

**M468. Proposed by Gheorghe Ghită, M. Eminescu National College, Buzău, Romania.**

Determine all pairs \((p, q)\) of prime numbers for which each of \(p + q, p + q^2, p + q^3,\) and \(p + q^4\) is a prime number.

**M469. Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.**

Prove that, for all real numbers \(x\), we have \((2\sin x + 2\cos x)^2 \geq 2^2 - \sqrt{2}\).

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### Mayhem Solutions

We acknowledge a correct solution to problem M413 by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain, and a correct solution to problem M419 by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Our apologies for these oversights.

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**M426. Proposed by the Mayhem Staff.**

Determine the number of positive integers less than or equal to 1000000 that are divisible by all of the integers 2, 3, 4, 5, 6, 7, 8, 9, and 10.

**Solution by Winda Kirana, student, SMPN 8, Yogyakarta, Indonesia.**

A positive integer is divisible by all of the integers from 2 to 10 if it is divisible by the least common multiple (lcm) of these numbers.

We can write this list of integers in terms of their prime factorizations as \(2, 3, 2^2, 5, 2 \times 3, 7, 2^3, 5^2, 2 \times 5\). Therefore, \(\text{lcm}(2, 3, 4, 5, 6, 7, 8, 9, 10) = 2^3 \times 3^2 \times 5 = 2520\).

Now the largest integer less than or equal to 1000000 that is divisible by 2520 is 2520 \(\times 396\). This is because the quotient when 1000000 is divided by 2520 is 396 and the remainder is 2080.

Thus, there are 396 positive integers less than or equal to 1000000 that are divisible by all of the integers from 2 to 10. (These 396 integers are the multiples of 2520 from 2520 \(\times 1\) to 2520 \(\times 396\).)

Additional solutions by JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, St. Mary’s Xaverius I, Palembang, Indonesia; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain; AFIFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; RICARDO PEIRO, IES “Abastos”, Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC, JOHN WYNN, student, Auburn University, Montgomery, AL, USA; and INGEStI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia. One incorrect solution was submitted.
**M427. Proposed by the Mayhem Staff.**

A semicircle has diameter \(AB\). Equilateral triangle \(ABC\) is drawn on the same side of \(AB\) as the semicircle. Determine the area of the region that lies inside the triangle and outside the semicircle.

*Solution by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania, modified by the editor.*

Suppose that \(r\) is the radius of the semicircle. Let \(O\) be the centre of the semicircle and points \(M\) and \(N\) where the semicircle intersects \(AC\) and \(BC\), respectively. Join \(OM\), \(ON\), and \(MN\).

Note that \(OA = OM = ON = OB\), since each is a radius. Since \(\triangle ABC\) is equilateral, \(\angle ABC = \angle ACB = \angle BAC = 60^\circ\).

Since \(OA = OM\), then \(\triangle OMA\) is isosceles and \(\angle AMO = \angle MAO = 60^\circ\). This tells us in fact that \(\triangle OMA\) is equilateral, because its third angle also equals \(60^\circ\). Similarly, \(\triangle ONB\) is equilateral.

Now \(\angle MON = 180^\circ - \angle MOA - \angle NOB = 180^\circ - 60^\circ - 60^\circ = 60^\circ\). Since \(OM = ON\), then in fact \(\triangle OMN\) is also equilateral since the remaining two angles are equal and add to \(120^\circ\).

Note that \(\angle CMN = 180^\circ - \angle AMO - \angle OMN = 180^\circ - 60^\circ - 60^\circ = 60^\circ\). Similarly, \(\angle CNM = 60^\circ\), so \(\triangle CMN\) is also equilateral.

Since each of \(\triangle OMA\), \(\triangle ONB\), \(\triangle OMN\), and \(\triangle CMN\) is equilateral, and each shares a side with one of the others, then these four equilateral triangles all have the same side length and so are all congruent.

The area inside \(\triangle ABC\) but outside the semicircle is equal to the area of rhombus \(MONC\) minus the area of sector \(MON\).

Now rhombus \(MONC\) is made up of the two congruent equilateral triangles \(MON\) and \(CMN\). Each is an equilateral triangle with side length \(r\) (the radius of the semicircle), and so each has area \(\frac{\sqrt{3}}{4}r^2\). (We could calculate this by constructing an altitude in one of these triangles.) Therefore, the area of rhombus \(MONC\) is \(2 \cdot \frac{\sqrt{3}}{4}r^2 = \frac{\sqrt{3}}{2}r^2\).

Sector \(MON\) has angle \(60^\circ\), and so has area \(\frac{60^\circ}{360^\circ} \cdot \pi r^2 = \frac{1}{6} \pi r^2\).

Therefore, the area of the region is \(\frac{\sqrt{3}}{2}r^2 - \frac{1}{6} \pi r^2\).

*Also solved by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia; Geoffrey A. Randall, Hamden, CT, USA; Winda Kirana, student, SMPN 8, Yogyakarta, Indonesia; Hugo Luyo Sanchez, Pontificia Universidad Catolica del Peru, Lima, Peru; Rafael Martinez Calafat, I.E.S. La Plana, Castellon, Spain; RICARD PEIRO, IES "Ahastos", Valencia, Spain; Bruno Salgueiro Fanego, Viveiro, Spain; Everest SHI, student, Burnaby North Secondary School, Burnaby, BC, and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution was submitted.*

Determine all integers \(x\) for which

\[(4-x)^4-x + (3-x)^3-x + 20 = 4^x + 3^x.\]

Solution by John Wynn, student, Auburn University, Montgomery, AL, USA, modified by the editor.

We will examine three cases to show that there is only one integer \(x\) that satisfies the equation.

Case 1: \(x \geq 3\). We note first in this case that if \(x = 3\) or if \(x = 4\), the left side will include a term of the form \(0^0\). We could sensibly adopt the convention that this is undefined, that it equals 0, or that it equals 1. Using any of these conventions, we first show that neither \(x = 3\) nor \(x = 4\) is a solution.

Substituting \(x = 3\), we see that the left side equals \(1^1 + 0^0 + 20\), which is either undefined or equal to 21 or 22. When \(x = 3\), the right side equals \(4^3 + 3^3\), which equals 91. Therefore, the equation is not satisfied, no matter which convention we adopt.

Substituting \(x = 4\), we see that the left side equals \(0^0 + (-1)^{-1} + 20\), which is either undefined or equal to 19 or 20. When \(x = 4\), the right side equals \(4^4 + 3^4\), which equals 337. Therefore, the equation is not satisfied, no matter which convention we adopt.

When \(x \geq 5\), we have that \(4^x + 3^x \geq 4^5 + 3^5 = 1267\).

Also note that when \(x \geq 5\), we have \(4-x \leq -1\) and \(3-x \leq -2\) and so \(|4-x| \geq 1\) and \(|3-x| \geq 2\). Therefore, \(|4-x|^{x-4} \geq 1\) and \(|3-x|^{x-3} \geq 2^2 = 4\). Thus, \((4-x)^{4-x} = \frac{1}{(4-x)^{x-4}} \leq \frac{1}{|4-x|^{x-4}} \leq 1\) and \((3-x)^{3-x} = \frac{1}{(3-x)^{x-3}} \leq \frac{1}{|3-x|^{x-3}} < 1\). Therefore, when \(x \geq 5\), the right side is at least 1267 and the left side is at most 22, so no such value of \(x\) satisfies the equation.

Case 2: \(x \leq 1\). When \(x \leq 1\), we have \(4^x + 3^x \leq 4^1 + 3^1 = 7\). Also, when \(x \leq 1\), we have that \(4-x \geq 3\) and \(3-x \geq 2\), so \((4-x)^{4-x} \geq 3^3 = 27\) and \((3-x)^{3-x} \geq 2^2 = 4\). Therefore, the left side is at least \(27 + 4 + 20 = 51\) and the right side is at most 7. Thus, there are no solutions in this case.

Case 3: \(x = 2\). Here, the left side equals \(2^2 + 1^1 + 20 = 25\) and the right side equals \(4^2 + 3^2 = 25\), so \(x = 2\) is a solution.

In summary, we see that \(x = 2\) is the only integer solution.

Also solved by HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incomplete solutions were submitted.
**M429.** Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

Determine all triples \((a, b, c)\) of positive integers with \(a^{(b^c)} = (a^b)^c\).

*Solution by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.*

The equation \(a^{(b^c)} = (a^b)^c\) is equivalent to the equation \(a^{(b^c)} = a^{bc}\). We examine a number of different cases.

**Case 1:** \(a = 1\). Then the equation is true regardless of the values of \(b\) and \(c\). Therefore, \((1, b, c)\) is a solution for all positive integers \(b\) and \(c\).

**Case 2:** \(a > 1\). In this case, \(a^{(b^c)} = a^{bc}\) is equivalent to \(b^c = bc\), which is equivalent to \(b^{c-1} = c\) since \(b > 0\). We consider subcases where \(c = 1\), \(c = 2\), and \(c > 2\).

**Subcase 2(a):** \(a > 1\) and \(c = 1\). If \(c = 1\), then we have \(b^0 = 1\), which is true for all positive integers \(b\). Therefore, \((a, b, 1)\) is a solution for all positive integers \(a > 1\) and all positive integers \(b\).

**Subcase 2(b):** \(a > 1\) and \(c = 2\). If \(c = 2\), then the equation \(b^{c-1} = c\) becomes \(b = 2\). Therefore, \((a, 2, 2)\) is a solution for all positive integers \(a > 1\).

**Subcase 2(c):** \(a > 1\) and \(c > 2\). If \(c > 2\), then \(b\) cannot equal 1, so \(b \geq 2\).

Using the fact that \(2^{c-1} > c\) for \(c \geq 3\) (proved at the end of this solution), we see that \(b^{c-1} > 2^{c-1} > c\), so \(b^{c-1} = c\) has no solutions in this case.

In conclusion, the solutions are all triples \((a, b, c)\) of positive integers with (i) \(a = 1\), or (ii) \(a > 1\) and \(c = 1\), or (iii) \(a > 1\) and \(b = c = 2\).

To finish, we must show that \(2^{c-1} > c\) for all positive integers \(c \geq 3\).

We prove this by mathematical induction on \(c\).

If \(c = 3\), the inequality becomes \(4 = 2^2 > 3\), which is true.

Suppose that \(2^{c-1} > c\) for \(c = k\) for some positive integer \(k \geq 3\).

Consider \(c = k + 1\). Since \(2^{k-1} > k\) by the induction hypothesis, then \(2^k = 2 \cdot 2^{k-1} > 2k\). Since \(k \geq 3\), then \(2k > k + 1\), so \(2^k > k + 1\), or \(2^{(k+1)-1} > k + 1\), as required. This completes the proof by induction.

*Also solved by RAFAEL MARTÍNEZ CALAFAT, I.E.S. La Plana, Castellón, Spain; RICARD PEIRO, IES “Alastros”, Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. Seven incorrect solutions were submitted.*

**M430.** Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Let \(p_n\) be the \(n^{th}\) prime number. Prove that \(p_n > 3n\) for all \(n \geq 12\).

*Solution by Bruno Salgueiro Fanego, Viveiro, Spain.*

We prove the result by induction on \(n\). First, we note that if \(n = 12\), then \(p_n = p_{12} = 37\) and \(3n = 36\), so \(p_n > 3n\) when \(n = 12\).
Next, we assume that \( p_k > 3k \) for some positive integer \( k \geq 12 \). We will prove that \( p_{k+1} > 3(k + 1) \).

Note that the first prime larger than \( p_k \) is \( p_{k+1} \) so \( p_{k+1} \geq p_k + 1 \). Since \( p_k \) is an odd prime (the only even prime is 2), then \( p_k + 1 \) is even and so cannot be prime. Thus, \( p_{k+1} \geq p_k + 2 \).

Also, note that since \( p_k > 3k \) and \( p_k \) is an integer, then \( p_k \geq 3k + 1 \).

Altogether, we obtain \( p_{k+1} \geq p_k + 2 \geq 3k + 1 + 2 = 3k + 3 = 3(k + 1) \).

But \( 3(k + 1) \) cannot be a prime number since it is divisible by 3 and it is at least 39, and \( p_{k+1} \) is a prime number, so \( p_{k+1} > 3(k + 1) \), as required.

Therefore, by induction, \( p_n > 3n \) for all positive integers \( n \geq 12 \).

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; JOSE HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; GEOFFREY A. RANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; and RAFAEL MARTINEZ CALAFAT, I. E. S. La Plana, Castellon, Spain. One incomplete solution was submitted.

**M431. Proposed by Shailesh Shirali, Rishi Valley School, India.**

In acute triangle \( ABC \), the foot of the perpendicular from \( A \) to \( BC \) is \( D \), and the foot of the perpendicular from \( D \) to \( AC \) is \( E \). Point \( F \) is located on line segment \( DE \) such that \( \frac{DF}{FE} = \frac{\cot C}{\cot B} \). Prove that \( AF \) and \( BE \) are perpendicular.

**Solution by the proposer, modified by the editor.**

Let \( L \) be the point where lines \( AF \) and \( BE \) intersect each other, and let \( K \) be the foot of the perpendicular from \( B \) to \( AC \). Draw \( BK \) and \( DL \).

Now \( \triangle BKC \) is similar to \( \triangle DEC \) since each is right-angled and the triangles share the angle at \( C \). Therefore, \( \frac{BC}{KC} = \frac{KC}{EC} \), and so we have \( \frac{DC + DB}{DC} = 1 + \frac{DB}{EC} = \frac{EK}{EC} \), or \( \frac{1 + \frac{DB}{EC}}{EC} = \frac{EK}{EC} \).

Since \( AD \), \( BC \) are perpendicular, \( \cot B = \frac{DB}{AD} \) and \( \cot C = \frac{DC}{AD} \).

Therefore, \( \cot B = \frac{DB}{AD} \), and we then have \( \frac{EC}{EK} = \frac{\cot C}{\cot B} = \frac{DF}{FE} \).

Note that \( \angle DAE = \angle CBK = 90^\circ - \angle ACB \). Thus, \( \triangle AED \) and \( \triangle BKC \) are similar since each has a right angle and a second equal angle. Therefore, in these similar triangles, points \( F \) and \( E \) divide the corresponding sides \( ED \) and \( KC \) in the same ratio. Also, from the similarity of these two triangles, we have \( \frac{ED}{EA} = \frac{KC}{KB} \).
We will show that this implies that $\angle EAF = \angle KBE$. This will mean that $\angle DAF = \angle CBE$ since $\angle CBK = \angle DAE$. This in turn will tell us that $\angle DAL = \angle DBL$. From this, we can conclude that points $A$, $B$, $D$, and $L$ form a cyclic quadrilateral. Hence, $\angle ALB = \angle ADB = 90^\circ$, and so $AF$ and $BE$ are perpendicular, as required.

It remains to show that $\angle EAF = \angle KBE$. Note that both angles are acute. Also, $\frac{FE}{ED} = \frac{FE}{FE + DF} = \frac{1}{1 + \frac{DF}{FE}} = \frac{1}{1 + \frac{EC}{EK}} = \frac{EK}{KC}$. Therefore,

$$\tan(\angle EAF) = \frac{FE}{EA} = \frac{ED \cdot \frac{EK}{KC}}{EA} = \frac{ED}{EA} \cdot \frac{EK}{KC} = \frac{KC \cdot EK}{KB \cdot KC} = \frac{EK}{KB} = \tan(\angle KBE).$$

Since acute angles with equal tangents are equal, then $\angle EAF = \angle KBE$, as required, thus completing the proof.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEOFFREY A. KANDALL, Hamden, CT, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.*

**Problem of the Month**

Ian VanderBurgh

This month, we investigate numbers expressed in bases other than 10.

**Problem** (1986 Canadian Invitational Mathematics Challenge) Find a base 7 three-digit number which has its digits reversed when expressed in base 9.

Let's review (or learn!) about numbers in different bases. Since the problem talks about three-digit numbers, we'll focus on three-digit numbers. All of what we look at can be extended to numbers with more digits.

When we write the three-digit integer two hundred seventy-three as 273, we normally mean that this is the base 10 representation of this integer. Writing 273 is a way of representing the integer equal to $2 \times 10^2 + 7 \times 10 + 3$. We could write this as $(273)_{10}$ to emphasize that we are thinking of a base 10 number.

Let's look at base 7. Any digit in base 7 must be less than 7, so the possible digits are 0, 1, 2, 3, 4, 5, and 6. The notation $(326)_7$ is an example of a three-digit integer in base 7. (The subscript of 7 indicates the base.) This is the base 7 representation of the integer equal to $3 \times 7^2 + 2 \times 7 + 6$, which equals one hundred sixty-seven. In other words, $(326)_7 = (167)_{10}$.

Let's look at a general base $b$, where $b$ is an integer with $b > 1$. In base $b$, the possible digits are from 0 to $b - 1$, inclusive. An example of a