Therefore, the solution set to the equation \( [x] \cdot \{x\} = x \) is \( x = 0 \) or \( x = -\frac{k^2}{k+1} \) for some positive integer \( k \). These can be combined to give \( x = -\frac{k^2}{k+1} \) for some nonnegative integer \( k \).

*Also solved by* CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; CARL LIBIS, Cumberland University, Lebanon, TN, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRO, IES “Abastos”, Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; GUSNADIWIYO, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incorrect solutions submitted.

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**Problem of the Month**

Ian VanderBurgh

Some of the best problems are ones that are simple to understand, do not require a whole lot of mathematical background, but that send you nicely down the “garden path”.

**Problem** (*2010 Pascal Contest*) The product of \( N \) consecutive four-digit positive integers is divisible by \( 2010^2 \). What is the least possible value of \( N \)?

(A) 5  (B) 12  (C) 19  (D) 6  (E) 7

Since we want to find a product of consecutive integers divisible by \( 2010^2 \), it makes good sense to find the prime factors of 2010. (It’s a good thing that we didn’t ask this in 2011.) This isn’t that difficult since 2010 is divisible by 10 (since its units digit is 0) and 3 (since the sum of its digits is 3 which is a multiple of 3). Therefore,

\[
2010 = 10 \times 201 = 10 \times 3 \times 67 = 2 \times 3 \times 5 \times 67.
\]

But we want to find a product of consecutive integers divisible by \( 2010^2 \), so we’d better write out the prime factorization of \( 2010^2 \):

\[
2010^2 = 2^2 \times 3^2 \times 5^2 \times 67^2.
\]

Now, let’s try to solve the problem.

**Solution 1.** Since we are looking for a set of \( N \) consecutive four-digit positive integers whose product is divisible by \( 2010^2 \), then we need to look for integers that are divisible by the prime factors of \( 2010^2 \).

We start with the largest prime factor of \( 2010^2 \), namely 67. We want the product of the integers in our set to include two factors of 67, and so
either two different integers in the set are multiples of 67 or one integer in the set has two factors of 67. In the first case, our set of consecutive integers would contain two different multiples of 67 which must be at least 67 apart from each other; this would mean that the set contains at least 68 positive integers. Since none of the available answers are anywhere close to 68, this must not be the case.

Since the integers in the set are four-digit positive integers, then our set should contain $67^2 = 4489$. We also need the product of the integers in the set to contain two multiples of 5. There is no multiple of $5^2$ that is close to 4489, so we try expanding the set to include 4490 and 4485, the two closest multiples of 5 to 4489. (These are also the two multiples of 5 that we can include to minimize the total number of integers in the set thus far.)

Up to this point our set is

$$\{4485, 4486, 4487, 4488, 4489, 4490\}.$$  

The product of these integers includes two factors of 67 (since $4489 = 67^2$), two factors of 5 (one in 4485 and one in 4490), two factors of 2 (one in 4486 and one in 4488), and also two factors of 3 (one in 4485 and one in 4488). We can check this last fact using a calculator (don't be tempted!) or by noting that the sum of the digits of 4485 is 21 and the sum of the digits of 4488 is 24; each of these sums is divisible by 3 so each of the integers is divisible by 3.

Therefore, the product of the four-digit integers in the set

$$\{4485, 4486, 4487, 4488, 4489, 4490\}$$

is divisible by $2010^2$ as required. This set contains 6 integers, so the answer to the problem is (D). 

This solution is not too difficult and makes good sense. In other words, it is a great solution except for one small problem. Solution 1 is wrong! Can you see what is wrong with it? Take a few minutes and read through it critically to see if you can spot the flaw. Don't be too alarmed if you can't find the flaw—a number of quite good mathematicians have missed this already!

The crucial mistake is in the sentence "Since the integers in the set are four-digit positive integers, then our set should contain $67^2 = 4489." Can you see the flaw now? The sentence can be corrected by re-writing it as "Since the integers in the set are four-digit positive integers, then our set should contain a four-digit integer that is divisible by $67^2 = 4489." Let's start Solution 2, picking up from Solution 1 in the third paragraph.

Solution 2. Since the integers in the set are four-digit positive integers, then our set should contain a four-digit integer that is divisible by $67^2 = 4489$. The four-digit multiples of 4489 are 4489 and $2 \times 4489 = 8978$. (Note that $3 \times 4489$ is too large, since it has five digits.)

In Solution 1, we saw that if the set includes 4489 and has the desired property, then the minimal size of the set is 6. So let's consider a set that includes 8978.
Let's look for multiples of 5 to include in the set. As in Solution 1, if we include two different multiples of 5, then the set includes at least 6 integers. (Can you see why?) Is it possible to include a multiple of 5² in our set in this case that is close to 8978? Yes! We can include 8975, which is divisible by 5².

So let's expand our set by including the intermediate integers to get

\{8975, 8976, 8977, 8978\}.

How are we doing so far with respect to the desired property? The product of these integers includes two factors of 67 (since 8978 is divisible by 67²), two factors of 5 (since 8975 is divisible by 5²), and two factors of 2 (at least one in each of 8976 and 8978). How about factors of 3? Since the sum of the digits of 8976 is 30, then 8976 is divisible by 3. However, 8976 is not divisible by 3² (since the sum of its digits is not divisible by 9) and none of the other three integers is divisible by 3. (Check the sum of the digits of each.)

Therefore, we need to expand the set to include a second multiple of 3. Is either 8974 or 8979 divisible by 3? Yes, 8979 is divisible by 3. Therefore, the set

\{8975, 8976, 8977, 8978\}

has the property that the product of its elements is divisible by 2010². Based on our reasoning, this set is the smallest set of four-digit integers with this property. Therefore, the answer to the problem is (A).

I really like this problem, because it really tests reasoning skills without requiring a whole lot of formal knowledge. There was quite a debate among the contest creation team as to whether to include the possible answer "(D) 6", given the trap that it sets. Leaving it in undoubtedly trapped a number of contestants, but really rewarded those who saw through this. On the other hand, leaving this answer out might have served as a good "teaching point" for students who got 6 as the answer and then found that their answer wasn't there and as a result persevered to find the real answer. Food for thought! In the end, in the context of the entire paper, the contest creation team decided to leave the answer (D) in.