Problem of the Month

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Are you someone who likes to memorize formulas or someone who prefers to remember how to derive them?

Problem 1 (2002 Canadian Open Mathematics Challenge) A regular pentagon is a five-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, TREND is a regular pentagon, PEA is an equilateral triangle, and OPEN is a square. Determine the size of $\angle EAR$.

When working with polygons, there is some important information that you need to be able to remember or derive. One of the most important pieces of information that you need to know to do this problem is that the sum of the angles in a pentagon is $540^\circ$. Some of us are able to remember this sort of thing. How could you figure it out if you didn't know this?

Imagine connecting $R$ to $N$ and $D$. This divides pentagon TREND into three triangles. What is the sum of the angles in each of these triangles? It is $180^\circ$. Therefore, the sum of the angles in the entire pentagon is $3 \times 180^\circ$ or $540^\circ$. Can you see why this is true? In a regular pentagon, all angles are equal, so each must be equal to one-fifth of their total. We can now solve Problem 1.

Solution to Problem 1. To determine $\angle EAR$, we first look at the angles around $E$. We know that $\angle AER + \angle REN + \angle NEP + \angle PEA = 360^\circ$.

Since $\angle PEA$ is an angle in an equilateral triangle, $\angle PEA = 60^\circ$. Since $\angle NEP$ is an angle in a square, $\angle NEP = 90^\circ$. Since $\angle REN$ is an angle in a regular pentagon, $\angle REN = \frac{1}{5}(540^\circ) = 108^\circ$.

Therefore,

$$\angle AER = 360^\circ - \angle REN - \angle NEP - \angle PEA = 360^\circ - 108^\circ - 90^\circ - 60^\circ = 102^\circ.$$ 

Now since $PEA$ is an equilateral triangle, OPEN is a square, and TREND is a regular pentagon, then their side lengths must all be the same, since OPEN and TREND share a side, and since OPEN and PEA share a side. In particular, $AE = ER$.

Therefore, $\triangle ARE$ is isosceles, and so

$$\angle EAR = \frac{1}{2}(180^\circ - \angle AER) = \frac{1}{2}(180^\circ - 102^\circ) = 39^\circ.$$
What about a general polygon with \( n \) sides? There are three types of angles that can be important:

- **Interior Angles**: Some of you may remember that the sum of the interior angles in such a polygon is \((n - 2)180^\circ\). Can you see how to derive this formula using a similar method to method above for the pentagon? Pick one of the \( n \) vertices and call it \( A \). Vertex \( A \) is already connected to the vertex on either side. Connect it to the \( n - 3 \) remaining vertices. (These are all of the \( n \) vertices, except for \( A \) and its two adjacent vertices.) This forms \( n - 2 \) triangles, each of which has angles that add to \( 180^\circ \). Therefore, the sum of the angles in the polygon is \((n - 2)180^\circ\). If the polygon is regular, then all \( n \) interior angles are equal and so each must equal \( \frac{(n - 2)180^\circ}{n} \).

- **Exterior Angles**: Sometimes it is much simpler (and more useful) to remember that the sum of the exterior angles in any convex polygon is \( 360^\circ \). The exterior angles are formed by extending each side and looking at the angles formed outside the polygon. You can see a picture in Solution 1 below. You can derive this total using the sum of the interior angles above. Can you come up with another way of deriving this directly by trying to combine all of the exterior angles visually in some way? Try drawing a pentagon, for example, identifying the exterior angles and seeing how you might be able to move all of these angles together.

- **Central Angles**: By central angle, we mean the angle formed at the centre of the polygon by the two ends of each side. The total central angle must be \( 360^\circ \), and if the polygon is regular, this is divided into \( n \) equal pieces each equal to \( \frac{360^\circ}{n} \).

Here is a second problem involving a polygon with an unknown number of sides, and two solutions to this problem. Solution 1 uses the exterior angles of the polygon, and Solution 2 uses the central angles.

**Problem 2** (2009 Sun Life Financial Canadian Open Mathematics Challenge) A polygon is called regular if all of its sides are equal in length and all of its interior angles are equal in size. In the diagram, a portion of a regular polygon is shown. If \( \angle ACD = 120^\circ \), how many sides does the polygon have?

**Solution 1 to Problem 2.** Suppose that the polygon has \( n \) sides. Extend \( CB \) outside of the polygon to a point \( E \). Since the sum of the exterior angles in a polygon is \( 360^\circ \), then
\(\angle ABE = \frac{360^\circ}{n}\), because there will be \(n\) equal exterior angles. Thus, \(\angle ABC = 180^\circ - \frac{360^\circ}{n}\) and this will also be the measure of \(\angle BCD\), since the polygon is regular.

Since the polygon is regular, then \(AB = BC\), so \(\triangle ABC\) is isosceles, which means that we have \(\angle BAC = \angle BCA\). Therefore,

\[
\angle BCA = \frac{1}{2}(180^\circ - \angle ABC) = \frac{1}{2}\left(180^\circ - \left(180^\circ - \frac{360^\circ}{n}\right)\right) = \frac{180^\circ}{n}
\]

But \(\angle BCD = \angle BCA + \angle ACD\), so

\[
180^\circ - \frac{360^\circ}{n} = \frac{180^\circ}{n} + 120^\circ \iff 60^\circ = \frac{540^\circ}{n} \iff n = 9.
\]

Therefore, the polygon has 9 sides.

**Solution 2 to Problem 2.** Suppose that the polygon has \(n\) sides. Let \(O\) be the centre of the polygon. Join \(O\) to each of \(A, B, C,\) and \(D\). Since the polygon is regular, then the angle subtended at \(O\) by each of the \(n\) sides will be equal, and these angles all add to \(360^\circ\). Since there are \(n\) equal central angles, then \(\angle AOB = \angle BOC = \angle COD = \frac{360^\circ}{n}\). This also tells us that \(\angle AOC = \angle AOB + \angle BOC = 2 \cdot \frac{360^\circ}{n} = \frac{720^\circ}{n}\).

Since the polygon is regular, then we have that \(OA = OC = OD\), which tells us that \(\triangle AOC\) and \(\triangle COD\) are both isosceles. Thus,

\[
\angle ACO = \frac{1}{2}(180^\circ - \angle AOC) = \frac{1}{2}\left(180^\circ - \frac{720^\circ}{n}\right) = 90^\circ - \frac{360^\circ}{n}
\]

and

\[
\angle DCO = \frac{1}{2}(180^\circ - \angle COD) = \frac{1}{2}\left(180^\circ - \frac{360^\circ}{n}\right) = 90^\circ - \frac{180^\circ}{n}.
\]

Now, \(\angle ACD = \angle ACO + \angle DCO\), so

\[
120^\circ = 90^\circ - \frac{360^\circ}{n} + 90^\circ - \frac{180^\circ}{n} \iff \frac{540^\circ}{n} = 60^\circ \iff n = 9.
\]

Therefore, the polygon has 9 sides.

Problems involving polygons arise frequently. Try to remember the facts and strategies here, but more importantly, remember how to rederive these facts.

What other solutions to this problem can you find?