34]\) with length 23. We have \( M = 33, m = 9 \); therefore \( M - m = 24 \).

We prove that generally \( M - m \geq 24 \).

Let \([a, b]\) be the minimal closed interval that contains all the given intervals.

If the rightmost \( \frac{1}{3} \) of each interval is deleted, then an interval \([a, r]\) of length 31 remains. Thus,

\[ r - a = 31. \]

At least one of the intervals with right end point in the interval \([r, b]\) will be reduced by a segment not greater than \( b - r \). The length of such an interval is not greater than \( 3(b - r) \), which implies that

\[ m \leq 3(b - r). \]

If the leftmost \( \frac{1}{3} \) of each interval is deleted, then an interval \([\ell, b]\) of length 23 remains. Thus,

\[ b - \ell = 23. \]

The initial collection of intervals contains an interval with left bound \( a \). Its left bound after the deletion of the left \( \frac{1}{3} \) is not less than \( \ell \). Hence, its length is not less than \( 3(\ell - a) \), which implies that

\[ M \geq 3(\ell - a). \]

We conclude

\[ M - m \geq 3(\ell - a) - 3(b - r) = 3[(r - a) - (b - \ell)] = 3(31 - 23) = 24, \]

which completes the proof.