SKOLIAD  No. 130

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by September 15, 2011. A copy of CRUX with Mayhem will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Niels Henrik Abel Mathematics Contest, 2009–2010, Second Round. Our thanks go to Øyvind Bakke, Norwegian University of Science and Technology, Trondheim, Norway, for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, University College of Saint Boniface, Winnipeg, MB, for translating this contest from English into French.

Niels Henrik Abel Mathematics Contest, 2009–2010
2nd Round
100 minutes allowed

1. A four-digit whole number is interesting if the number formed by the leftmost two digits is twice as large as the number formed by the rightmost two digits. (For example, 2010 is interesting.) Find the largest whole number, \(d\), such that all interesting numbers are divisible by \(d\).

2. A calculator performs this operation: It multiplies by 2.1, then erases all digits to the right of the decimal point. For example, if you perform this operation on the number 5, the result is 10; if you begin with 11, the result is 23. Now, if you begin with the whole number \(k\) and perform the operation three times, the result is 201. Find \(k\).

3. The pentagon \(ABCDE\) consists of a square, \(ACDE\), with side length 8, and an isosceles triangle, \(ABC\), such that \(AB = BC\). The area of the pentagon is 90. Find the area of \(\triangle BEC\).

4. In how many ways can one choose three different integers between 0.5 and 13.5 such that the sum of the three numbers is divisible by 3?

5. If \(a\) and \(b\) are positive integers such that \(a^3 - b^3 = 485\), find \(a^3 + b^3\).

6. If \(a\) and \(b\) are positive integers such that \(a^3 + b^3 = 2ab(a + b)\), find \(a^{-2}b^2 + a^2b^{-2}\).
7. Let $D$ be the midpoint of side $AC$ in $\triangle ABC$. If $\angle CAB = \angle CBD$ and the length of $AB$ is 12, then find the square of the length of $BD$.

8. If $x$, $y$, and $z$ are whole numbers and $xyz + xy + 2yz + xz + x + 2y + 2z = 28$ find $x + y + z$.

9. Henrik’s math class needs to choose a committee consisting of two girls and two boys. If the committee can be chosen in 3630 ways, how many students are there in Henrik’s math class?

10. Let $S$ be $1!(1^2 + 1 + 1) + 2!(2^2 + 2 + 1) + 3!(3^2 + 3 + 1) + \cdots + 100!(100^2 + 100 + 1)$. Find $\frac{S + 1}{101!}$. (As usual, $k! = 1 \cdot 2 \cdot 3 \cdots \cdot (k - 1) \cdot k$.)

Concours Mathématique Niels Henrik Abel, 2009–2010
2i`eme ronde
100 minutes sont accordées

1. Un entier à quatre chiffres est intéressant si l’entier formé par les deux chiffres à l’extrême gauche est deux fois plus grand que l’entier formé par les deux chiffres à l’extrême droite. (Par exemple, 2010 est intéressant.) Déterminer le plus grand entier, $d$, tel que tous les nombres intéressants sont divisibles par $d$.

2. Une calculatrice effectue cette opération : elle multiplie par 2,1, puis elle efface tous les chiffres à droite de la décimale. Par exemple, si on effectue cette opération à partir de 5, le résultat est 10 ; à partir de 11, le résultat est 23. Or, si on commence avec un entier $k$ et qu’on effectue cette opération trois fois, le résultat est 201. Déterminer $k$.

3. Le pentagone $ABCDE$ consiste d’un carré, $ACDE$, de côtés de longueur 8, puis d’un triangle isocèle, $ABC$, tel que $AB = BC$. La surface du pentagone est 90. Déterminer la surface de $\triangle BEC$.

4. De combien de façons pouvons-nous choisir trois entiers différents entre 0, 5 et 13, 5, tels que la somme des trois entiers soit divisible par 3 ?

5. Si $a$ et $b$ sont des entiers positifs tels que $a^3 - b^3 = 485$, déterminer $a^3 + b^3$.

6. Si $a$ et $b$ sont des entiers positifs tels que $a^3 + b^3 = 2ab(a + b)$, déterminer $a^2b^2 + a^2b^{-2}$.

7. Soit $D$ le mipoint du côté $AC$ dans $\triangle ABC$. Si $\angle CAB = \angle CBD$ et si la longueur de $AB$ est 12, déterminer le carré de la longueur de $BD$.

8. Si $x$, $y$ et $z$ sont des entiers et si $xyz + xy + 2yz + xz + x + 2y + 2z = 28$, déterminer $x + y + z$.
9. La classe de mathématiques d’Henri a besoin de choisir un comité formé de deux filles et de deux garçons. Si ce comité peut être formé de 3630 façons, combien d’étudiants y a-t-il dans la classe de mathématiques d’Henri ?

10. Soit $S$ égal à $1!(1^2 + 1 + 1) + 2!(2^2 + 2 + 1) + 3!(3^2 + 3 + 1) + \cdots + 100!(100^2 + 100 + 1)$. Déterminer $\frac{S+1}{101!}$. (Comme d’habitude, $k! = 1 \cdot 2 \cdot 3 \cdots \cdots (k-1) \cdot k$.)
so $4b = 5a$, so $4b = 5a$. Clearly $b$ is divisible by 5, and since $b = 0 = a$ does not yield a two-digit number, $b$ must be 5. Hence $a = 4$ and the number is 45.

3. Three squares are placed side-by-side inside a right-angled triangle as shown in the diagram. The side length of the smallest of the three squares is 16. The side length of the largest of the three squares is 36. What is the side length of the middle square?

Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

Impose a coordinate system as in the figure. If the middle square has side length $n$, then the coordinates are as indicated. Since the slanted line passes through $(0, 16)$, the equation of the line is $y = mx + 16$ for some slope, $m$.

Using the two other points yields

$$n = 16m + 16 \quad \text{and} \quad 36 = (16 + n) + 16.$$ Therefore $36 = m(16 + 16m + 16) + 16 = 16m^2 + 32m + 16$, so

$$0 = 16m^2 + 32m - 20 = 4(2m + 5)(2m - 1), \quad \text{so} \quad x = -\frac{5}{2} \quad \text{or} \quad x = \frac{1}{2}.$$ In the figure, the slope is clearly positive, so $m = \frac{1}{2}$, and $n = 16m + 16 = 24$.

Also solved by LISA WANG, student, Port Moody Secondary School, Port Moody, BC. You can also solve this problem using similar triangles.

4. Friends Maya and Naya ordered finger food in a restaurant, Maya ordering chicken wings and Naya ordering bite-size ribs. Each wing cost the same amount, and each rib cost the same amount, but one wing was more expensive than one rib. Maya received 20% more pieces than Naya did, and Maya paid 50% more in total than Naya did. The price of one wing was what percentage higher than the price of one rib?

Solution by Lisa Wang, student, Port Moody Secondary School, Port Moody, BC.

Say Naya gets $n$ pieces at $\$r$ each. Then Maya gets 1.2$n$ pieces at, say, $\$w$ each. Then Naya pays $nr$ and Maya pays $1.2nw$. Since Maya pays 50% more than Naya, $1.2nw = 1.5nr$, so $w = \frac{1.5}{1.2}r = 1.25r$, so one wing is 25% more expensive than one rib.
5. A $9 \times 12$ rectangular piece of paper is folded once so that a pair of diagonally opposite corners coincide. What is the length of the crease?

Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

If you fold the paper as instructed and unfold it again, you obtain the figure below where the section outlined with thick lines used to overlap and the dashed line is the crease. The Pythagorean Theorem now yields that

\[
9^2 + x^2 = (12 - x)^2 \\
81 + x^2 = 144 - 24x + x^2 \\
24x = 63 \\
x = \frac{63}{24}
\]

Then redraw the diagram as in the figure below. Use the Pythagorean Theorem again:

\[
c^2 = 9^2 + (12 - 2x)^2.
\]

Since \(x = \frac{63}{24}\), \(c^2 = \frac{2025}{16}\); so \(c = \frac{45}{4}\).

6. In calm weather, an aircraft can fly from one city to another 200 kilometres north of the first and back in exactly two hours. In a steady north wind, the round trip takes five minutes longer. Find the speed (in kilometres per hour) of the wind.

Solution by the editors.

The airspeed of the plane is \(\frac{400}{2} = 200\) kilometres per hour. Let \(w\) denote the speed of the wind. Then, if you fly with the wind, the ground speed is \(200 + w\); and if you fly against the wind, the ground speed is \(200 - w\). Therefore, the plane takes \(\frac{200}{200 + w}\) hours to fly with the wind and \(\frac{200}{200 - w}\) hours to fly against the wind. If you add these two expressions, you get the total time for the round trip,
but this time is given to be $2 + \frac{5}{60}$ hours, so

\[
\frac{200}{200 - w} + \frac{200}{200 + w} = \frac{25}{12}
\]

\[
\frac{8(200 + w) + 8(200 - w)}{(200 - w)(200 + w)} = \frac{1}{12}
\]

\[
\frac{3200}{200^2 - w^2} = \frac{1}{12}
\]

\[
w = \pm 40.
\]

Therefore the speed of the wind is 40 kilometres per hour.

7. A rectangular floor, 24 feet $\times$ 40 feet, is covered by squares of sides 1 foot. A chalk line is drawn from one corner to the diagonally opposite corner. How many tiles have a chalk line segment on them?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Since $\frac{24}{8} = 3$ and $\frac{40}{8} = 5$, consider instead the $3 \times 5$ rectangle on the left in the figure. You can easily count that the diagonal crosses seven squares.

Now tile the $24 \times 40$ rectangle with $3 \times 5$ rectangles as in the righthand side of the figure. The diagonal of the $24 \times 40$ rectangle is also the diagonal of each of eight of the $3 \times 5$ rectangles. Therefore the diagonal crosses 56 of the $1 \times 1$ squares.

This issue’s prize of one copy of *Crux Mathematicorum* for the best solutions goes to Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

As *Skoliad* editors we are quite pleased to see envelopes with “exotic” stamps in the mail, but receiving more Canadian solutions would be wonderful. The address is on the inside of the back cover. You do not have to solve the entire featured contest; a well-presented solution to a single problem is enough. The test for “well-presented” is that your classmates at school can understand it. You can send your solution(s) by mail or electronically—even if that means that we miss out on the exotic stamps.