THE OLYMPIAD CORNER

No. 288

R.E. Woodrow

We begin this number with a collection of examination sets from Peru that were sent to us by Hugo Luyo, trainer for the Mathematics Competitions in Peru. They were translated with the assistance of Leda Sanchez, Executive Assistant to the Vice-Provost (International) at the University of Calgary. My thanks go to both of them for making these available for the Corner.

OLIMPIADA NACIONAL ESCOLAR DE MATEMÁTICA 2009
Level 1
November, 2009

1. If $P$, $E$, $R$, and $U$ are pairwise distinct nonzero digits such that $\text{PER} + \text{PRU} + \text{PUE} + 2009 = \text{PERU}$, find all the values that $P + E + R + U$ can take.

2. Saladin (‘unlucky’ one) and Suertudo (‘lucky’ one) are playing with a die. Each time a player rolls a 6 he gets one point. Suertudo is so lucky that he always gets at least one point in every five consecutive rolls. On the other hand Saladin gets at most one point in every six consecutive rolls. The first one to accumulate four points wins, with Suertudo starting the game and players throwing the die alternately.

(a) Show a sequence of play in which Suertudo wins.

(b) Show a sequence of play in which Saladin wins.

3. Andrés and Bertha play on a $4 \times 4$ table with tetrominoes as shown.

```
+---+---+---+---+
|   |   |   |   |
+---+---+---+---+
|   |   |   |   |
+---+---+---+---+
|   |   |   |   |
+---+---+---+---+
```

Andrés begins the game placing 4 tetrominoes of the same shape on the table without overlaps and leaving no empty space. Then Bertha must write on each square of the table one of the numbers 1, 2, 3 or 4 in such a way that each row and column has no two numbers repeated. Bertha wins if each of the tetrominoes on the table covers 4 distinct numbers.

(a) Show that Bertha can always win the game.

(b) Andrés fills the table with 4 tetrominoes where at least 2 are different. Is it true that in this situation, playing with the same rules, Bertha can always win?
4. Let $k > 1$ be an integer. A positive integer $N$ is a bimultiple of $k$ if $N$ is a multiple of $k$ and when the order of the digits of $N$ is reversed, then the resulting number is also a multiple of $k$. Mario writes a 7-digit number on the board, all digits nonzero. Show that one can erase three of the digits of $N$ so that the remaining 4-digit number is a bimultiple of some $k > 1$.

**Level 2**

1. Let $a$, $b$, $c$, and $d$ be four integers whose sum is 0. Let

$$M = (bc - ad)(ac - bd)(ab - cd).$$

Show that there is an integer $P$ such that $P^2 = M$.

2. An equilateral triangle of side length 6 is divided into 36 small equilateral triangles of side length 1. The resulting chart is covered by $m$ markers of type $A$ and $n$ markers of type $B$ without doubling or leaving empty spaces. Markers of type $A$ are formed by two equilateral triangles of side length 1 and markers of type $B$ are formed from 3 small triangles, as shown in the figure. Determine all possible values of $m$.

3. For each positive integer $n$ let $d$ be the largest divisor of $n$ with $d \leq \sqrt{n}$, and define $a_n = \frac{n}{d} - d$. Show that in the sequence $a_1, a_2, a_3, \ldots$ each nonnegative integer $k$ appears infinitely often.

4. On a circle $N \geq 5$ points are marked so that the $N$ arcs formed have the same length. A coin is placed on each point, and Ricardo and Tomás play a game with the following rules:

- They play alternately.
- Ricardo starts.
- A player may take a coin only if that coin forms an acute triangle with at least two other coins.

A player loses when he cannot take any coin during his turn.

Does either player have a winning strategy? If so, what is it?

**Level 3**

1. For each positive integer $N$ let $c(n)$ be the number of decimal digits of $N$. Let $A$ be a set of positive integers such that if $a$ and $b$ are two distinct elements of $A$, then $c(a + b) + 2 > c(a) + c(b)$. Find the largest number of elements that $A$ can have.
2. In a quadrilateral $ABCD$, a circle is drawn that is tangent to the sides $AB$, $BC$, $CD$, and $DA$ at the points $M$, $N$, $P$, and $Q$ respectively. Prove that if

$$AM \cdot CP = BN \cdot DQ,$$

then $ABCD$ can be inscribed in a circle.

3. (a) There are 8 points placed on a circle. We say that Juliana performs "operation $T$" if she chooses 3 such points and paints the sides of the triangle they determine in such a way that each painted triangle has at most one vertex in common with a previously painted triangle.

What is the greatest number of operations $T$ that Juliana can make?

(b) If in part (a) you have 7 points instead of 8 points, then what is the greatest number of operations $T$ Juliana can make?

4. Let $n$ be a positive integer. A rectangular $4 \times n$ array is tiled by $2n$ dominoes, and each point lying underneath a corner of a domino is painted red. What is the smallest number of red points that can be obtained?

Next we give the problems of the selection test for the Swiss Olympiad Team for 2006, Sélection OIM 2006. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for obtaining them for us.

**SÉLECTION OIM 2006**

29 et 30 avril 2006, 13 et 14 mai 2006

**Premier jour — 4.5 heures**

1. Dans le triangle $ABC$ soit $D$ le milieu du côté $BC$ et $E$ la projection de $C$ sur $AD$. On suppose que $\angle ACE = \angle ABC$. Montrer que le triangle $ABC$ est soit isocèle, soit rectangle.

2. Soit $n \geq 5$ un nombre entier. Déterminer le plus grand entier $k$ tel qu'il existe un $n$-gone avec exactement $k$ angles intérieurs de $90^{\circ}$. (Le $n$-gone n’a pas besoin d’être convexe.)

3. Soit $n$ un nombre naturel. Chacun des nombres $\{1, 2, \ldots, n\}$ est coloré soit en blanc, soit en noir. On choisit un nombre et on change sa couleur, tout comme la couleur des nombres avec lesquels il a un diviseur commun. Au départ tous les nombres sont blancs. Pour quels $n$ peut-on arriver à une configuration où tous les nombres sont noirs en un nombre fini de changements?
Deuxième jour — 4.5 heures

4. Soient $1 = d_1 < d_2 < \cdots < d_k = n$ les diviseurs positifs de $n$. Déterminer tous les $n$ tels que

$$2n = d_n^2 + d_o^2 - 1.$$ 

5. Soit $ABC$ un triangle et $D$ un point à l’intérieur. Soit $E$ un point sur la droite $AD$ différent de $D$. Soient $\omega_1$ et $\omega_2$ les cercles circonscrits des triangles $BDE$, respectivement $CDE$. Soit $F$ et $G$ les intersections intérieures respectives de $\omega_1$ et $\omega_2$ avec le côté $BC$. Soit $X$ le point d’intersection de $DG$ avec $AB$ et $Y$ le point d’intersection de $DF$ avec $AC$. Montrer que $XY$ est parallèle à $BC$.

6. Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que pour tout $x, y \in \mathbb{R}$ on ait l’égalité suivante

$$f(f(x) - y^2) = f(x)^2 - 2f(x)y^2 + f(f(y)).$$

Troisième jour — 4.5 heures

7. Les trois zéros réels du polynôme $P(x) = x^3 - 2x^2 - x + 1$ sont $a > b > c$. Trouver la valeur de l’expression

$$a^2b + b^2c + c^2a.$$ 

8. On aligne les nombres 1, 2, ..., 2006 le long d’un cercle dans un ordre quelconque. Un coup consiste à échanger deux nombres voisins. Après un nombre fini de coups tous les nombres se trouvent diamétralement opposés à leur position de départ. Montrer qu’au moins une fois on a échangé deux nombres dont la somme valait 2007.


Quatrième jour — 4.5 heures

10. Soient $a, b, c$ des nombres réels positifs avec $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Démontrer l’inégalité suivante :

$$\sqrt{ab + c} + \sqrt{bc + a} + \sqrt{ca + b} \geq \sqrt{abc} + \sqrt{a} + \sqrt{b} + \sqrt{c}.$$ 

11. Trouver tous les nombres naturels $k$ tels que $3^k + 5^k$ est la puissance d’un nombre naturel d’exposant $\geq 2$. 
12. Un aéroport contient 25 terminaux qui sont deux à deux reliés par des tunnels. Il y a exactement 50 tunnels principaux qui peuvent être parcourus dans les deux sens, les autres sont à sens unique. Un groupe de quatre terminaux est appelé connexe si de chacun d'entre eux on peut accéder à tous les autres en utilisant uniquement les six tunnels qui les relient. Déterminer le nombre maximal de groupes de terminaux connexes.

Next we present the First Round of the 17th Japanese Mathematical Olympiad, written January 8, 2007. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for us.

17th JAPANESE MATHEMATICAL OLYMPIAD
First Round
January 8, 2007

1. Let $ABCD$ be a convex quadrilateral with $AB = 3$, $BC = 4$, $CD = 5$, $DA = 6$, and $\angle ABC = 90^\circ$. Find the area of $ABCD$.

2. Determine the ten's digit of $11^{12^{13}}$ (the $12^{13}$th power of 11, not the $13^{th}$ power of $11^{12}$).

3. The segment $AB$ and point $P$ lie in a plane, $AB$ is 7 units long, and $P$ is 3 units away from the line $AB$. Find the smallest possible value of $AP \cdot BP$.

4. The ten's digit of the 4-digit integer $n$ is nonzero. If we take the first 2 digits and the last 2 digits of $n$ as two 2-digit integers, then their product is a divisor of $n$. Determine all $n$ with this property.

5. Three rectangles lie in a plane such that any two of them have parallel sides. They divide the plane into several regions. Determine the maximum possible number of such regions. (The area contained in no rectangles is one of the regions, so a single rectangle divides the plane into two regions.)

6. We have 15 cards numbered 1, 2, ..., 15. How many ways are there to choose some (at least 1) cards so that all numbers on these cards are larger than or equal to the number of cards chosen?

7. In how many ways can 100 be written as a sum of nonnegative powers of 3? (Two ways are the same if they differ only in the order of the powers.)

8. How many ways are there to cut a cube $S$ into tetrahedra $T_1$, $T_2$, ..., $T_k$ with the following properties?

   (i) Every vertex of $T_1$, $T_2$, ..., $T_k$ is one of the vertices of $S$.

   (ii) For every $i \neq j$, the intersection of $T_i$ and $T_j$ is a common face, a common edge, a common vertex, or empty.
9. How many pairs of integers \((a, b)\) satisfy \(a^2b^2 = 4a^5 + b^3\)?

10. A set of cards with positive integers on them is given, and the sum of these integers is 2007. For each integer \(k = 1, 2, \ldots, 2006\) there is only one way to choose some of these cards so that the sum of the numbers on them is \(k\). (Cards with the same number are considered identical). How many such sets of cards are there?

11. In a mathematical competition, gold medals are given to \(\left\lfloor \frac{n}{a} \right\rfloor\) people, silver medals to \(\left\lfloor \frac{n}{b} \right\rfloor\) people, and bronze medals to \(\left\lfloor \frac{n}{c} \right\rfloor\) people \((a \geq b \geq c\) are integer constants and \(n\) is the number of participants). No one gets two or more medals. Determine all triples \((a, b, c)\) with the following property: For all integers \(k \geq 3\), there are exactly two values for \(n\) such that the number of people without medals is \(k\). Here \([x]\) is the largest integer that does not exceed \(x\).

12. There is a village with a population of 2007. This village has no name. You are God of this village and you want villagers to decide the name of this village. Every villager has one idea of the village’s name.

   Each villager can send a letter to each villager (including himself), and every villager can send any number of letters every day. Letters are collected in the evening and delivered at once the next morning every day. The villager who sends the letter can decide to whom the letter should be delivered. Each villager can send a letter to tell the idea of the name of the village to God only one time. This idea does not need to be the same as the idea which he and the other villagers had thought at first. And every villager’s action is only writing a letter.

   Every villager can be classified into an honest person or a liar. You and every villager do not know who is an honest person, and who is a liar. But you know that the number of liars is less than or equal to \(T\), and there is one honest person at least in this village.

   You can give instructions to every villager only once at noon of one day. An honest person necessarily follows the instruction, but you do not know if a liar follows the instruction. Find the maximum \(T\) for which there exists an instruction which fulfills the conditions below.

   (i) At last, every honest person sends a letter to God and every honest person sends the same idea of the village’s name.

   (ii) If every honest person had thought the same idea of the name of the village at first, every honest person sends this idea to God.

And to complete that set we give the Final Round of the 17th Japanese Mathematical Olympiad, written February 11, 2007. Thanks again to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for obtaining them.
17th JAPANESE MATHEMATICAL OLYMPIAD
Final Round
February 11, 2007 (Time: 4 hours)

1. Let \( n \) be a positive integer. Two people \( P, Q \) play a game in which they alternately call integers \( m \) with \( 1 \leq m \leq n \). Player \( P \) calls the first number, and once a number is called it cannot be called again. The game ends when all \( n \) numbers have been called. If the sum of the numbers that \( P \) has called is divisible by 3, then \( P \) wins, otherwise \( Q \) wins. Find all \( n \) such that \( P \) can win the game no matter what \( Q \) does.

2. Find all functions \( f \), defined on the positive real numbers and taking real values, such that

\[
f(x) + f(y) \leq \frac{f(x + y)}{2} \quad \text{and} \quad \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x + y)}{x + y}
\]

for all positive real numbers \( x \) and \( y \).

3. Let \( \Gamma \) be the circumcircle of triangle \( ABC \). Let \( \Gamma_A \) be the circle tangent to \( AB, AC \) and tangent internally to \( \Gamma \), and let \( \Gamma_B \) and \( \Gamma_C \) be defined similarly. Let \( \Gamma_A, \Gamma_B, \Gamma_C \) be tangent to \( \Gamma \) at \( A', B', C' \), respectively. Prove that the lines \( AA', BB', CC' \) are concurrent.

4. A band of width \( d \) in the plane is a set of points whose distance from a line is at most \( \frac{d}{2} \). Any three of the points \( A, B, C, D \) in the plane lie in a band of width 1. Prove that all of them lie in a band of width \( \sqrt{2} \).

5. Let \( \lfloor r \rfloor \) be the largest integer not exceeding the real number \( r \). For real positive numbers \( x \) let \( A(x) = \{ \lfloor nx \rfloor : n \text{ is a positive integer} \} \). Find all irrational numbers \( \alpha > 1 \) with the property that whenever a positive real number \( \beta \) satisfies \( A(\alpha) \supset A(\beta) \), then \( \frac{\beta}{\alpha} \) is an integer.

Next we give our readers' solutions to problems of the 32nd Austrian Mathematical Olympiad Regional Competition (Qualifying Round) given at [2009 : 290–291].

1. Let \( 0 < x < y \) be real numbers and

\[
H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad \text{and} \quad Q = \sqrt{\frac{x^2+y^2}{2}}
\]

be the harmonic, geometric, arithmetic, and quadratic means of \( x \) and \( y \), respectively. It is well known that \( H < G < A < Q \) holds. Order the intervals \( [H, G], [G, A], \) and \( [A, Q] \) by length.
Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We use the write-up of Curtis.

By homogeneity, we may assume that \( y = 1 \) and \( x = t \), with \( 0 < t < 1 \), and we consider the functions

\[
H(t) = \frac{2t}{1 + t}, \quad G(t) = \sqrt{t}, \quad A(t) = \frac{t + 1}{2}, \quad Q(t) = \frac{\sqrt{t^2 + 1}}{2}.
\]

Set \( u(t) = G(t) - H(t) \), \( v(t) = A(t) - G(t) \), and \( w(t) = Q(t) - A(t) \). For each fixed \( t \in (0, 1) \) we claim that \( u(t) < w(t) < v(t) \), so that

\[
\text{length}([H, G]) < \text{length}([A, Q]) < \text{length}([G, A]).
\]

Case 1. \( u(t) < w(t) \)

The inequality is successively equivalent to

\[
\sqrt{t} - \frac{2t}{1 + t} < \frac{\sqrt{t^2 + 1}}{2} - \frac{t + 1}{2},
\]

\[
\sqrt{t} + \frac{(t - 1)^2}{2(t + 1)} < \frac{\sqrt{t^2 + 1}}{2},
\]

\[
t + \frac{(t - 1)^4}{4(t + 1)^2} + 2\sqrt{t} \cdot \frac{(t - 1)^2}{2(t + 1)} < \frac{t^2 + 1}{2},
\]

\[
\sqrt{t} < \frac{t^2 + 6t + 1}{4(t + 1)},
\]

\[
0 < \frac{(\sqrt{t} - 1)^4}{4(t + 1)},
\]

which is clearly true.

Case 2. \( w(t) < v(t) \)

The inequality is successively equivalent to

\[
\sqrt{\frac{t^2 + 1}{2}} - \frac{t + 1}{2} < \frac{t + 1}{2} - \sqrt{t},
\]

\[
\sqrt{\frac{t^2 + 1}{2}} < (t + 1) - \sqrt{t},
\]

\[
\frac{t^2 + 1}{2} < t^2 + 3t + 1 - 2\sqrt{t}(t + 1),
\]

\[
0 < t^2 - 4t\sqrt{t} + 6t - 4\sqrt{t} + 1,
\]

\[
0 < (\sqrt{t} - 1)^4.
\]

Since this last inequality holds, the claim is proved.
2. Let \( n > 1 \) be an integer and \( a \) a real number. Determine all real solutions \((x_1, x_2, \ldots, x_n)\) of the following system of equations:

\[
\begin{align*}
  x_1 + ax_2 &= 0, \\
  x_2 + a^2x_3 &= 0, \\
  x_3 + a^3x_4 &= 0, \\
    &\vdots \\
  x_{n-1} + a^{n-1}x_n &= 0, \\
  x_n + a^nx_1 &= 0.
\end{align*}
\]

_Solved by Mohammed Aassila, Strasbourg, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Aassila._

Clearly \((x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\) is a solution.

So we suppose that \((x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)\), in which case \(a \neq 0\) and \(x_i \neq 0\) for each \(i\). Then from the system of equations we have

\[
\frac{x_1}{x_2} = -a, \quad \frac{x_2}{x_3} = -a^2, \quad \ldots, \quad \frac{x_n}{x_1} = -a^n. 
\]

Multiplying these equalities we get

\[
\begin{cases}
  a^{\frac{n(n+1)}{2}} = +1 & \text{if } n \text{ is even}, \\
  a^{\frac{n(n+1)}{2}} = -1 & \text{if } n \text{ is odd}.
\end{cases}
\]

**Case 1.** \( n \) is even. 

Then, if \( a = 1 \) we obtain the solution

\[
\begin{cases}
  x_i = k & \text{for } i \text{ odd}, \\
  x_i = -k & \text{for } i \text{ even},
\end{cases}
\]

where \( k \) is a free parameter. If \( a = -1 \) and \( 4 \mid n \), then we obtain the solution

\[
\begin{cases}
  x_i = k & \text{for } i \equiv 1, 2 \pmod{4}, \\
  x_i = -k & \text{for } i \equiv 0, 3 \pmod{4},
\end{cases}
\]

where again \( k \) is a free parameter.

**Case 2.** \( n \) is odd. 

Here we must have \( a = -1 \) and \( n \equiv 1 \pmod{4} \), so that

\[
\begin{cases}
  x_i = k & \text{for } i \equiv 1, 2 \pmod{4}, \\
  x_i = -k & \text{for } i \equiv 0, 3 \pmod{4},
\end{cases}
\]

is the solution, where \( k \) is a free parameter.
4. Let $\{h_n\}_{n=1}^{\infty}$ be a harmonic sequence of positive rational numbers. In other words, each $h_n$ is the harmonic mean of its neighbours:

$$h_n = \frac{2h_{n-1}h_{n+1}}{h_{n-1} + h_{n+1}}.$$

Prove that if some term $h_j$ of the sequence is the square of a rational number, then the sequence contains an infinite number of terms $h_k$ that are each squares of rational numbers.

Solved by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the write-up by Curtis.

Solving the recursion given in the problem statement for $h_{n+1}$ gives

$$h_{n+1} = \frac{h_{n-1}h_n}{2h_{n-1} - h_n}. \quad (1)$$

For each $n$ let $x_n = \frac{1}{h_n}$. Then (1) implies that

$$x_{n+1} = 2x_n - x_{n-1}, \quad (2)$$

which in turn implies that $x_{n+1} - x_n = x_n - x_{n-1}$, so that the sequence $\{x_n\}$ is arithmetic. Let $k$ be its common difference. Note that $h_n$ is the square of a rational number if and only if $x_n$ is the square of a rational number. By reindexing, we may assume that $x_0 = \left(\frac{a}{b}\right)^2$, where $a$ and $b$ are positive integers. Also, $k = \frac{c}{d}$ for some positive integers $c$ and $d$. Thus,

$$x_n = x_0 + nk = \frac{a^2}{b^2} + n \left(\frac{c}{d}\right) = \frac{a^2d + nb^2c}{b^2d}.$$

Let $t$ be a positive integer, and set $m = a + tb^2c$. Set $l = \frac{m^2 - a^2}{b^2c}$ and $n = ld$. Then

$$l = \frac{2tab^2c + t^2b^4c^2}{b^2c} = 2ta + t^2b^2c,$$

and

$$x_n = x_{ld} = \frac{a^2d + ldb^2c}{b^2d} = \frac{a^2 + ldb^2c}{b^2} = \frac{a^2 + (2ta + t^2b^2c)b^2c}{b^2} = \left(\frac{a + b^2ct}{b}\right)^2,$$

providing an infinite subsequence of $\{x_n\}$ consisting of squares of rational numbers. The corresponding terms of the sequence $\{h_n\}$ are also squares of rational numbers.
Next we turn to problems of the 37th Austrian Mathematical Olympiad, National Competition, Final Round, Part 1, given at [2009 : 291].

2. Prove that the sequence \( \left\{ \frac{(n + 1)^{n^2 - n}}{7n^2 + 1} \right\}_{n=0}^{\infty} \) is strictly increasing.

Solved by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's solution, modified by the editor.

We write

\[
\frac{(n + 1)^{n^2 - n}}{7n^2 + 1} = \frac{(n + 1)^n}{n^n} \cdot \frac{n^2}{7n^2 + 1} = \left(1 + n^{-1}\right)^n \left(7 + n^{-2}\right)^{-1}.
\]

The portion \( (7 + n^{-2})^{-1} \) is increasing with \( n \), since \( f(n) = n^{-2} \) is a decreasing function of \( n \).

It is known that the other portion \( (1 + n^{-1})^n \) is increasing, and here is a proof: Starting with the AM–GM Inequality, we successively deduce that

\[
\left(1 + \left(1 + n^{-1}\right) + \cdots + \left(1 + n^{-1}\right)^{n-1}\right) > (n + 1) \sqrt[1+n]{(1 + n^{-1})^n};
\]

\[
1 + n + n \cdot n^{-1} > (n + 1) \sqrt[1+n]{(1 + n^{-1})^n};
\]

\[
\left(\frac{n + 2}{n + 1}\right)^{n+1} > \left(1 + n^{-1}\right)^n;
\]

\[
\left(1 + (n + 1)^{-1}\right)^{n+1} > \left(1 + n^{-1}\right)^n.
\]

Since both portions are positive and strictly increasing, the given sequence is also strictly increasing.

3. The incircle of triangle \( ABC \) touches the lines \( BC \) and \( AC \) at \( D \) and \( E \), respectively. Prove that if \( AD \) and \( BE \) are of the same length, then the triangle is isosceles.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figueres, Mallorca, Spain; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Amengual Covas.

In \( \triangle ABC \) let \( BC = a, CA = b, AB = c \), and let \( s = \frac{1}{2}(a + b + c) \) denote the semiperimeter. By the Law of Cosines in \( \triangle ADC \) with \( DC = s - c \) we have

\[ AD^2 = b^2 + (s - c)^2 - 2b(s - c) \cos C, \]

and similarly in \( \triangle BCE \) with \( CE = s - c \) we have

\[ BE^2 = a^2 + (s - c)^2 - 2a(s - c) \cos C. \]
Using the Law of Cosines once more in \( \triangle ABC \) and combining this with
the preceding results, we have
\[
AD^2 - BE^2 = b^2 - a^2 - 2(s - c)(b - a) \cos C
\]
\[
= (b - a) \left[ b + a - (a + b - c) \cdot \frac{a^2 + b^2 - c^2}{2ab} \right]
\]
\[
= \frac{b - a}{2ab} \left( ab^2 + a^2b + bc^2 + b^2c + ca^2 + c^2a - a^3 - b^3 - c^3 \right)
\]
\[
= \frac{b - a}{2ab} \left[ a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \right].
\]

Now, \( b + c - a, c + a - b, \) and \( a + b - c \) are all positive, therefore
\[ a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \]
is positive. It follows that \( AD^2 - BE^2 = 0 \) if and only if \( b - a = 0, \) that is, \( AD = BE \) if and only if
\( \triangle ABC \) is isosceles with \( a = b. \)


1. Find the number of nonnegative integers \( n \leq N \) with the property that the decimal expansion of some multiple of \( n \) contains only the digits 2 and 6 (not necessarily the same number of each).

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We give Zvonaru’s write-up.

Since none of 22, 26, 62, 66 is divisible by 4, it follows that no integer ending in these digits can be divisible by 4. Hence, if \( n \) is a multiple of 4, then there is no multiple of \( n \) containing only the digits 2 and 6.

If \( n \) is a multiple of 5, then any multiple of \( n \) ends in 0 or 5, so no multiple of \( n \) can contain only the digits 2 and 6.

If \( n \) is coprime to 10, then there is a number \( 11 \ldots 1 \) with all digits 1
[a rep-unit] that is divisible by \( n. \) [Ed.: By the Pigeon Hole Principle, two
numbers in the sequence 1, 11, 111, \ldots leave the same remainder modulo \( n. \) Thus, \( n \) divides their difference, which is of the form 111\ldots 0, and hence \( n \) divides 111\ldots 1 since \( n \) is coprime to 10.]

Now, \( n \) divides a \( t \)-digit rep-unit \( 11 \ldots 1, \) hence \( n \) divides the \( 2t \)-digit number \( \overbrace{22 \ldots 2}^{\frac{t}{2}} \overbrace{66 \ldots 6}^{\frac{t}{2}} \) and this multiple of \( n \) contains only digits 2 and 6.

If \( n = 2k \) with \( k \) coprime to 10, then \( k \) divides some \( t \)-digit rep-unit
\( 11 \ldots 1, \) and then \( n \) divides the \( 2t \)-digit number \( \overbrace{22 \ldots 2}^{\frac{t}{2}} \overbrace{66 \ldots 6}^{\frac{t}{2}} \).

We deduce that the number of nonnegative integers \( n \leq N \) with the required property is
\[
N - \left\lfloor \frac{N}{4} \right\rfloor - \left\lfloor \frac{N}{5} \right\rfloor + \left\lfloor \frac{N}{20} \right\rfloor.
\]
2. Prove that
\[ 3(a + b + c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}} \]
for all positive real numbers \(a, b, \) and \(c\). Determine when equality holds.

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Díaz-Barrero.

Applying mean inequalities, we have
\[
8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}} \leq 9 \sqrt[3]{\frac{24abc + a^3 + b^3 + c^3}{27}} = 3\sqrt[3]{24abc + a^3 + b^3 + c^3}.
\]
It then suffices to prove any of the following inequalities
\[
\sqrt[3]{24abc + a^3 + b^3 + c^3} \leq a + b + c,
24abc + a^3 + b^3 + c^3 \leq (a + b + c)^3,
\]
and the last inequality follows from the AM–GM inequality. Indeed, we have
\[
a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 \geq 6abc,
\]
and equality holds if and only if \(a = b = c\), and we are done.

3. Given triangle \(ABC\), let point \(R\) be on the extension of \(AB\) beyond \(B\) with \(BR = BC\), and let point \(S\) be on the extension of \(AC\) beyond \(C\) with \(CS = CB\). Let the diagonals of \(BRSC\) intersect in the point \(A'\), and construct the points \(B'\) and \(C'\) similarly. Prove that the area of the hexagon \(AC'BA'CB'\) is the sum of the areas of triangles \(ABC\) and \(A'B'C'\).

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kendall, Hamden, CT, USA; and Titu Zvonaru, Comănești, Romania. We use Bataille's submission.

Let \([\cdot]\) denote the area of the figure it encloses. We will show that
\[
[A'B'C'] + [B'CA] + [C'AB] = [ABC]
\]
and
\[
[A'B'C'] = [ABC].
\]
This clearly implies the required result.
Let \( BC = a, \ CA = b, \) and \( AB = c. \)

Since \( bC\overline{S} = aA\overline{C} \) and \( c\overline{B}R = a\overline{AB}, \) we have \( b\overline{S} = -aA + (a + b)C \)
and \( c\overline{R} = -aA + (a + c)B. \)

It follows that

\[
b\overline{S} + (a + c)B = -aA + (a + c)B + (a + b)C
\]

and hence

\[
2sA' = b\overline{S} + (a + c)B = c\overline{R} + (a + b)C
\]

(1)

where \( 2s = a + b + c. \) In particular, we obtain the first relation below and the other two follow similarly:

\[
\frac{[A'BC]}{[ABC]} = \frac{a}{2s}, \quad \frac{[B'C'A]}{[ABC]} = \frac{b}{2s}, \quad \frac{[C'AB]}{[ABC]} = \frac{c}{2s};
\]

and hence

\[
[A'BC] + [B'C'A] + [C'AB] = \frac{a + b + c}{2s} [ABC] = [ABC].
\]

In addition, from (1) we also deduce that

\[
2s(A + A') = (b + c)A + (c + a)B + (a + b)C.
\]

Clearly, the same result \((b + c)A + (c + a)B + (a + b)C\) will be obtained for \(2s(B + B')\) and \(2s(C + C'),\) proving that \(AA', BB', CC'\) have the same midpoint. This means that \(\triangle ABC\) and \(\triangle A'B'C'\) are symmetrical about some point, and hence they have the same area. This completes the proof.

4. Determine all rational numbers \(x\) such that \(1 + 105 \cdot 2^x\) is the square of a rational number.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Geupel's solution.

It is readily checked that \(-8, -6, -4, 3,\) and \(4\) are solutions and we shall prove there are no others.

Let \(x\) be a solution. Then there are integers \(a, b, c,\) and \(d\) such that \(b, d\) are positive, \(\gcd(a, b) = \gcd(c, d) = 1, \) \(x = \frac{a}{b}, \) and \(1 + 105 \cdot 2^{a/b} = \frac{c^2}{d^2}. \) We
obtain $2^a = \left( \frac{c^2 - d^2}{105d^2} \right)^b$. The multiplicity of the factor 2 in the right term is a multiple of $b$. Hence, $b | a$, which implies that $b = 1$ and

$$c^2 = d^2 \left( 1 + 105 \cdot 2^n \right).$$

Since 106 is not a square, $a \neq 0$, and we consider the cases $a < 0$ and $a > 0$.
First, let $a < 0$ with $m = -a$. From (1), we have

$$2^m c^2 = d^2 \left( 2^m + 105 \right).$$

If $m$ were odd, then the multiplicity of the factor 2 in (2) would be odd on the left but even on the right, a contradiction. Thus, $m$ is even, say $m = 2n$. We obtain $(2^m c)^2 = d^2 \left( 4^n + 105 \right)$. Thus, $4^n + 105$ is a square of an integer $q > 0$, and the equation $3 \cdot 5 \cdot 7 = (q - 2^n)(q + 2^n)$ yields a factorization of 105 into two positive integers. There is no solution with $q - 2^n = 1$. If $q - 2^n = 3$, then $a = -8$, while if $q - 2^n = 5$ or $q - 2^n = 7$ then $a = -6$ or $a = -4$, respectively. Otherwise we have $q - 2^n \geq 15$, which yields $q + 2^n \leq 7 < q - 2^n$, a contradiction.

Second, let $a > 0$. From (1) we see that $d | c$, therefore $d = 1$ and

$$2^a \cdot 105 = (c - 1)(c + 1).$$

Checking the cases $1 \leq a \leq 8$, we find the solutions $a = 3$ and $a = 4$. Assume $a \geq 9$. Without loss of generality let $c > 0$. One of the numbers $c - 1$ and $c + 1$ must be divisible by $2^{a - 1}$ and is therefore not less than 256. The other number is not greater than $2 \cdot 105 = 210$, a contradiction.

This completes the proof.

5. Find all monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\neg f(x)) = f(f(x)) = f(x)^2.$$

(A function $f$ is monotonic if either $f(a) \leq f(b)$ for all $a < b$ or $f(a) \geq f(b)$ for all $a < b$.)

Solved by Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give Bataille's solution.

Let $f$ be a monotonic function such that $f(\neg f(x)) = f(f(x)) = f(x)^2$ for all real numbers $x$, and let $d(x) = x(f(x) - f(0))$. Since $f$ is monotonic, we have $d(x) \cdot d(y) \geq 0$ for all real $x, y$.

By a simple calculation, $d(f(0)) \cdot d(\neg f(0)) = -f(0)^4(f(0) - 1)^2$, so $f(0) = 0$ or $f(0) = 1$.

First, suppose $f(0) = 0$, so that $d(x) = xf(x)$. For any $x$, the number

$$d(f(x)) \cdot d(\neg f(x)) = f(x)f(f(x)) \cdot (\neg f(x)f(\neg f(x)) = -f(x)^6$$

is nonnegative, hence $f(x) = 0$. Thus, $f$ is the constant function $x \mapsto 0$. 
Second, suppose \( f(0) = 1 \), so that \( d(x) = x(f(x) - 1) \). Then,
\[
d(f(x)) \cdot d(-f(x)) = -f(x)^2(f(x)^2 - 1)^2,
\]
hence \( f(x) \in \{0, 1, -1\} \) for all \( x \). But if for some \( x_0 \) we had \( f(x_0) = 0 \), we would have \( 0 = f(x_0)^2 = f(f(x_0)) = f(0) \), contradicting \( f(0) = 1 \). It follows that \( f(x) = 1 \) or \(-1\) for all \( x \). Since \( f(-1) = f(-f(0)) = f(0)^2 = 1 \) and similarly \( f(1) = 1 \), the monotonicity of \( f \) implies \( f(x) = 1 \) for all \( x \in [-1, 1] \) and \( f(x) = 1 \) for \( x \geq 1 \) if \( f \) is increasing, \( f(x) = 1 \) for \( x \leq 1 \) if \( f \) is decreasing.

Now, suppose that \( f \) is increasing and different from the constant function \( x \mapsto 1 \). Let the set \( \{ x \in \mathbb{R} : x < -1 \text{ and } f(x) = -1 \} \) have \( a \) as its lowest upper bound. We have \( a \leq 1 \) and if \( x < a \), then \( f(x') = 1 \) for some \( x' \) with \( x < x' < a \), so that \( f(x) = 1 \). As a result, \( f(x) = 1 \) for \( x < a \) and \( f(x) = 1 \) for \( x > a \) and \( f \) is one of the functions \( \phi, \phi_a, \psi_a \) defined by \( \phi(x) = -1 \) \( (x < -1) \) and \( \phi(x) = 1 \) \( (x \geq -1) \); \( \phi_a(x) = -1 \) \( (x < a) \) and \( \phi_a(x) = 1 \) \( (x \geq a) \); \( \psi_a(x) = -1 \) \( (x \leq a) \) and \( \psi_a(x) = 1 \) \( (x > a) \), where \( a < -1 \) in each instance.

Similarly, if \( f \) is decreasing, then \( f \) is one of the functions \( \lambda, \lambda_b, \mu_b \) defined by \( \lambda(x) = 1 \) \( (x \leq 1) \) and \( \lambda(x) = -1 \) \( (x > 1) \); \( \lambda_b(x) = 1 \) \( (x \leq b) \) and \( \lambda_b(x) = -1 \) \( (x > b) \); \( \mu_b(x) = 1 \) \( (x < b) \) and \( \mu_b(x) = -1 \) \( (x \geq b) \), where \( b > 1 \) in each instance.

Conversely, the constant functions \( x \mapsto 0 \) and \( x \mapsto 1 \) and the functions \( \phi, \phi_a, \psi_a, \lambda, \lambda_b, \mu_b \) where \( a < -1 \) and \( b > 1 \) are monotonic on \( \mathbb{R} \) and satisfy \( f(-f(x)) = f(f(x)) = f(x)^2 \) for all \( x \) (the nonconstant ones satisfy \( f(x) = \pm 1 \) and \( f(1) = f(-1) = 1 \)).

We conclude that these are all of the solutions to the given equation.

6. Let \( A \) be a nonzero integer. Find all integer solutions of the following system of equations:

\[
\begin{align*}
x + y^2 + z^3 &= A, \\
x^{-1} + y^{-2} + z^{-3} &= A^{-1}, \\
xyz &= A^2.
\end{align*}
\]

Solved by Mohammed Assila, Strasbourg, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Tito Zvonaru, Comănești, Romania. We give the write-up of Wang and Zhao.

We solve the more general problem where \( x, y, z, \) and \( A \) are real numbers and \( A \neq 0 \). We shall show that if \( A > 0 \), then there are no solutions, while if \( A < 0 \) then there are exactly four solutions given by \( (x, y, z) = (A, \sqrt[3]{-A}, -\sqrt[3]{-A}) \) or \( (-\sqrt[3]{A}, \sqrt[3]{-A}, \sqrt[3]{-A}) \). If we restrict \( x, y, z, \) and \( A \) to be integers, then there are four solutions if \( A = -n^{12} \) for some positive integer \( n \) and no solutions otherwise.
We first set \( s = y^2 \) and \( t = z^3 \). Then the given system becomes
\[
\begin{align*}
x + s + t &= A, \quad (1) \\
x^{-1} + s^{-1} + t^{-1} &= A^{-1}, \quad (2) \\
xst &= A^2. \quad (3)
\end{align*}
\]

From (2) and (3) we obtain
\[
xst + tx = A. \quad (4)
\]

From (1), (3), and (4) we see that \( x, s, \) and \( t \) are the roots of the cubic equation
\[
f(u) = u^3 - Au^2 + Au - A^2 = 0.
\]

Since \( f(u) = (u - A)(u^2 + A) \) the roots of \( f(u) \) are \( A \) and \( \pm \sqrt{-A} \).

Hence, there is only one real solution if \( A > 0 \), while if \( A < 0 \) then \( x, s, t \) are just \( A, \sqrt{-A}, -\sqrt{-A} \) in some order.

Since \( s = y^2 \) is nonnegative, then we must have \( y^2 = \sqrt{-A} \), so that \( (x, s, t) = (A, \sqrt{-A}, -\sqrt{-A}) \) or \( (-\sqrt{-A}, \sqrt{-A}, A) \). That is \((x, y, z) = (A, \pm \sqrt[3]{-A}, -\sqrt[3]{-A}) \) or \((-\sqrt[3]{-A}, \pm \sqrt[3]{-A}, \sqrt[3]{A}) \).

Finally, if \( A \) is an integer, then clearly \( x, y, \) and \( z \) are all integers if and only if \( -A = n^{12} \) for some natural number \( n \).

This completes our proof.

Next we open our file of readers' solutions to problems of the Brazilian Mathematical Olympiad 2005, given in the Corner at [2009 : 292-293].

1. A positive integer is a **palindrome** if reversing its digits leaves it unchanged (for example, 481184, 131, and 2 are palindromes). Find all pairs \((m, n)\) of positive integers such that \( 111 \ldots 1 \times 111 \ldots 1 \) is a palindrome.

**Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.**

If \( m \geq 10 \) and \( n \geq 10 \), the product can be computed as the sum of \( m \) rows of \( n \) ones each, with each row shifted one digit to the right of the preceding one. Hence, the eighth digit from the right will be the sum of eight ones, and is thus 8. The eighth digit from the left will be the sum of eight ones plus another one from a carry, and is thus 9. In this case, therefore, the product is not a palindrome.

Otherwise, we may assume by symmetry that \( m \leq 9 \). Again regarding the product as the sum of \( m \) rows of \( n \) ones each, there are no carries, so the product is \( 123 \cdots mnm \cdots mnm \cdots 321 \), a palindrome.

2. Determine the smallest real number \( C \) such that
\[
C(x_1^{2005} + x_2^{2005} + \cdots + x_5^{2005}) \geq x_1x_2x_3x_4x_5 (x_1^{125} + x_2^{125} + \cdots + x_5^{125})^{16}
\]
for all positive real numbers \( x_1, x_2, x_3, x_4, \) and \( x_5 \).
Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille’s generalization.

The smallest suitable $C$ is 5\(^{15}\).

Taking $x_1 = x_2 = x_3 = x_4 = x_5 = 1$, we see that $C \geq 5^{15}$. Thus, it suffices to prove the inequality for all positive $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ when $C = 5^{15}$. In fact we prove a generalization: If $m$, $n$ and $x_1$, $x_2$, ..., $x_k$ are positive real numbers and $p > 1$, then

$$x_1^m \cdots x_k^m (x_1^n + \cdots + x_k^n)^p \leq k^{p-1} (x_1^{km+np} + \cdots + x_k^{km+np}).$$

Let $R$ denotes the righthand side. We have

$$R = \left( \frac{x_1^m x_2^m \cdots x_k^m}{x_1^n + \cdots + x_k^n} \right)^p = \left( \left( \frac{x_1^{m+n} x_2^m \cdots x_k^m}{x_1^n x_2^m \cdots x_k^m} \right)^{\frac{1}{p}} + \left( \frac{x_1^m x_2^{m+n} x_3^m \cdots x_k^m}{x_1^m x_2^m \cdots x_k^m} \right)^{\frac{1}{p}} + \cdots + \left( \frac{x_1^m x_2^m \cdots x_{k-1}^m x_k^{m+n}}{x_1^m x_2^m \cdots x_k^m} \right)^{\frac{1}{p}} \right)^p \leq k^{p-1} (x_1^{m+n} x_2^m \cdots x_k^m + x_1^m x_2^{m+n} x_3^m \cdots x_k^m + \cdots + x_1^m x_2^m \cdots x_{k-1}^m x_k^{m+n}) \quad (1)$$

where (1) follows from the inequality of means:

$$\left( \frac{a_1^{1/p} + \cdots + a_k^{1/p}}{k} \right)^p \leq \frac{a_1 + \cdots + a_k}{k}$$

for positive $a_1$, $a_2$, ..., $a_k$.

Now, using the weighted AM–GM Inequality we have, for example,

$$x_1^{m+n} x_2^m \cdots x_k^m \leq \frac{m+np}{km+np} x_1^{km+np} + \frac{np}{km+np} x_2^{km+np} + \cdots + \frac{m}{km+np} x_k^{km+np}.$$ 

Treating the other terms of (1) in the same way and adding up gives

$$R \leq k^{p-1} (x_1^{km+np} + x_2^{km+np} + \cdots + x_k^{km+np}),$$

as desired.

\textbf{5.} Let $ABC$ be an acute triangle and let $F$ be its Fermat point, that is, the interior point of $ABC$ such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each of the triangles $ABF$, $BCF$, and $CAF$, draw its Euler line, that is, the line connecting its circumcentre and its centroid.

Prove that these three lines are concurrent.

Comment by Mohammed Aassila, Strasbourg, France.

This is problem A323 from Kömal (September 2003 issue).
6. Let \( b \) be an integer and let \( a \) and \( c \) be positive integers. Prove that there exists a positive integer \( x \) such that \( a^x + x \equiv b \pmod{c} \), that is, prove there exists a positive integer \( x \) such that \( c \) divides \( a^x + x - b \).

Comment by Mohammed Aassila, Strasbourg, France.


Lastly, we give a solution to a problem from the Croatian Mathematical Olympiad 2006, National Competition, 4th Grade, at [2009: 293–294].

1. Prove that three tangents to a parabola always form the sides of a triangle whose altitudes intersect on the directrix of the parabola.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give Bataille’s presentation.

Let \( M_1, M_2, M_3 \) be the points of contact of the three given tangents \( t_1, t_2, t_3 \) to the parabola and let \( H_1, H_2, H_3 \) be their orthogonal projections onto the directrix \( L \) (see figure at right). If \( F \) denotes the focus of the parabola, then \( t_i \) is the perpendicular bisector of \( FH_i \), for \( i = 1, 2, 3 \) (a well-known result). It follows that the orthogonal projections of \( F \) onto \( t_1, t_2, t_3 \) are three points on the tangent \( t \) to the parabola at its vertex (the image of \( L \) under the homothety with centre \( F \) and factor \( \frac{1}{2} \)). As a result, \( F \) is on the circumcircle of \( \triangle P_1 P_2 P_3 \) and the tangent \( t \) is the Simson line of \( F \) relative to this triangle. We know this Simson line bisects the segment \( FH \), where \( H \) is the orthocentre of \( \triangle P_1 P_2 P_3 \). It follows that \( H \) is on the image of \( t \) under the homothety with centre \( F \) and factor 2, that is, \( H \) is on \( L \) and \( L \) is the Steiner line of \( F \) relative to \( \triangle P_1 P_2 P_3 \).

That’s all the material for this number. Send me your nice solutions!