M462. *Proposé par Alex Song, Detroit Country Day School, Detroit, MI, É-U et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

On désigne par $\lfloor x \rfloor$ le plus grand entier n'excédant pas x et par $\lceil x \rceil$ le plus petit entier plus grand ou égal à x. Par exemple, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor -1.4 \rfloor = -2$, et $\lceil -1.4 \rceil = -1$. Déterminer tous les nombres réels x pour lesquels $\lfloor x \rfloor \lceil x \rceil = x^2$.

Mayhem Solutions

M420. *Proposed by the Mayhem Staff.*

Riley is a poor starving university student, but is mathematically astute. He notices that five suppers in residence cost the same as seven lunches. After one week of skipping supper most nights, he notices that five lunches and one supper cost $\$48$ in total. How much do 16 suppers cost?

Solution by Oscar Xia, student, St. George's School, Vancouver, BC.

Let one lunch cost x dollars and one supper cost y dollars.

Since five suppers cost the same as seven lunches, then $5y = 7x$. Since five suppers and one lunch cost $\$48$, then $5x + y = 48$.

From the first equation, $x = \frac{5}{7}y$ and so we obtain $5\left(\frac{5}{7}y\right) + y = 48$, or $\frac{32}{7}y = 48$, or $y = \frac{21}{2}$.

Therefore, 16 suppers cost $16\left(\frac{21}{2}\right) = 168$ dollars.

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; LEWIS HUGHES, Auburn University, Montgomery, AL, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M421. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $\lfloor x \rfloor$ be the greatest integer less than or equal to the real number x. Determine all real numbers x such that

$$\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor = 4 .$$

Solution by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA.

Since x cannot be 0, then we let $y = \frac{1}{x}$. Therefore, we are looking for all real nonzero solutions to $\lfloor y \rfloor + \lfloor 3y \rfloor = 4$. Note that y cannot be negative, because if it were, then the left side of the equation would be negative. Therefore, y is positive.

If $y < 1$, then $3y < 3$ and so $\lfloor y \rfloor + \lfloor 3y \rfloor < y + 3y < 1 + 3 = 4$. Therefore, $y \geq 1$.

If $y \geq \frac{4}{3}$, then $3y \geq 4$. In this case, $\lfloor y \rfloor \geq 1$ and $\lfloor 3y \rfloor \geq 4$, so $\lfloor y \rfloor + \lfloor 3y \rfloor \geq 5$. Therefore, $y < \frac{4}{3}$.

Combining what we know, we have $1 \leq y < \frac{4}{3}$. For these y, we have $\lfloor y \rfloor = 1$. Also, we have $3 \leq 3y < 4$ and so $\lfloor 3y \rfloor = 3$. Therefore, for all of these y, we have $\lfloor y \rfloor + \lfloor 3y \rfloor = 4$.

Therefore, the values of y that satisfy the equation are $1 \leq y < \frac{4}{3}$, and so the values of x that satisfy the original equation are $\frac{3}{4} < x \leq 1$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; OSCAR XIA, student, St. George's School, Vancouver, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution and two incomplete solutions were submitted.

M422. *Proposed by Adnan Arapovic, student, University of Waterloo, Waterloo, ON.*

Prove that

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Since $\binom{k+1}{2} = \frac{k(k+1)}{2}$, then the given equation is equivalent to

$$\sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3}.$$

The total number of 3-element subsets of $\{1, 2, 3, \ldots, n+1, n+2\}$ is equal to $\binom{n+2}{3}$.

These subsets can also be counted in the following way. First, choose the largest element, M, in the subset. Note that M can take any value from 3 to $n+2$. Next, choose the remaining two elements from among the numbers 1 to $M-1$. Let $M = k+2$, where $k = 1, 2, \ldots, n$. For a fixed value of k, there are $\binom{k+1}{2}$ ways of choosing the remaining two elements. Summing over all possible values of k, we see that the total number of 3-element subsets of $\{1, 2, 3, \ldots, n+1, n+2\}$ is also equal to $\sum_{k=1}^{n} \binom{k+1}{2}$.

Therefore, $\sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3}$.

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; KONSTANTINOS AL. NAKOS, Agrinio, Greece; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON (second and third solutions); JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; OSCAR XIA, student, St. George's School, Vancouver, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incomplete solutions were submitted.

M423. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The tens digit of a perfect square S is three greater than the ones digit of S. Determine all possible remainders when S is divided by 100.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Every perfect square has its units digit in the set $\{0,\ 1,\ 4,\ 5,\ 6,\ 9\}$. (This can be seen by squaring the integers from 0 to 9 and observing that the pattern of units digits continues for larger squares.)

Since the tens digit is to be 3 greater than the units digit, then units digit cannot be greater than 6, so the only possible remainders when S is divided by 100 (that is, the possible pairs of last two digits of S) are 30, 41, 74, 85, and 96. That is, S must have one of the following forms

$$100k + 30\,, \quad 100k + 41\,, \quad 100k + 74\,, \quad 100k + 85\,, \quad 100k + 96$$

for some integer k.

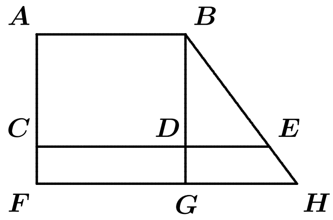
Note that even perfect squares must be divisible by 4, because an even perfect square is the square of an even integer, say $2n$ for some integer n, so the square is $4n^2$. Any integer of the form $100k + 30$ or $100k + 74$ is not divisible by 4, so cannot be a perfect square. Thus, S cannot end in 30 or 74.

Also, S cannot end in 85. If S did end in 85, then its square root would end in 5, so S would be of the form $(10a + 5)^2$ for some integer a. In this case, $S = 100a^2 + 100a + 25$, which ends in 25. So S cannot end in 85.

Therefore, the only possibilities for the last two digits of S are 41 and 96. Each of them is possible since $21^2 = 441$ and $14^2 = 196$.

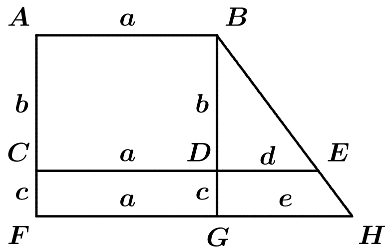
Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. One incomplete solution and one incorrect solution were submitted.

M424. *Proposed by Margo Kondratieva, Memorial University of Newfoundland, St. John's, NL.*



In the diagram, line segments AB, CDE, and FGH are parallel. Also, line segments ACF and BDG are perpendicular to AB. Suppose that the area of rectangle $ABDC$ is x, the area of rectangle $CDGF$ is y, and the area of $\triangle BDE$ is z. Determine the area of $DEHG$ in terms of x, y, and z.

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.



Since line segments AB, CDE, and FGH are parallel, and line segments ACF and BDG are perpendicular to AB, then AB, CD, and FGH are each perpendicular to both ACF and BDG. Therefore, $ABDC$ and $CDGF$ are rectangles, and $\triangle BDE$ is similar to $\triangle BGH$, since they share a common angle at B.

Suppose that $AB = CD = FG = a$, $AC = BD = b$, $CF = DG = c$, $DE = d$, and $GH = e$.

Since $\triangle BDE$ is similar to $\triangle BGH$, we then have $\dfrac{DE}{GH} = \dfrac{BD}{BG}$, and so $\dfrac{d}{e} = \dfrac{b}{b+c}$, which gives $e = \dfrac{d(b+c)}{b}$.

Since the area of $ABDC$ is x, then $ab = x$. Since the area of $CDGF$ is y, then $ac = y$. Since the area of $\triangle BDE$ is z, then $bd = 2z$.

From the first two equations, $\dfrac{x}{y} = \dfrac{ab}{ac} = \dfrac{b}{c}$. From the second and third equations, $abcd = 2yz$, which gives $cd = \dfrac{2yz}{ab} = \dfrac{2yz}{x}$.

Since $DEHG$ is a trapezoid, then

$$\begin{aligned}
&\text{Area of } DEHG \\
&= \frac{1}{2}(d+e)c = \frac{1}{2}cd + \frac{1}{2}ce = \frac{1}{2}cd + \frac{1}{2}c \cdot \frac{d(b+c)}{b} \\
&= \frac{1}{2}cd + \frac{1}{2}cd + \frac{1}{2} \cdot \frac{dc^2}{b} = cd + \frac{c^2 d}{b} = cd + \frac{1}{2}cd \cdot \frac{c}{b} \\
&= \frac{2yz}{x} + \frac{1}{2} \cdot \frac{2yz}{x} \cdot \frac{y}{x} = \frac{2xyz + y^2 z}{x^2}.
\end{aligned}$$

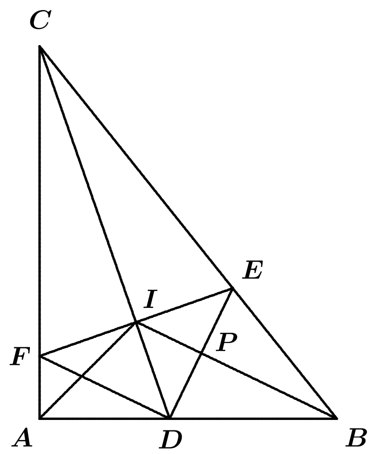
Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; G.C. GREUBEL, Newport News, VA, USA; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES

"Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution was submitted.

M425. *Proposed by Titu Zvonaru, Comănești, Romania.*

In $\triangle ABC$, $\angle BAC = 90°$ and I is the incentre. The interior bisector of angle C meets AB at D. The line segment through D perpendicular to BI intersects BC at E. The line segment through D parallel to BI meets AC at F. Prove that E, I, and F are collinear.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.



Since $\triangle ABC$ is right-angled at A, then $\angle ABC + \angle ACB = 90°$.

Consider $\triangle CIB$. We see that $\angle DIB$ is an external angle of this triangle, so $\angle DIB = \angle IBC + \angle ICB$. Since CI and BI bisect $\angle ACB$ and $\angle ABC$, respectively, then $\angle IBC = \frac{1}{2}\angle ABC$ and $\angle ICB = \frac{1}{2}\angle ACB$. Therefore, $\angle DIB = \frac{1}{2}(\angle ABC + \angle ACB) = \frac{1}{2}(90°) = 45°$.

Since DF is parallel to BI, then $\angle FDI = \angle DIB = 45°$. Since AI bisects $\angle BAC$, then $\angle FAI = \angle DAI = 45°$. Thus, FI subtends equal angles at A and at D. Therefore, quadrilateral $FADI$ is cyclic.

In addition, chord DI in cyclic quadrilateral subtends both $\angle DAI$ and $\angle DFI$. Thus, $\angle DFI = \angle DAI = 45°$. We then see that in $\triangle FID$, there are two $45°$ angles, so $\angle FID = 90°$.

Suppose that BI and DE intersect at P. Since BI and DE are perpendicular and $\angle EBP = \angle DBP$ (because BI is an angle bisector) and BP is a common side in $\triangle EBP$ and $\angle DBP$, then these two triangles are congruent. Therefore, $DP = EP$.

This tells us that $\triangle IDP$ and $\triangle IEP$ are congruent, since the triangles have the side IP in common, $DP = EP$, and $\angle IPD = \angle IPE = 90°$. Therefore, $\angle DIP = \angle EIP$.

Thus, $\angle DIE = 2\angle DIP = 2\angle DIB = 90°$.

Finally, $\angle FIE = \angle FID + \angle DIE = 90° + 90° = 180°$. This tells us that E, I, and F are collinear, as required.

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incorrect solutions were submitted.