SKOLIAD No. 129

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Please send your solutions to problems in this Skoliad by June 1, 2011. A copy of CRUX with Mayhem will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest for this month is the City Competition of the Croatian Mathematical Society, 2010, secondary level, grade 1. Our thanks go to Željko Hanjić, University of Zagreb, Croatia, for providing us with this contest and for permission to publish it.

Compétition 2010 de la Société mathématique croate
Niveau secondaire, première année

1. Soit $n$ un entier positif et $a$ un nombre réel non nul. Simplifier la fraction
\[
\frac{a^{3n+1} - a^4}{a^{2n+3} + a^{n+4} + a^5}.
\]

2. Trouver un entier positif qui, multiplié par 9 donne un entier compris entre 1100 et 1200, et lorsque multiplié par 13 donne un entier compris entre 1500 et 1600.

3. Dans le plan, on donne trois cerces de rayon 2, de sorte que le centre de chacun d’eux se trouve à l’intersection des deux autres. Trouver l’aire de l’intersection des trois disques limités par ces cercles.

4. On considère l’entier $n$. Soit $m$ l’entier obtenu à partir de $n$ en y biffant le chiffre des unités. Si $n - m = 2010$, trouver $n$.

5. Un sac contient un grand nombre de balles rouges, blanches et bleues. Chaque enfant d’un groupe donné sort du sac au hasard trois balles. Quel est le nombre minimal d’enfants dans ce groupe permettant que deux d’entre eux aient la même combinaison de balles, c.-à-d. le même nombre de balles de chaque couleur ?

6. Si $a^2 + 2b^2 = 3c^2$, montrer que
\[
\left(\frac{a+b}{b+c} + \frac{b-c}{b-a}\right) \cdot \frac{a+2b+3c}{a+c}
\]
est un entier positif.
7. Un triangle rectangle $ABC$, d'angle droit en $B$ et dont les côtés de l'angle droit mesurent 15 et 20, est congruent à un triangle $BDE$ avec l'angle droit en $D$. Le point $C$ est situé strictement à l'intérieur du segment $BD$, et les points $A$ et $E$ sont situés du même côté de la droite $BD$.

(a) Trouver la distance entre les points $A$ et $E$.
(b) Trouver l'aire de l'intersection des triangles $ABC$ and $BDE$.

8. Soit $p$ et $q$ deux nombres premiers impairs distincts. Montrer que l'entier $(pq + 1)^4 - 1$ possède au moins quatre diviseurs premiers différents.

City Competition of the Croatian Mathematical Society, 2010
Secondary level, Grade 1

1. Let $n$ be a positive integer and $a$ a non-zero real number. Reduce the fraction

$$\frac{a^{3n+1} - a^4}{a^{2n+3} + a^{n+4} + a^5}.$$

2. Find a positive integer which when multiplied by 9 gives an integer between 1100 and 1200, and when multiplied by 13 gives an integer between 1500 and 1600.

3. Three circles, each with radius 2, are given in the plane such that the centre of each lies on the intersection of the other two. Determine the area of the intersection of the three disks bounded by those circles.

4. Consider the integer $n$. Let $m$ be the integer obtained from $n$ by removing its ones digit. If $n - m = 2010$, find $n$.

5. A bag contains a sufficient number of red, white, and blue balls. Each child in a given group takes three balls at random from the bag. What is the smallest number of children in the group that ensures that two of them have taken the same combination of balls, that is, the same number of balls of each colour?

6. If $a^2 + 2b^2 = 3c^2$, prove that

$$\left(\frac{a+b}{b+c} + \frac{b-c}{b-a}\right) \cdot \frac{a+2b+3c}{a+c}$$

is a positive integer.

7. A right triangle, $\triangle ABC$, with legs of lengths 15 and 20 and the right angle at vertex $B$ is congruent to a triangle, $\triangle BDE$, with the right angle at vertex $D$. The point $C$ lies strictly inside the segment $BD$, and the points $A$ and $E$ are on the same side of the straight line $BD$. 

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(a) Find the distance between points $A$ and $E$.
(b) Find the area of the intersection of $\triangle ABC$ and $\triangle BDE$.

8. Let $p$ and $q$ be different odd prime numbers. Prove that the integer $(pq+1)^2-1$ has at least four different prime divisors.

Next we give the solutions to the City Competition of the Croatian Mathematical Society, 2009, Secondary Level, Grade 1, given in Skoliad 123 at [2010 : 67–68].

1. Reduce the fraction

$$\frac{a^4 - 2a^3 - 2a^2 + 2a + 1}{(a + 1)(a + 2)}.$$

Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

First, factor the numerator:

$$a^4 - 2a^3 - 2a^2 + 2a + 1 = (a^4 - 2a^2 + 1) - 2a^3 + 2a$$

$$= (a^2 - 1)^2 - 2a(a^2 - 1) = (a^2 - 1)(a^2 - 2a - 1)$$

Therefore,

$$\frac{a^4 - 2a^3 - 2a^2 + 2a + 1}{(a + 1)(a + 2)} = \frac{(a + 1)(a - 1)(a^2 - 2a - 1)}{(a + 1)(a + 2)}$$

$$= \frac{(a - 1)(a^2 - 2a - 1)}{a + 2}.$$

Also solved by Natalia Desy, student, SMA Xaverius I, Palembang, Indonesia.

Note that the denominator is already factored as $(a + 1)(a + 2)$. Therefore, the only candidates for reducing are $a + 1$ and $a + 2$. If you make $a = -1$ in the numerator, you get 21. So the expression cannot be reduced by $a + 2$. If you make $a = -1$ in the numerator, you get 0, so reducing by $a + 1$ is possible. You can now obtain the answer by polynomial division.

2. If you write the digit 3 on the left side of a two-digit number, you obtain, of course, a three-digit number. If twice the three-digit number equals 27 times the two-digit number, what is the original two-digit number?

Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

Let $x$ be the original two-digit number. When the digit 3 is inserted in front of $x$, the resulting three-digit number is $300 + x$. The given relationship between the two numbers is then that $2(300 + x) = 27x$. Solving this equation yields that $x = 24$. 
Also solved by ELLEN CHEN, student, Burnaby North Secondary School, Burnaby, BC; LENA CHOIL, student, Ecole Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius I, Palembang, Indonesia; and GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

3. Find the largest integer $n$ such that $3 \left( n - \frac{5}{3} \right) - 2(4n + 1) > 6n + 5$.

Solution by Ellen Chen, student, Burnaby North Secondary School, Burnaby, BC.

If $3(n - \frac{5}{3}) - 2(4n + 1) > 6n + 5$, then $3n - 5 - 8n - 2 > 6n + 5$, so $-5n - 7 > 6n + 5$, so $-12 > 11n$. Thus $n < -\frac{12}{11} \approx -1.09$, so the largest integer value for $n$ is $-2$.

Also solved by MATTHEW NG, student, St. Francis Xavier Secondary School, Mississauga, ON.

4. Find the number of divisors of 288.

Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

The prime factorisation of 288 is $2^5 \cdot 3^2$. Therefore, any divisor of 288 has the form $2^a \cdot 3^b$, where $a$ and $b$ are integers such that $0 \leq a \leq 5$ and $0 \leq b \leq 2$. You have 6 choices for $a$ and 3 choices for $b$, for a total of $6 \cdot 3 = 18$ choices. These are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144, and 288.

Also solved by LENA CHOIL, student, Ecole Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius I, Palembang, Indonesia; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC.

Our solver's method for counting divisors is much easier than listing divisors systematically. If you were not familiar with it, read the solution again.

5. In the figure, $ABCDEF$ is a regular hexagon while $EFGHI$ is a regular pentagon. Determine the angle $\angle GAF$.

Solution by Natalia Desy, student, SMA Xaverius I, Palembang, Indonesia.

The angle sum of an $n$-gon is $180(n - 2)$, so the angle sum of a hexagon is $720^\circ$ and the angle sum of a pentagon is $540^\circ$. Since the polygons in the problem are regular, $\angle AFE = 120^\circ$ and $\angle GFE = 108^\circ$. Therefore, $\angle AFG = 360^\circ - 120^\circ - 108^\circ = 132^\circ$. Since $FG = EF = AF$, $\triangle AFG$ is isosceles, so

$$\angle GAF = \frac{180^\circ - 132^\circ}{2} = 24^\circ.$$
6. In a trapezoid $ABCD$, the angle at $B$ is a right angle, and the diagonal $BD$ is perpendicular to the leg $AD$. The length of the leg $BC$ is 5, and the length of the diagonal $BD$ is 13. Find the area of the trapezoid $ABCD$.

Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

For diagonal $BD$ to be perpendicular to $AD$, the parallel sides of the trapezoid must be $AB$ and $CD$, as in the figure. Thus, $\angle ABC = \angle BCD = \angle ADB = 90^\circ$. It now follows from the Pythagorean Theorem that $CD = \sqrt{13^2 - 5^2} = 12$. Moreover, $\angle ABD = \angle BDC$, so $\triangle ABD$ is similar to $\triangle BDC$. Therefore, $\frac{AB}{BD} = \frac{BD}{DC}$, so $\frac{13}{12} = \frac{169}{12}$. The area of trapezoid $ABCD$ is thus

$$\frac{AB + CD}{2} \cdot BC = \frac{169 + 12}{2} \cdot 5 = 1565 \cdot \frac{24}{24}.$$ 

Also solved by NATALIA DESY, student, SMA Xaverius 1, Pallembang, Indonesia.

7. At Tihana’s birthday party, the first guest arrived the first time the bell rang. Each time the bell rang thereafter the number of guests arriving was two more than the number that had arrived the previous time the bell rang. If the bell rang $n$ times, how many guests attended the party?

Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

From the pattern in the table below it is easy to see that $2n - 1$ guests arrived when the bell rang the $n^{th}$ time:

<table>
<thead>
<tr>
<th>Time the bell rang</th>
<th>1\textsuperscript{st}</th>
<th>2\textsuperscript{nd}</th>
<th>3\textsuperscript{rd}</th>
<th>4\textsuperscript{th}</th>
<th>...</th>
<th>$n\textsuperscript{th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guests arriving</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>...</td>
<td>$2n - 1$</td>
</tr>
</tbody>
</table>

The total number of guests is then the sum of the numbers in the second row in the table, $1 + 3 + 5 + 7 + \cdots + (2n - 1)$. But this is an arithmetic sum with first term 1, last term $2n - 1$, and $n$ terms. Therefore, the sum is $\frac{1 + (2n - 1)}{2} \cdot n = \frac{2n - 1}{2} \cdot n = n^2$.

If you are not familiar with our solver’s formula for the sum of an arithmetic sequence, you can use Gauss’ trick:

$$\begin{align*}
1 & + 3 + \cdots + (2n - 3) + (2n - 1) = S \\
(2n - 1) & + (2n - 3) + \cdots + 3 + 1 = S \\
\text{so that} & \\
\frac{2n + 2n + \cdots + 2n + 2n}{n \text{ copies}} & = 2S
\end{align*}$$

Thus, $2n^2 = 2S$ and $S = n^2$. 

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8. Determine all positive integers \( n \) such that \( n^2 - 440 \) is the square of an integer.

Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

If \( n^2 - 440 = k^2 \), where \( k \) is a positive integer, then

\[
440 = n^2 - k^2 = (n + k)(n - k).
\]

Therefore, \( n + k \) and \( n - k \) must both be (positive, integer) divisors of 440. Since 440 = \( 2^3 \cdot 5 \cdot 11 \), the only divisors are 1, 2, 4, 5, 8, 10, 11, 20, 22, 40, 44, 55, 88, 110, 220, and 440. [Ed.: To find the divisors, see the solution to Problem 4 above.] To reduce the number of cases to check, note that \( n + k \) is larger than \( n - k \) and that they have the same parity (that is, they are either both even or both odd). That leaves just four cases:

- If \( n + k = 220 \) and \( n - k = 2 \), then \( n = 111 \) and \( k = 109 \).
- If \( n + k = 110 \) and \( n - k = 4 \), then \( n = 57 \) and \( k = 53 \).
- If \( n + k = 44 \) and \( n - k = 10 \), then \( n = 27 \) and \( k = 17 \).
- If \( n + k = 22 \) and \( n - k = 20 \), then \( n = 21 \) and \( k = 1 \).

Thus, the only possible values for \( n \) are 21, 27, 57, and 111.

Also solved by NATALIA DESY, student, SMA Xavier 1, Palembang, Indonesia.

This issue’s prize of one copy of CRUX with MAYHEM for the best solutions goes to Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

We hope that our readers will enjoy the featured contest and that they will share their joy by submitting one or more solutions for publication.

NOTICE TO CRUX READERS

The CMS is in the process of appointing a new Editor-in-Chief for 2011 as well as finding a number of section editors. The situation is causing severe production problems with the journal and has caused delays in 2010 and is expected to cause delays in the delivery of issues in 2011.

The CMS apologizes for this disruption and delay in service.

Johan Rudnick,
Managing Editor and CMS Executive Director.