THE OLYMPIAD CORNER

No. 289

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We begin this number of the *Corner* with problems from the 3rd and 4th grade of the Croatian National Mathematical Competition. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam and to Željko Hanjš, Zagreb, for making them available for our use.

CROATIAN MATHEMATICAL COMPETITION 2007
National Competition

3rd Grade

1. Let n be a positive integer such that $n+1$ is divisible by 24.

(a) Prove that n has an even number of divisors (including 1 and n itself).

(b) Prove that the sum of all divisors of n is divisible by 24.

(Simplified from Putnam Competition 1969)

2. In the triangle ABC, with $\angle BAC = 120^\circ$, the bisectors of the angles $\angle BAC$, $\angle ABC$, $\angle BCA$ intersect the opposite sides in the points D, E, F, respectively. Prove that the circle with diameter EF passes through D.

(British Mathematical Olympiad 2005)

3. In triangle ABC the vertex A is equidistant from the circumcentre and the orthocentre. Find the angle $\alpha = \angle BAC$.

(USA proposal for IMO 1989)

4. Ten integers 1, 4, 7, $\dots$, 28 (an arithmetic progression with common difference 3) are arranged in a circle. Let N be the maximum of the 10 sums obtained by adding to any integer its two neighbours on the circle. What is the minimum possible value of N?

4th Grade

1. The same as problem 1 of 3rd Grade.

2. A sequence of positive integers $(a_n)_{n\geq 0}$ is defined recursively by $a_0 = 3$ and $a_n = 2 + a_0 a_1 \cdots a_{n-1}$ for $n \geq 1$.

(a) Prove that any two distinct terms of the sequence are relatively prime.

(b) Determine a_{2007}.

3. In a $5 \times n$ table, where n is a positive integer, each 1×1 cell is painted either red or blue. Find the smallest possible n such that, for any painting of the table, one can always choose three rows and three columns for which the 9 cells in their intersection all have the same colour.

4. In acute triangle ABC let A_1, B_1 and C_1 be the midpoints of the sides BC, CA, and AB, respectively. The circumcircle of ABC has centre O and radius 1. Prove that

$$\frac{1}{|OA_1|} + \frac{1}{|OB_1|} + \frac{1}{|OC_1|} \geq 6\,.$$

Next we look to the 51st National Mathematics Olympiad in Slovenia and the Selection Examinations for IMO 2007. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for collecting them for us.

51st NATIONAL MATHEMATICAL OLYMPIAD IN SLOVENIA

Selection Examinations for the IMO 2007

First Selection Examination, December 2006

1. Show that the inequality $(1+a^2)(1+b^2) \geq a(1+b^2)+b(1+a^2)$ holds for any pair of real numbers a and b.

2. Prove that any triangle can be decomposed into n isosceles triangles for every positive integer $n \geq 4$.

3. Let ABC be a triangle with $AC < BC$ and denote its circumcircle by Γ. Let E be the midpoint of the arc AB that contains the point C and let D be a point on the segment BC, such that $BD = AC$. The line DE meets the circle Γ again in F. Prove that A, B, C, and F are the vertices of an isosceles trapezoid.

Second Selection Examination, February 2007

1. Every point in the plane with positive integer coordinates (x, y) such that $x \leq 19$ and $y \leq 4$ is coloured green, red, or blue. Prove that there exists a rectangle with sides parallel to the coordinate axes and with all four vertices of the same colour.

2. The circles Γ_1 and Γ_2 of different radii meet at A_1 and A_2. Let t be the common tangent of the two circles, such that the distance from t to A_1 is shorter than the distance from t to A_2. Let B_1 and B_2 be the points at which t touches Γ_1 and Γ_2, respectively.

Let Γ_3 and Γ_4 be the circles with radii $|A_1B_1|$ and $|A_1B_2|$ and the centre A_1. The circles Γ_1 and Γ_3 meet again at C_1, while the circles Γ_2 and Γ_4 meet again at C_2. Denote the intersection of the lines B_1C_1 and B_2C_2 by D and let E be the intersection of B_1C_1 and Γ_4 which lies on the same side of the line B_2C_2 as C_1.

Show that A_1D is perpendicular to EC_2.

3. Find a positive integer n such that n^2-1 has exactly 10 positive divisors. Show that n^2-4 cannot have exactly 10 positive divisors for any positive integer n.

Third Selection Examination, March 2007

1. Find all functions $f:\mathbb{R}\to\mathbb{R}$, such that

$$f(x^2(z^2+1)+f(y)(z+1)) \;=\; 1-f(z)(x^2+f(y))-z(1+z)x^2+2f(y)$$

for all real numbers x, y, and z.

2. Let

$$x \;=\; 0.a_1a_2a_3a_4\ldots \quad \text{and} \quad y \;=\; 0.b_1b_2b_3b_4\ldots$$

be the decimal representations of two positive real numbers. The equality $b_n=a_{2^n}$ holds for all positive integers n. Given that x is a rational number, show that y is also a rational number.

3. Let $ABCD$ be a trapezoid with AB parallel to CD and $|AB|>|CD|$. Let E and F be the points on the segments AB and CD, respectively, such that $\dfrac{|AE|}{|EB|}=\dfrac{|DF|}{|FC|}$. Let K and L be two points on the segment EF such that

$$\angle AKB \;=\; \angle DCB \quad \text{and} \quad \angle CLD \;=\; \angle CBA\,.$$

Show that K, L, B, and C are concyclic.

Next we look at some problems of the Correspondence Mathematical Competition in Slovakia 2006/2007. These are arranged by students of Comenius University in Bratislava, with support of the Slovak Mathematical Olympiad Committee. In a year there are two rounds of competitions, each round consisting of three series of problems, some for first and second year high school students (1–7 in a set), some for older students (5–11), and some intended as IMO preparation (10–14). We give the fourteen problems of the first set from the first round. The organizers often use problems from other contests and note the source where possible. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

CORRESPONDENCE MATHEMATICAL COMPETITION IN SLOVAKIA 2006/7

First Round

First Set

1. Some pigeons and sparrows are sitting on a fence. Five sparrows fly away and there remain two pigeons for each sparrow. Then 25 pigeons fly away and there remain three sparrows for each pigeon. Find the initial numbers of sparrows and pigeons.

2. There are six dominoes on the table. They are arranged to form a perimeter of a square of size 4×4. Fnd the smallest possible number of spots on all the dominoes together. *Remark*: There are from 0 to 6 spots on each half of the domino. The full set of dominoes contains all 28 possible pieces. [*Ed.: This is how we received this problem and an interpretation of the problem yielding an interesting solution would be much appreciated.*]

3. We have eight cubes with digits 1, 2, 3, 4, 5, 6, 7, 9 (each cube has one digit written on one of its faces). In how many ways can we create four two-digit primes from the cubes?

4. A nine-member committee was formed to select a chief of the KMS. There are three candidates for the chief. Each member of the committee orders the candidates and gives 3 points to the first one, 2 points to the second one and 1 point to the last one. After summing the points of the candidates no two candidates had the same number of points, hence the order of the candidates was clear. Someone noticed that if every member of the committee selected only one candidate, then the resulting order of the candidates was reversed. How many points did the candidates get?

5. (a) Find all positive integers n such that both of the numbers $2^n - 1$ and $2^n + 1$ are primes.

(b) Find all primes p such that both of the numbers $4p^2 + 1$ and $6p^2 + 1$ are primes.

6. Find all positive integers n such that $n + 200$ and $n - 269$ are cubes of integers.

7. There were 33 children at a camp. Every child answered two questions: "How many other children at camp have the same first name as you?" and "How many other children at camp have the same family name as you?" Among the answers each of the numbers from 0 to 10 occurred at least once. Show that there were at least two children at camp with the same first name and the same family name.

(Mathematical Contests 1997–1998, 1.18 Russia, 29/95)

8. There are $2n$ white and $2n$ black balls in a row. Prove that, whatever order they are in, we can always find $2n$ consecutive balls of which exactly n are white.

9. Find all triples of integers x, y, z satisfying

$$2^x + 3^y = z^2 .$$

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10. Numismatist Christian has 241 coins with total value of 360 talers. (The value of a coin in talers is a positive integer.) Can Christian be sure that he can divide his coins into three piles all of equal value?

(Ukraine 2005)

11. There are n people living on an island. One day their leader decided that all islanders (including himself) will make and wear a necklace composed of one-coloured stones (at least zero stones per necklace). Two islanders are to have at least one stone of the same colour in their necklaces if and only if they are friends.

(a) Prove that the islanders can fulfill their leader's orders.

(b) At least how many colours of stones are needed to fulfill the order, regardless of what the friendships on the island are?
(Belarus 2001)

12. We are given an acute triangle ABC with circumcentre O. Let T be the circumcentre of AOC. Let M be the midpoint of AC. The points D and E lie on the lines AB and CB respectively in such a way that the angles MDB and MEB are equal to the angle ABC. Prove that the lines BT and DE are perpendicular.

13. A line passing through the centroid T of the triangle ABC meets the side AB at P and the side CA at Q. Prove that

$$4 \cdot PB \cdot QC \leq PA \cdot QA .$$

(R.B. Manfrino: Inequalities, 111/3.29, Spain 1998)

14. Prove that if integers x, y each greater than 1 satisfy $2x^2 - 1 = y^{15}$, then 5 divides x. Can you find such integers x and y?

(Russia 2004/05)

Next we turn to the 57th Latvian Mathematical Olympiad 2007 and the problems for Grade 11 and Grade 12 for the 3rd round. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for obtaining them for the *Corner*.

57th LATVIAN MATHEMATICAL OLYMPIAD 2007

Problems of the 3rd Round

Grade 11

1. For a positive integer n

(a) can the sums of digits of n and $n + 2007$ be equal?

(b) can the sums of digits of n and $n + 199$ be equal?

2. Do there exist three quadratic trinomials such that each of them has at least one root, but the sum of any two quadratic trinomials doesn't have any roots?

3. Each side of a sheet of paper is partitioned into 3 polygons. On one side one of the polygons is coloured white, another red, and the third one green. Prove that on the other side of the sheet it is possible to colour one of the polygons white, another red, and the third one green in such a way that at least one third of the area of the paper sheet is coloured with the same colour on both sides.

4. In $\triangle ABC$ the point K lies on median AM and $\angle BAC + \angle BKC = 180°$. Prove that $AB \cdot KC = AC \cdot KB$.

5. For a sequence of real numbers a_1, a_2, a_3, $\ldots$ we have $a_{11} = 4$, $a_{22} = 2$, and $a_{33} = 1$. In addition the relation

$$\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}}$$

holds for each n. Prove that

(a) $a_i \neq 0$ for each i,

(b) the sequence is periodic,

(c) $a_1^k + \cdots + a_{100}^k$ is a square of an integer for each positive integer k.

Grade 12

1. What can be the values of nonnegative real numbers a and b, if it is known that equations $x^2 + a^2x + b^3 = 0$ and $x^2 + b^2x + a^3 = 0$ have a common real root?

2. At each vertex of an n-gonal prism the number $+1$ or -1 is written, and the product of the numbers on each face of the prism is -1. Can $n = 4$? Can $n = 10$?

3. Solve the system of equations

$$\left\{\begin{array}{rcl}\sin^2 x + \cos^2 y & = & y^2\,,\\ \sin^2 y + \cos^2 x & = & x^2\,.\end{array}\right.$$

4. Two circles w_1 and w_2 intersect in points A and B. Line t_1 is drawn through point B with the other intersection point with w_1 being C and the other intersection point with w_2 being E. Line t_2 is drawn through point B the with other intersection point with w_1 being D and the other intersection point with w_2 being F. Point B lies between C and E and between D and F. Midpoints of segments CE and DF are denoted by M and N. Prove that triangles ACD, AEF, and AMN are similar.

5. The set of all positive integers is partitioned into several parts so that each positive integer belongs to exactly to one part and each part contains infinitely many integers. Can this be done so that one part contains a multiple of each positive integer? Give the answer if

(a) there are a finite number of parts,

(b) there are an infinite number of parts.

Next we turn to the Final Round of the Finnish National High School Mathematics Competition 2007. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for collecting them for our use.

FINNISH NATIONAL HIGH SCHOOL MATHEMATICS COMPETITION 2007

Final Round

Helsinki, February 2, 2007

1. Show that when a prime number is divided by 30, the remainder is either a prime number or 1. Is a similar claim true when the divisor is 60 or 90?

2. Determine the number of real roots of the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} \;=\; 0\,.$$

3. There are five points in the plane, no three of which are collinear. Show that some four of these points are the vertices of a convex quadrilateral.

4. The six offices of the City of Salavaara are to be conected to each other by a communication network which utilizes modern picotechnology. Each of the offices is to be connected to all the other ones by direct cable connections.

Three operators compete to build the connections, and there is a separate competition for every connection. When the network is finished one notices that the worst has happened: the systems of the three operators are incompatible. So the city must reject connections built by two of the operators, and these are to be chosen so that the damage is minimized. What is the minimal number of offices which still can be connected to each other, possibly through intermediate offices, in the worst possible case?

5. Show that there exists a polynomial $P(x)$ with integer coefficients such that the equation $P(x) = 0$ has no integer solutions but for each positive integer n there is an integer m such that $n \mid P(m)$.

To complete the problem sets for this number we give the IX Olimpiada Mathemático de Centroamérica y El Caribe 2007. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for collecting them for us.

IX OLIMPIADA MATHEMÁTICO DE CENTROAMÉRICA Y EL CARIBE 2007

First Day (Tuesday, June 5, 2007)

1. The OMCC is an annual mathematical competition. The ninth olympiad takes place in the year 2007. Which positive integers n divide the year in which the n^{th} olympiad takes place?

2. Let ABC be a triangle; D, E points on the sides AC, AB, respectively, such that the lines BD, CE, and the angle bisector of angle A concur at an interior point P of the triangle. Prove that there is a circle tangent to the four sides of the quadrilateral $ADPE$ if and only if $AB = AC$.

3. Let S be a finite set of integers. For any two integers p, q in S with $p \neq q$, there are integers a, b, c in S, not necessarily distinct and with $a \neq 0$, such that the polynomial $F(x) = ax^2 + bx + c$ satisfies $F(p) = F(q) = 0$. Determine the maximum number of elements the set S can have.

Second Day (Wednesday, June 6, 2007)

4. The inhabitants of a certain island speak a language in which every word can be written with the following letters: a, b, c, d, e, f, g. A word is said to *produce* another one if the second word can be formed from the first one by applying any of the following rules as many times as needed:

(i) Replace a letter by two letters according to one of the substitutions

$$a \to bc\,,\ b \to cd\,,\ c \to de\,,\ d \to ef\,,\ e \to fg\,,\ f \to ga\,,\ g \to ab\,.$$

(ii) If only one letter is between two letters that are the same, these two letters can be eliminated. For example, $dfd \to f$.

As another example, $cefed$ produces $bfed$, since $cafed \to cbcfed \to bfed$.

Prove that every word on this island produces any other word.

5. Given two nonnegative integers m and n with $m > n$, we say that m *ends in* n if one can erase some consecutive digits from the left of m to obtain n. For example, 329 ends in 9 and in 29. Determine how many three-digit numbers end in the product of their digits.

6. Let A and B be points on the circle Γ such that the lines PA and PB are tangent to Γ for an exterior point P. Let M be the midpoint of AB. The perpendicular bisector of AM intersects Γ at C which is interior to $\triangle ABP$, the line AC intersects the line PM at G, and the line PM intersects Γ at D, which is exterior to the triangle $\triangle ABP$. If BD is parallel to AC, prove that G is the point in which the medians of $\triangle ABP$ concur.

We pick up again with solutions to problems of the Croatian Mathematical Olympiad 2006, National Competition, 4$^{\text{th}}$ Grade [2009 : 293–294].

2. Let k and n be positive integers. Prove that

$$(n^4 - 1)(n^3 - n^2 + n - 1)^k \ + \ (n+1)n^{4k-1}$$

is divisible by $n^5 + 1$.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's solution.

We fix n and use induction on k.

Let $P(k) = (n^4-1)(n^3-n^2+n-1)^k + (n+1)n^{4k-1}$. We have that $(n^5+1) \mid P(1)$, since

$$\begin{aligned} P(1) &= (n^4-1)(n^3-n^2+n-1) + (n+1)n^3 \\ &= (n^5+1)(n^2-n+1)\,, \end{aligned}$$

Now assume that $(n^5+1) \mid P(k)$ for some integer $k > 0$. Then

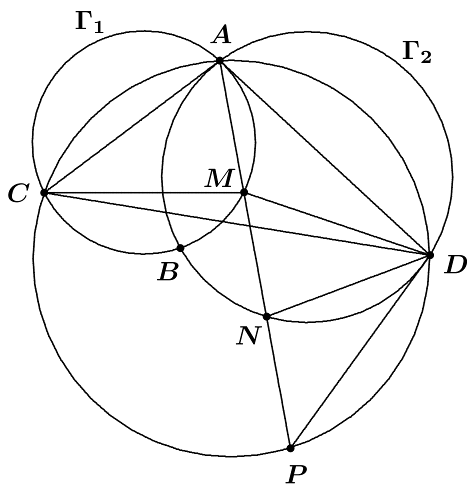
$$\begin{aligned} &P(k+1) \\ &\quad = (n^4-1)(n^3-n^2+n-1)^k(n^3-n^2+n-1) + (n+1)n^{4k+3} \\ &\quad = [P(k) - (n+1)n^{4k-1}](n^3-n^2+n-1) + (n+1)n^{4k+3} \\ &\quad = P(k)(n^3-n^2+n-1) + (n+1)n^{4k-1}(n^4-n^3+n^2-n+1) \\ &\quad = P(k)(n^3-n^2+n-1) + (n^5+1)n^{4k-1}\,, \end{aligned}$$

and consequently $(n^5+1) \mid P(k+1)$.

We conclude that $(n^5+1) \mid P(k)$ for each integer $k \geq 1$.

3. The circles Γ_1 and Γ_2 intersect at the points A and B. The tangent line to Γ_2 through the point A meets Γ_1 again at C and the tangent line to Γ_1 through A meets Γ_2 again at D. A half-line through A, interior to the angle $\angle CAD$, meets Γ_1 at M, meets Γ_2 at N, and meets the circumcircle of $\triangle ACD$ at P. Prove that $|AM| = |NP|$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution of Amengual Covas.



We use the theorem that the angle between a tangent and a chord of a circle is equal to the angle in the segment on the opposite side of the chord. Thus, $\angle ACM = \angle DAM = \angle DAN$ and $\angle CAN = \angle ADN$, making $\triangle ACM$ similar to $\triangle DAN$, so that $\dfrac{AM}{AC} = \dfrac{DN}{AD}$.

Since quadrilateral $ACPD$ is cyclic, on chord DA we have $\angle APD = \angle ACD$ and on chord CP we have

$$\begin{aligned} \angle PDC &= \angle CAP \\ &= \angle CAN = \angle ADN\,. \end{aligned}$$

Hence, $\angle PDC - \angle NDC = \angle ADN - \angle NDC$, that is, $\angle PDN = \angle CDA$. It follows that $\triangle NPD$ is similar to $\triangle ACD$, hence $\dfrac{NP}{AC} = \dfrac{DN}{AD}$.

From $\dfrac{AM}{AC} = \dfrac{DN}{AD}$ and $\dfrac{NP}{AC} = \dfrac{DN}{AD}$ we get $AM = NP$, as desired.

Next we turn to the Balkan Mathematical Olympiad 2006, Nicosia, Cyprus, Greece, given at [2009 : 294].

1. (*Greece*) Let a, b, and c be real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}\,.$$

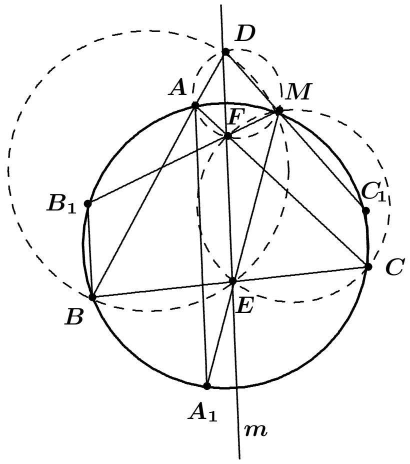
Solved by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Comǎneşti, Romania. Comments by Mohammed Aassila, Strasbourg, France; and Michel Bataille, Rouen, France. We give Bataille's comment.

The inequality does not hold if we take $a = b = -2$ and $c = 1$. Thus we assume that the intended hypothesis is a, b, $c > 0$. If this is actually the case, then the problem is not new: it is problem 2362 [1998 : 304; 1999 : 375].

2. (*Greece*) Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The lines through points A, B, C and parallel to the line m intersect the circumcircle of triangle ABC again at the points A_1, B_1, C_1, respectively. Prove that the lines A_1E, B_1F, and C_1D are concurrent.

Solution by Michel Bataille, Rouen, France.

More generally, the result holds whenever m is any transversal of $\triangle ABC$ (see the figure).



Since D, E, F are points on the lines AB, BC, CA, distinct from the vertices, Miquel's theorem tells us that the circumcircles of $\triangle ADF$, $\triangle BDE$, and $\triangle CEF$ have a common point, M. Applying the same theorem to the points B, C, E on the lines AD, AF, DF, respectively, we see that M must be a point on the circumcircle of $\triangle ABC$.

Now, denoting by $\angle(\ell, \ell')$ the directed angle between the lines ℓ and ℓ', we have

$$\begin{aligned}\angle(MA_1, ME) &= \angle(MA_1, m) + \angle(m, ME)\\ &= \angle(A_1M, A_1A) + \angle(EF, EM)\\ &= \angle(CM, CA) + \angle(CF, CM),\end{aligned}$$

where we have used the fact that A, C, M, A_1 are concyclic and also the fact that E, C, F, M are concyclic.

Thus, $\angle(MA_1, ME) = 0$, which means that A_1E passes through M. Similarly, B_1F and C_1D pass through M, and the result follows.

3. (*Romania*) Find all triples of positive rational numbers (m, n, p) such that each of the numbers

$$m + \frac{1}{np}, \qquad n + \frac{1}{pm}, \qquad p + \frac{1}{mn}$$

is an integer.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We use Bataille's write-up.

We show that the solutions for (m, n, p) are the triples $(\frac{1}{2}, \frac{1}{2}, 4)$, $(\frac{1}{2}, 1, 2)$ and their permutations, and the triple $(1, 1, 1)$.

It is easily checked that these triples are solutions.

Conversely, for a solution (m,n,p) define positive integers a, b, c by

$$a = m + \frac{1}{np}\,, \qquad b = n + \frac{1}{pm}\,, \qquad c = p + \frac{1}{mn}\,.$$

Since $mnp+1 = anp = bpm = cmn$, we have $abc(mnp)^2 = (mnp+1)^3$. Upon setting $mnp = \frac{u}{v}$ where u, v are coprime positive integers, we have $abcu^2v = (u+v)^3$.

Now, any prime number dividing u^2v must divide u or v, hence cannot divide $u+v$ (since otherwise it would divide both u and v). Thus, u^2v and $(u+v)^3$ are coprime and since $abc = \frac{(u+v)^3}{u^2v}$, it follows that $(u+v)^3 = abc$ and $u^2v = 1$. Thus, $u = v = 1$, $abc = 8$, $mnp = 1$.

Assuming $a \le b \le c$, the only possibilities for (a,b,c) are $(1,1,8)$, $(1,2,4)$, $(2,2,2)$ which lead to $(\frac{1}{2},\frac{1}{2},4)$, $(\frac{1}{2},1,2)$, $(1,1,1)$ for (m,n,p). The result follows.

Now we turn to solutions from our readers to problems of the Finnish Mathematical Olympiad 2006, Final Round, given at [2009 : 295].

1. Determine all pairs (x,y) of positive integers such that

$$x + y + xy \;=\; 2006\,.$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's write-up.

Adding 1 to both sides of the given relation, we obtain

$$(1+x)(1+y) \;=\; 2007\,.$$

The prime factorization of 2007 is $3^2 \cdot 223$, so 2007 has exactly six positive divisors: 1, 3, 9, 223, 669, 2007. Consequently, the only admissible products $(1+x)(1+y)$ are $3\cdot 669$, $9\cdot 223$, $223\cdot 9$ and $669\cdot 3$. Thus, the only solutions (x,y) are $(2,668)$, $(8,222)$, $(222,8)$, and $(668,2)$.

2. For all real numbers a, prove that

$$3\left(1+a^2+a^4\right) \;\ge\; \left(1+a+a^2\right)^2$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Curtis.

Since $a^4 + a^2 + 1 = (a^2 + a + 1)(a^2 - a + 1)$, the claimed inequality is equivalent to

$$2(a-1)^2(a^2+a+1) \ \geq \ 0\,,$$

which is true, as $a^2 + a + 1 = (a + \frac{1}{2})^2 + \frac{3}{4}$ for all real numbers a.

3. The numbers p, $4p^2 + 1$, and $6p^2 + 1$ are primes. Determine p.

Solved by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give Alt's write-up.

First consider primes $p = 2$, 3, and 5.
If $p = 2$ then $4p^2 + 1 = 17$ is prime, but $6p^2 + 1 = 25$ is not prime.
If $p = 3$ then $4p^2 + 1 = 37$ is prime, but $6p^2 + 1 = 55$ is not prime.
If $p = 5$ then $4p^2 + 1 = 101$ and $6p^2 + 1 = 151$ are both primes.
Now let p be a prime greater than 5. Since

$$\begin{aligned} 4p^2+1 &= 5p^2-(p^2-1) \ \equiv \ -(p^2-1) \pmod 5\,, \\ 6p^2+1 &= 5(p^2-p-1)+(p+2)(p+3) \\ &\equiv (p+2)(p+3) \pmod 5 \end{aligned}$$

and

$$-(p-1)p(p+1)(p+2)(p+3) \ \equiv \ 0 \pmod 5\,,$$

it follows that

$$p(4p^2+1)(6p^2+1) \ \equiv \ 0 \pmod 5\,.$$

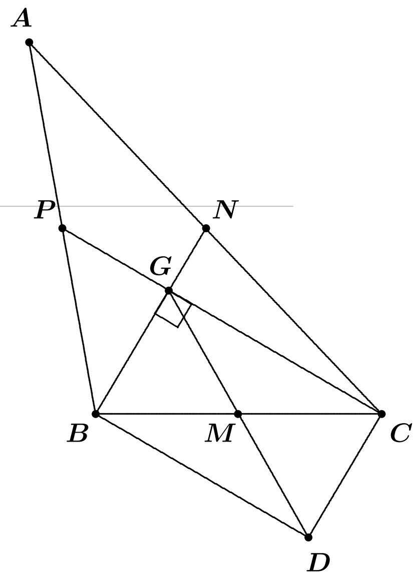
Then $(4p^2+1)(6p^2+1) \equiv 0 \pmod 5$, because p and 5 are coprime. Hence, $4p^2+1$ or $6p^2+1$ is a composite number, because each is greater than 5 and one of them is divisible by 5.

Thus, the only solution to the problem is $p = 5$.

4. Prove that if two medians of a triangle are perpendicular, then the triangle whose sides are congruent to the medians of the original triangle is a right triangle.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give the two solutions by Amengual Covas.

First Solution: Let ABC be the given triangle, and let M, N, and P be the midpoints of the sides BC, CA, and AB, respectively. Let G be the centroid of $\triangle ABC$ and let D be symmetric to G with respect to M.



Since segments BC and GD bisect each other, the quadrilateral $BDCG$ is a parallelogram, so we have

$$\begin{aligned} GD &= 2 \cdot GM = \frac{2}{3}AM\,, \\ DC &= BG = \frac{2}{3}BN\,, \\ CG &= \frac{2}{3}CP\,. \end{aligned}$$

Hence, $\triangle CGD$ is similar to the triangle whose sides are congruent to the medians of $\triangle ABC$.

Thus, it suffices to prove that $\triangle CGD$ is a right triangle if two medians of $\triangle ABC$ are perpendicular.

Without loss of generality take $BN \perp CP$. Then $\angle BGC = 90^\circ$, so $BDCG$ is a rectangle.

Then $\angle GCD = 90^\circ$ and $\triangle GCD$ is a right triangle, as required.

Second Solution: Let m_a, m_b, m_c be the medians of the triangle to the sides a, b, c, respectively. Assume without loss of generality that m_b and m_c are perpendicular and we use the fact that this is equivalent to the relation $b^2 + c^2 = 5a^2$ to obtain

$$\begin{aligned} m_b^2 + m_c^2 &= \frac{1}{4}(2c^2 + 2a^2 - b^2) + \frac{1}{4}(2a^2 + 2b^2 - c^2) \\ &= a^2 + \frac{1}{4}(b^2 + c^2) = \frac{9}{4}a^2 = \frac{1}{4}(2(b^2 + c^2) - a^2) = m_a^2\,. \end{aligned}$$

The conclusion now follows from the converse of the Pythagorean Theorem.

To complete this *Corner* we look at a solution to a problem of the Estonian Mathematical Olympiad 2005–2006, Final Round [2009 : 375–376].

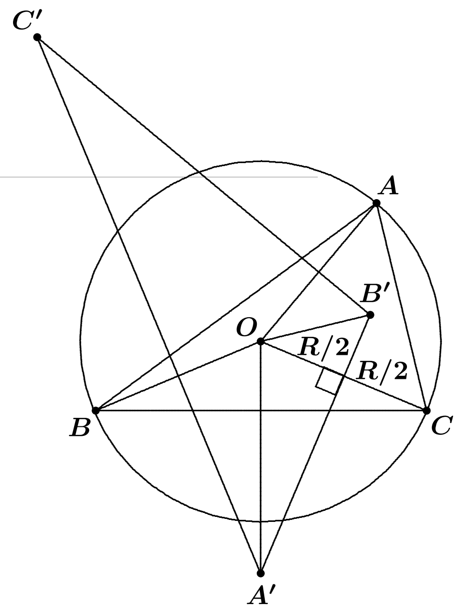
4. The acute triangle ABC has circumcentre O and triangles BCO, CAO, and ABO have circumcentres A', B', and C', respectively. Prove that the area of triangle ABC does not exceed the area of triangle $A'B'C'$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give the write-up of Amengual Covas.

Let R be the circumradius of $\triangle ABC$. Since $\triangle ABC$ is acute, O is inside $\triangle ABC$, so $\angle A'OC = \angle BAC$, $\angle COB' = \angle ABC$, and $\angle AOC' = \angle BCA$. We also have that $A'B'$ is the perpendicular bisector of segment OC.

Hence, $OA' = \dfrac{R/2}{\cos\angle A'OC} = \dfrac{R}{2\cos A}$ and $OB' = \dfrac{R/2}{\cos\angle COB'} = \dfrac{R}{2\cos B}$.

Therefore, with brackets denoting the area of the enclosed figure,



$$\begin{aligned}[OA'B'] &= \frac{1}{2}OA'\cdot OB'\cdot\sin\angle A'OB' \\ &= \frac{1}{2}\cdot\frac{R}{2\cos A}\cdot\frac{R}{2\cos B}\cdot\sin(A+B) \\ &= \frac{1}{8}R^2\frac{\sin C}{\cos A\cos B},\end{aligned}$$

since $A + B = 180^\circ - C$.

Similarly, we obtain

$$\begin{aligned}[OB'C'] &= \frac{1}{8}R^2\frac{\sin A}{\cos B\cos C}, \\ [OC'A'] &= \frac{1}{8}R^2\frac{\sin B}{\cos C\cos A}.\end{aligned}$$

The following are known:

$$\begin{aligned}\sin 2A + \sin 2B + \sin 2C &= 4\sin A\sin B\sin C, & (1)\\ [ABC] &= 2R^2\sin A\sin B\sin C, & (2)\\ \cos A\cos B\cos C &\le \frac{1}{8}, & (3)\end{aligned}$$

(see, respectively, Formula 120, p. 105 and Formula 254, p. 178 of *Relations entre les éléments d'un triangle*, Librairie Nony, Paris, 1893, and item 2.23, p. 25, of O. Bottema et al., *Geometric Inequalities*, Groningen, 1969).

We thus obtain

$$\begin{aligned}[A'B'C'] &= [OA'B'] + [OB'C'] + [OC'A'] \\ &= \frac{1}{8}R^2\cdot\frac{\sin A\cos A + \sin B\cos B + \sin C\cos C}{\cos A\cos B\cos C} \\ &= \frac{1}{16}R^2\cdot\frac{\sin 2A + \sin 2B + \sin 2C}{\cos A\cos B\cos C} \\ &= \frac{1}{4}R^2\cdot\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C} = \frac{1}{8}\frac{[ABC]}{\cos A\cos B\cos C} \ge [ABC],\end{aligned}$$

as desired. Equality occurs if and only if $\triangle ABC$ is equilateral.

That completes another (somewhat short!) *Corner*, and we're aiming to clear a backlog of material in the next issue. As always we welcome your nice solutions and generalizations.