A Solution to Gibson's and Rodgers' Problem in 3 Dimensions

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1 Introduction

Peter M. Gibson and Michael H. Rodgers [1] posed problem 844 in CRUX Mathematicorum on iterated triangles inscribed in a circle and a higher dimensional analogue. The first part of their problem is as follows:

(a) A triangle $A_0B_0C_0$ with centroid $G_0$ is inscribed in a circle $\Gamma$ with centre $O$. The lines $A_0G_0$, $B_0G_0$, $C_0G_0$ meet $\Gamma$ again in $A_1$, $B_1$, $C_1$, respectively, and $G_1$ is the centroid of triangle $A_1B_1C_1$. A triangle $A_2B_2C_2$ with centroid $G_2$ is obtained in the same way from $A_1B_1C_1$, and the procedure is repeated indefinitely, producing triangles with centroids $G_3$, $G_4$, ..., If $g_n = OG_n$, prove that the sequence \{g_0, g_1, g_2, \ldots\} is decreasing and converges to 0.

This part was solved by R.B. Killgrove and Dan Sokolowsky [3]. The second part of problem 844 was to determine if a similar result holds for a tetrahedron inscribed in a sphere, or, more generally, for an $n$-simplex inscribed in an $n$-sphere. This latter problem is hitherto unsolved. Here we give a positive answer and a proof in the 3-dimensional case.

2 Notation and Preliminary Results

Throughout we will assume that all tetrahedra are nondegenerate or we shall prove that the tetrahedra which arise are nondegenerate.

For convenience we adopt certain notations. Let $S_A$, $S_B$, $S_C$, $S_D$ be the areas of the faces opposite the vertices $A$, $B$, $C$, $D$ of tetrahedron $ABCD$, let $(XYZ)$ be the plane through the three points $X$, $Y$, $Z$, and let $V(WXYZ)$ be the volume of tetrahedron $WXYZ$. For certain special sums, the following notation will be used:

$$\sum S_A^2L_A = S_A^2L_A + S_B^2L_B + S_C^2L_C + S_D^2L_D,$$

$$\sum AB^2 = AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2.$$

A dot "\cdot" will denote either multiplication of two numbers or the dot product of two vectors, depending on the context.
We now make some definitions. Let $ABCD$ be a tetrahedron. A plane through the edge $AB$ and the midpoint of the edge $CD$ is called the median plane through the edge $AB$ of the tetrahedron. A bisecting plane of the dihedral angle at the edge $AB$ of the tetrahedron is called the bisector plane through the edge $AB$ of the tetrahedron. The plane that is the reflection of the median plane through edge $AB$ in the bisector plane through the edge $AB$ is called the symmedian plane through the edge $AB$ of the tetrahedron.

Each tetrahedron has six edges and thus has six median planes, six bisector planes, and six symmedian planes.

It is known that the six median planes intersect in a common point which is the centroid of the tetrahedron, and the six bisector planes intersect in a common point which is the centre of the inscribed sphere. The six symmedian planes also intersect in a common point and we shall call this point the Lemoine point of the tetrahedron (we will prove this later).

Our main theorem has two parts, the second part being the positive answer to the problem posed by Peter M. Gibson and Michael H. Rodgers in three dimensions.

**Theorem** Let $A_0B_0C_0D_0$ be a tetrahedron with volume $V_0$ and centroid $G_0$ inscribed in a sphere $\Gamma$ with centre $O$. The lines $A_0G_0$, $B_0G_0$, $C_0G_0$, $D_0G_0$ intersect $\Gamma$ again in $A_1$, $B_1$, $C_1$, $D_1$, respectively, and $V_1$ and $G_1$ are the volume and the centroid of tetrahedron $A_1B_1C_1D_1$, respectively. A tetrahedron $A_2B_2C_2D_2$ with volume $V_2$ and centroid $G_2$ is obtained in a similar way from $A_1B_1C_1D_1$, and the procedure is repeated indefinitely, producing tetrahedra with volumes $V_3$, $V_4$, ... and centroids $G_3$, $G_4$, ... Then,

1. The sequence $\{V_n\}$ is nondecreasing, and
2. The sequence $\{OG_n\}$ is nonincreasing and converges to zero.

In order to prove Theorem 1 we need several lemmas.

**Lemma 1** If $M$ is inside tetrahedron $ABCD$, then $\sum V(MBCD)\overline{MA} = 0$.

**Proof:** Choose points $A'$, $B'$, $C'$, $D'$ on the rays $MA$, $MB$, $MC$, $MD$, respectively, so that $M$ is the centroid of tetrahedron $A'B'C'D'$. Note that the volume of each tetrahedron $MB'C'D'$, $MC'D'A'$, $MD'A'B'$, $MA'B'C'$ is one-fourth the volume of tetrahedron $A'B'C'D'$. We have

\[
\sum V(MBCD)\overline{MA} = \frac{1}{4} V(A'B'C'D') \sum V(MBCD) \overline{MA} \\
= \frac{1}{4} V(A'B'C'D') \sum V(MBCD) \overline{MA} \\
= \frac{1}{4} V(A'B'C'D') \frac{MA \cdot MB \cdot MC \cdot MD}{MA' \cdot MB' \cdot MC' \cdot MD'} \sum \frac{MA'}{MA} \overline{MA} \\
= \frac{1}{4} V(A'B'C'D') \frac{MA \cdot MB \cdot MC \cdot MD}{MA' \cdot MB' \cdot MC' \cdot MD'} \sum \overline{MA} = 0 .
\]
Lemma 2 Tetrahedron $ABCD$ is inscribed in sphere $(O)$. Let $M$ be a point in the interior of the tetrahedron. Let the lines $MA$, $MB$, $MC$, $MD$ meet $(O)$ again at $A'$, $B'$, $C'$, $D'$. Then

$$
\frac{V(ABCD)}{V(A'B'C'D')} = \frac{\text{MA} \cdot \text{MB} \cdot \text{MC} \cdot \text{MD}}{\text{MA'} \cdot \text{MB'} \cdot \text{MC'} \cdot \text{MD'}}.
$$

Proof: By Lemma 1, we have $\sum V(MBCD)\overrightarrow{MA} = \overrightarrow{0}$. Thus,

$$
\sum V(MBCD)\overrightarrow{MA} = \sum V(MBCD)\left(-\frac{\text{MA}}{\text{MA'}}\right)\overrightarrow{MA'}
$$

$$
= -\sum V(MBCD)\overrightarrow{MA} = \overrightarrow{0}.
$$

Since the numbers $V(MBCD)\frac{\text{MA}}{\text{MA'}}$, $V(MCDA)\frac{\text{MB}}{\text{MB'}}$, $V(MDAB)\frac{\text{MC}}{\text{MC'}}$, and $V(MABC)\frac{\text{MD}}{\text{MD'}}$ are positive, $M$ is inside the tetrahedron $A'B'C'B'$, and hence $V(A'B'C'D') = \sum V(MB'C'D')$.

Note that $\text{MA} \cdot \text{MA'} = \text{MB} \cdot \text{MB'} = \text{MC} \cdot \text{MC'} = \text{MD} \cdot \text{MD'} = R^2 - OM^2$, where $R$ is the radius of $(O)$. Thus,

$$
V(A'B'C'D')
= \sum \frac{V(MB'C'D')}{V(MBCD)} V(MBCD)
= \sum \frac{\text{MB'} \cdot \text{MC'} \cdot \text{MD'}}{\text{MB} \cdot \text{MC} \cdot \text{MD}} V(MBCD)
= \frac{\text{MA'} \cdot \text{MB'} \cdot \text{MC'} \cdot \text{MD'}}{\text{MA} \cdot \text{MB} \cdot \text{MC} \cdot \text{MD}} \cdot \frac{1}{\text{MA} \cdot \text{MA'}} \sum V(MBCD)\text{MA}^2
$$

$$
= \frac{\text{MA'} \cdot \text{MB'} \cdot \text{MC'} \cdot \text{MD'}}{\text{MA} \cdot \text{MB} \cdot \text{MC} \cdot \text{MD}} \cdot \frac{1}{R^2 - OM^2} \sum V(MBCD)\text{MA}^2. \quad (1)
$$

However, we also have

$$
V(ABCD)R^2
= \sum V(MBCD)\text{OA}^2 = \sum V(MBCD) \left| \overrightarrow{OM} + \overrightarrow{MA} \right|^2
= \left( \sum V(MBCD) \right) OM^2 + 2\overrightarrow{OM} \cdot \left( \sum V(MBCD)\overrightarrow{MA} \right) + \sum V(MBCD)\text{MA}^2
= V(ABCD)OM^2 + 2\overrightarrow{OM} \cdot \overrightarrow{0} + \sum V(MBCD)\text{MA}^2
= V(ABCD)OM^2 + \sum V(MBCD)\text{MA}^2. \quad (2)
$$

It follows that $\sum V(MBCD)\text{MA}^2 = V(ABCD)(R^2 - OM^2)$. The lemma now follows from the above identities (1) and (2).
**Lemma 3** The opposing edges (three pairs altogether) of a tetrahedron are of equal length if and only if its centroid coincides with the centre of its circumscribed sphere.

**Proof:** Let tetrahedron $ABCD$ have centroid $G$ and let $O$ be the centre of its circumscribed sphere.

Let $(\alpha)$, $(\alpha')$ be two parallel planes that contain $AB$, $CD$, respectively; let $(\beta)$, $(\beta')$ be two parallel planes that contain $AC$, $DB$, respectively; and let $(\gamma)$, $(\gamma')$ be two parallel planes that contain $AD$, $BC$, respectively. The pairs of planes $(\alpha)$, $(\alpha')$; $(\beta)$, $(\beta')$; and $(\gamma)$, $(\gamma')$ define a parallelepiped, which we denote by $ATBZ.YCXD$ (see the figure at right).

It is evident that $CD = TZ$, $DB = YT$, $BC = ZY$ and $G$ is the common midpoint of the diagonals of the parallelepiped $ATBZ.YCXD$. Hence, the following conditions are equivalent.

(a) $AB = CD$, $AC = DB$, $AD = BC$.
(b) $AB = TZ$, $AC = YT$, $AD = ZY$.
(c) $ATBZ$, $AYCT$, $AZDY$ are rectangles.
(d) $ATBZ.YCXD$ is a rectangular parallelepiped.
(e) $AX = BY = CZ = DT$.
(f) $GA = GB = GC = GD$.
(g) $G$ coincides with $O$.

A tetrahedron is said to be **quasiregular** if it satisfies one of the two equivalent conditions stated in Lemma 3.

**Lemma 4** Six symmedian planes of tetrahedron $ABCD$ intersect at one common point $L$ defined by

$$\sum S^2_A \overrightarrow{LA} = \overrightarrow{0}.$$

We note that this point is uniquely defined by the above equality and is referred to as the Lemoine point (as aforementioned).

More generally, for each quadruple of positive real numbers $(\alpha, \beta, \gamma, \delta)$ there exists a unique point $P$ in the interior of the tetrahedron such that $\sum \alpha \overrightarrow{PA} = \overrightarrow{0}$, and conversely for each point $P$ in the interior of the tetrahedron $ABCD$ there is a unique quadruple of positive real numbers $(\alpha, \beta, \gamma, \delta)$ such that $\alpha + \beta + \gamma + \delta = 1$ and $\sum \alpha \overrightarrow{PA} = \overrightarrow{0}$. 
Proof of Lemma 4: Let the median plane, the bisector plane, and the symmedian plane through the edge $AB$ of the tetrahedron meet the edge $CD$ at $M$, $N$, and $P$, respectively.

Let $(\pi)$ be a plane perpendicular to the line $AB$. Let $A'$ be the orthogonal projection of $A$, $B$ onto the plane $(\pi)$, and let $C'$, $D'$, $M'$, $N'$, $P'$ be the orthogonal projections of $C$, $D$, $M$, $N$, $P$ onto the plane $(\pi)$, respectively.

It is evident that in triangle $A'C'D'$ the segments $A'M'$, $A'N'$, $A'P'$ are, respectively, the median, the bisector, and the symmedian from the vertex $A'$. By the symmedian property [2], we have

$$\frac{P'C'}{P'D'} = \left(\frac{A'C'}{A'D'}\right)^2.$$

From this, and the fact that $CC'$, $DD'$, $PP'$ are parallel, we have

$$\frac{PC}{PD} = \left(\frac{A'C'}{A'D'}\right)^2.$$

Suppose that $E$, $F$ are respectively the orthogonal projections of $C$, $D$ on $AB$. It is easily seen that $A'C' = EC$, $A'D' = FD$.

Thus,

$$\frac{PC}{PD} = \left(\frac{EC}{FD}\right)^2 = \left(\frac{\frac{1}{2}AB \cdot EC}{\frac{1}{2}AB \cdot FD}\right)^2 = \frac{S_D^2}{S_E^2}.$$
This implies that \( S^2 \overrightarrow{PC} + S^2 \overrightarrow{PD} = 0 \). On the other hand, since
\[
\sum S^2 \overrightarrow{LA} = 0,
\]
we have
\[
S^2 \overrightarrow{LA} + S^2 \overrightarrow{LB} + S^2 \overrightarrow{(LP + PC)} + S^2 \overrightarrow{(LP + PD)} = 0.
\]
This means that
\[
S^2 \overrightarrow{LA} + S^2 \overrightarrow{LB} + (S^2 + S^2) \overrightarrow{LP} + (S^2 \overrightarrow{PC} + S^2 \overrightarrow{PD}) = 0.
\]
Consequently,
\[
S^2 \overrightarrow{LA} + S^2 \overrightarrow{LB} + (S^2 + S^2) \overrightarrow{LP} = 0,
\]
so that \( L \) lies in \((ABP)\), the symmedian plane through the edge \( AB \) of the tetrahedron \( ABCD \).

Therefore, \( L \) lies in all six symmedian planes of tetrahedron \( ABCD \).  

**Lemma 5** If \( M \) is in the interior of tetrahedron \( ABCD \) and \( H, K, I, J \) are the orthogonal projections of \( M \) onto the planes \((BCD), (CDA), (DAB), (ABC)\), respectively, then
\[
\sum S^2 \overrightarrow{MH} = 0.
\]

**Proof:** Let \( S(UVW) \) denote the area of triangle \( U VW \). Let the inscribed sphere of tetrahedron \( ABCD \) touch the planes \((BCD), (CDA), (DAB), (ABC)\) at \( X, Y, Z, T \), respectively. Let \( P, r \) be the centre and radius of the inscribed sphere, respectively.

From the planar analogue of Lemma 1 (see also [4]),
\[
S(XCD)\overrightarrow{XB} + S(XDB)\overrightarrow{XC} + S(XBC)\overrightarrow{XD} = 0,
\]
so it follows that
\[
S(XCD)(\overrightarrow{XP} + \overrightarrow{PB}) + S(XDB)(\overrightarrow{XP} + \overrightarrow{PC}) + S(XBC)(\overrightarrow{XP} + \overrightarrow{PD}) = 0.
\]
Hence, \( S_A \overrightarrow{PX} = S(XCD)\overrightarrow{PB} + S(XDB)\overrightarrow{PC} + S(XBC)\overrightarrow{PD} \), and also
\[
S_B \overrightarrow{PY} = S(YDA)\overrightarrow{PC} + S(YAC)\overrightarrow{PD} + S(YCD)\overrightarrow{PA},
\]
\[
S_C \overrightarrow{PZ} = S(ZAB)\overrightarrow{PD} + S(ZBD)\overrightarrow{PA} + S(ZDA)\overrightarrow{PB},
\]
\[
S_D \overrightarrow{PT} = S(TBC)\overrightarrow{PA} + S(TCA)\overrightarrow{PB} + S(TAB)\overrightarrow{PC}.
\]

Moreover, we note that
\[
S(ZAB) = S(TAB), \quad S(XCD) = S(YCD), \quad S(YAC) = S(TAC), \quad S(ZBD) = S(XDB), \quad S(ZDA) = S(YDA), \quad S(TBC) = S(XBC);
\]
so, by using Lemma 1, we have

\[
\sum \frac{S_A}{MH} \overrightarrow{MH} = \frac{1}{r} \sum S_A \frac{PX}{MH} \overrightarrow{MH} = \frac{1}{r} \sum S_A \overrightarrow{PA} = \frac{1}{r} \sum (S/XCD) \overrightarrow{PB} + S(XDB) \overrightarrow{PC} + S(XBC) \overrightarrow{PD}
\]

\[
= \frac{1}{r} \sum (S(YCD) + S(ZDB) + S(TBC)) \overrightarrow{PA} = \frac{1}{r} \sum S(XCD) + S(XDB) + S(XBC)) \overrightarrow{PA} = \frac{1}{r} \sum S_A \overrightarrow{PA} = \frac{3}{r^2} \sum \frac{1}{3} S_A \cdot PX \cdot \overrightarrow{PA}
\]

\[
= \frac{3}{r^2} \sum V(PBCD) \overrightarrow{PA} = \overrightarrow{0}.
\]

The planar analogue of the next lemma can be found in [4].

**Lemma 6** Suppose that any three of \(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}\) are not coplanar, that \(x, y, z, t, x', y', z', t'\) are nonzero, and that the equations \(x \overrightarrow{a} + y \overrightarrow{b} + z \overrightarrow{c} + t \overrightarrow{d} = \overrightarrow{0}\) and \(x' \overrightarrow{a} + y' \overrightarrow{b} + z' \overrightarrow{c} + t' \overrightarrow{d} = \overrightarrow{0}\) hold. Then

\[
\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{t}{t'}.
\]

**Proof:** By isolating \(\overrightarrow{d}\) we have

\[
\frac{x}{t} \overrightarrow{a} + \frac{y}{t} \overrightarrow{b} + \frac{z}{t} \overrightarrow{c} = -\overrightarrow{d} = \frac{x'}{t'} \overrightarrow{a} + \frac{y'}{t'} \overrightarrow{b} + \frac{z'}{t'} \overrightarrow{c}.
\]

Since \(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\) are not coplanar, it follows that

\[
\frac{x}{t} = \frac{x'}{t'}, \quad \frac{y}{t} = \frac{y'}{t'}, \quad \frac{z}{t} = \frac{z'}{t'},
\]

which implies that \(\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{t}{t'}\).

**Lemma 7** Let \(M\) be any point in the interior of tetrahedron \(ABCD\). Let \(H, K, I, J\) be the orthogonal projections of the point \(M\) onto the planes \((BCD), (CDA), (DAB), (ABC)\). Then \(M\) is the centroid of tetrahedron \(HKIJ\) if and only if \(M\) is the Lemoine point of the tetrahedron \(ABCD\).

**Proof:** We shall show the equivalence of the following statements.

(a) The point \(M\) is the centroid of tetrahedron \(HKIJ\).

(b) The equation \(\overrightarrow{MH} + \overrightarrow{MK} + \overrightarrow{MI} + \overrightarrow{MJ} = \overrightarrow{0}\) holds.

(c) The equation \(\frac{S_A}{MH} = \frac{S_B}{MK} = \frac{S_C}{MI} = \frac{S_D}{MJ}\) holds.
(d) The equation \[ \frac{S_A^2}{\frac{1}{3} M H \cdot S_A} = \frac{S_B^2}{\frac{1}{3} M K \cdot S_B} = \frac{S_C^2}{\frac{1}{3} M I \cdot S_C} = \frac{S_D^2}{\frac{1}{3} M J \cdot S_D} \]
holds.

(e) The equation \[ \frac{S_A^2}{V(MBCD)} = \frac{S_B^2}{V(MCD A)} = \frac{S_C^2}{V(MD A B)} = \frac{S_D^2}{V(M A B C)} \]
holds.

(f) The equation \[ \sum S_A^2 \overrightarrow{MA} = 0 \] holds.

(g) The point \( M \) is the Lemoine point of the tetrahedron \( ABCD \).

Parts (a) and (b) are equivalent by properties of the centroid. Lemma 5 and Lemma 6 imply the equivalence of (b) and (c). Clearly, (c), (d), and (e) are equivalent. Lemma 1 and Lemma 6 imply that (e) and (f) are equivalent, while Lemma 4 implies that (f) and (g) are equivalent.

Lemma 8 Let \( ABCD \) be a tetrahedron and \( X, Y, Z, T \) points on the planes \( (BCD), (CDA), (DAB), (ABC) \), respectively. The sum \( \sum XY^2 \) is minimized if and only if \( X, Y, Z, T \) are the orthogonal projections of the Lemoine point of \( ABCD \) onto the planes \( (BCD), (CDA), (DAB), (ABC) \).

Proof: Let \( M \) be the centroid of tetrahedron \( XYZT \) and \( H, K, I, J \) the orthogonal projections of \( M \) onto the planes \( (BCD), (CDA), (DAB), (ABC) \). We have

\[
\sum XY^2 = \sum \left[ \overrightarrow{MX} - \overrightarrow{MY} \right]^2 = 3 \sum MX^2 - 2 \sum \overrightarrow{MX} \cdot \overrightarrow{MY} \\
= 4 \sum MX^2 - \left( \sum \overrightarrow{MX} \right)^2 = 4 \sum MX^2 \geq 4 \sum MH^2 \\
= \frac{4}{S_A^2} \left( \sum MH^2 \right) \left( \sum S_A^2 \right) \geq \frac{4}{S_A^2} \left( \sum S_A MH \right)^2 \\
= \frac{4}{S_A^2} \left( \sum 3V(MBCD) \right)^2 \geq \frac{36}{S_A^2} V^2(ABCD).
\]

Therefore, \( \sum XY^2 \geq \frac{36}{S_A} V^2(ABCD) \), with equality if and only if the following three conditions are satisfied:

(a) The points \( X, Y, Z, T \) are the orthogonal projections of \( M \) onto the planes \( (BCD), (CDA), (DAB), (ABC) \).

(b) The equation \[ \frac{MH}{S_A} = \frac{MK}{S_B} = \frac{MI}{S_C} = \frac{MJ}{S_D} \] holds.

(c) The point \( M \) is in the interior of the tetrahedron \( ABCD \).

By Lemma 5 and Lemma 7, these conditions are satisfied if and only if \( X, Y, Z, T \) are the orthogonal projections of the Lemoine point of tetrahedron \( ABCD \) onto the planes \( (BCD), (CDA), (DAB), (ABC) \).
3 Proof of Theorem 1

Let $R$ be the radius of the circumsphere, $\Gamma$, of tetrahedron $A_0B_0C_0D_0$.

Proof of part (1): Note that

$$G_0A_0 \cdot G_0A_1 = G_0B_0 \cdot G_0B_1 = G_0C_0 \cdot G_0C_1 = G_0D_0 \cdot G_0D_1 = R^2 - OG_0^2,$$

$$\sum G_0A_0^2 = \left( \sum OA_0^2 \right) - 4OG_0^2 = 4 \left( R^2 - OG_0^2 \right) \quad (3)$$

Using (3), Lemma 2, and the AM-GM Inequality, we have

$$\frac{V_0}{V_1} = \frac{G_0A_0 \cdot G_0A_0 \cdot G_0B_0 \cdot G_0B_1 \cdot G_0C_0 \cdot G_0C_1 \cdot G_0D_0 \cdot G_0D_1}{G_0A_1 \cdot G_0B_1 \cdot G_0C_1 \cdot G_0D_1} \leq \left( \frac{1}{4} \sum G_0A_0 \right)^4$$

$$= \left( \frac{1}{4} \sum G_0A_0^2 \right)^4 = \left( \frac{1}{4(R^2 - OG_0^2)} \right)^4 \left( \sum G_0A_0^2 \right)^4$$

$$= \frac{1}{(4(R^2 - OG_0^2))^4} \left( \sum (OA_0^2 - 4OG_0^2) \right)^4$$

$$= \frac{1}{(4(R^2 - OG_0^2))^4} \left( 4(R^2 - OG_0^2) \right) = 1.$$

Thus, $V_0 \leq V_1$.

We remark that by (3) and Lemma 3, the following are equivalent.

(a) The volumes of successive tetrahedra are equal, that is, $V_0 = V_1$.

(b) The equation $\frac{G_0A_0}{G_0A_1} = \frac{G_0B_0}{G_0B_1} = \frac{G_0C_0}{G_0C_1} = \frac{G_0D_0}{G_0D_1}$ holds.

(c) The equation $\frac{G_0A_0^2}{G_0A_1} = \frac{G_0B_0^2}{G_0B_1} = \frac{G_0C_0^2}{G_0C_1}$ holds.

(d) The equation $G_0A_0 = G_0B_0 = G_0C_0 = G_0D_0$ holds.

(e) The centroid $G_0$ coincides with $O$.

(f) The tetrahedron $A_0B_0C_0D_0$ is quasiregular.

Repeating this procedure, we have $V_0 \leq V_1 \leq V_2 \leq \cdots$, and $\{V_n\}$ is a nondecreasing sequence.

Proof of part (2): Let $(\alpha)$, $(\beta)$, $(\gamma)$, and $(\delta)$ be the planes through the points $A_0$, $B_0$, $C_0$, $D_0$ respectively and perpendicular to $A_0G_0$, $B_0G_0$, $C_0G_0$, and $D_0G_0$ in that order.

Let $A'_0 = (\beta) \cap (\gamma)$, $B'_0 = (\gamma) \cap (\delta)$, $C'_0 = (\delta) \cap (\alpha)$, and $D'_0 = (\alpha) \cap (\beta)$.
Since \( G_0 \) is the centroid of tetrahedron \( A_0B_0C_0D_0 \), by Lemma 7 \( G_0 \) is the Lemoine point of the tetrahedron \( A'_0B'_0C'_0D'_0 \).

Let \( A'_1, B'_1, C'_1, D'_1 \) be the reflections of \( A_1, B_1, C_1, D_1 \) in \( O \). Then \( A'_1, B'_1, C'_1, D'_1 \) are on the planes \((\alpha), (\beta), (\gamma), (\delta)\), respectively.

By Lemma 8, \( \sum (A'_1B'_1)^2 \geq \sum (A_0B_0)^2 \). Since \( \sum (A'_1B'_1)^2 = \sum (A_1B_1)^2 \), we obtain
\[
\sum (A_1B_1)^2 \geq \sum (A_0B_0)^2 .
\]

Furthermore, we have
\[
\sum (A_0B_0)^2 = \sum |\overrightarrow{OA_0} - \overrightarrow{OB_0}|^2 = 12R^2 - 2 \sum \overrightarrow{OA_0} \cdot \overrightarrow{OB_0} = 16R^2 - \sum |\overrightarrow{OA_0}|^2 = 16R^2 - |4\overrightarrow{OG_0}|^2 = 16(R^2 - OG_0^2) ,
\]
and \( \sum (A_1B_1)^2 = 16(R^2 - OG_0^2) \) is deduced similarly.

Therefore, \( OG_0 \geq OG_1 \), and by Lemma 7 equality holds if and only if \( A'_1, B'_1, C'_1, D'_1 \) respectively coincide with \( A_0, B_0, C_0, D_0 \). In other words, \( G_0 \) coincides with \( O \). By Lemma 3 this occurs if and only if \( A_0B_0C_0D_0 \) is a quasiregular tetrahedron.

We now know that \( \{OG_n\} \) is a nonincreasing sequence bounded below by \( 0 \), so the following limit exists:
\[
\lim_{n \to \infty} OG_n .
\]

Let \( \bar{\Gamma} \) be the closed ball with boundary \( \Gamma \). Since \( \bar{\Gamma} \) is closed and bounded, by the Bolzano–Weierstrass Theorem there is an increasing sequence of positive integers \( \{n_k\} \) such that each of the sequences \( \{A_{n_k}\}, \{B_{n_k}\}, \{C_{n_k}\}, \{D_{n_k}\}, \{G_{n_k}\}, \{A_{n_k+1}\}, \{B_{n_k+1}\}, \{C_{n_k+1}\}, \{D_{n_k+1}\}, \{G_{n_k+1}\} \) is convergent in \( \bar{\Gamma} \). Let the respective limits of these sequences be \( A_0^*, B_0^*, C_0^*, D_0^*, G_0^* \), \( A_1^*, B_1^*, C_1^*, D_1^*, G_1^* \); that is, \( A_{n_k} \rightarrow A_0^*, B_{n_k} \rightarrow B_0^*, \) and so forth.

It is evident that
\[
OG_0^* = \lim_{k \to \infty} OG_{n_k}, \quad OG_1^* = \lim_{k \to \infty} OG_{n_k+1} .
\]

Since \( \lim_{n \to \infty} OG_n \) exists, it follows from (5) that
\[
OG_0^* = OG_1^* .
\]

Let \( V_n \) be the volume of \( A_nB_nC_nD_n \). The sequence \( \{V_n\} \) is nondecreasing by part (1), and is bounded above by the volume of \( \Gamma \) and bounded below by \( V_0 > 0 \). Therefore, \( \lim V_n \) exists and is positive, and it follows that \( \lim_{n \to \infty} V_n = \lim_{n \to \infty} V_{n+1} > 0 \).

If either tetrahedron \( A_0^*B_0^*C_0^*D_0^* \) or \( A_1^*B_1^*C_1^*D_1^* \) were degenerate, then we would have \( \lim_{n \to \infty} V_n = 0 \) or \( \lim_{n \to \infty} V_{n+1} = 0 \), a contradiction.

Thus, \( A_0^*B_0^*C_0^*D_0^* \) and \( A_1^*B_1^*C_1^*D_1^* \) are nondegenerate tetrahedra.
On the other hand, \( \Gamma \) is closed and bounded, so \( \Gamma \) contains \( A_n^*, B_n^*, C_n^*, D_n^* \). \( A_n^*, B_n^*, C_n^*, D_n^* \). Since \( G_{n_k} \) and \( G_{n_k+1} \) are the respective centroids of the tetrahedra \( A_{n_k}B_{n_k}C_{n_k}D_{n_k} \) and \( A_{n_k+1}B_{n_k+1}C_{n_k+1}D_{n_k+1} \) for all \( n_k \), we have that \( G_0^* \) and \( G_1^* \) are the respective centroids of tetrahedra \( A_0^*B_0^*C_0^*D_0^* \) and \( A_1^*B_1^*C_1^*D_1^* \). Since \( A_{n_k+1}, B_{n_k+1}, C_{n_k+1}, D_{n_k+1} \) are the respective intersections of the lines \( A_{n_k}G_{n_k}, B_{n_k}G_{n_k}, C_{n_k}G_{n_k}, D_{n_k}G_{n_k} \) with \( \Gamma \), it then follows that \( A_1^*, B_1^*, C_1^*, D_1^* \) are the respective intersections of the lines \( A_0^*G_0^*, B_0^*G_0^*, C_0^*G_0^*, D_0^*G_0^* \) with \( \Gamma \).

By the above remarks, the tetrahedra \( A_0^*B_0^*C_0^*D_0^* \) and \( A_1^*B_1^*C_1^*D_1^* \) are related to one another in the same way that the tetrahedra \( A_0B_0C_0D_0 \) and \( A_1B_1C_1D_1 \) are related to one another.

By the same reasoning as in the first part of the proof, \( OG_0^* \geq OG_1^* \), with equality only when \( A_0^*B_0^*C_0^*D_0^* \) is a quasiregular tetrahedron. However, we showed in (6) that equality does indeed hold. This implies that \( G_0^* \) coincides with the circumcentre \( O \) of the sphere. Then \( OG_0^* = 0 \), so that

\[
\lim_{n \to \infty} OG_n = \lim_{n \to \infty} OG_{n_k} = OG_0^* = 0.
\]

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References


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