SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Our apologies to Albert Stadler, Herrliberg, Switzerland, for misfiling his correct solution to #3449 (which was the only other solution submitted for that problem other than the proposer's).


Let \((X, \langle \cdot, \cdot \rangle)\) be a real or complex inner product space and let \(x, y,\) and \(z\) be nonzero vectors in \(X\). Prove that

\[
\sum_{\text{cyclic}} \left| \frac{\langle z, x \rangle - \langle x, y \rangle}{\|x\|} \right|^{1/2} \leq \sum_{\text{cyclic}} \left( \frac{\|x\| \|y\| \|z\|}{\|x\| \|y\| \|z\|} \right)^{1/2} |\langle y, z \rangle|.
\]

Solution by Albert Stadler, Herrliberg, Switzerland.

Put

\[
a = \sqrt{\|x\| \|y\| \|z\|}, \\
b = \sqrt{\|y\| \|z\| \|x\|}, \\
c = \sqrt{\|z\| \|x\| \|y\|}.
\]

By the Cauchy–Schwarz Inequality,

\[
|ab + bc + ca| \leq \sqrt{a^2 + b^2 + c^2} \sqrt{b^2 + c^2 + a^2} = a^2 + b^2 + c^2,
\]

or

\[
\sum_{\text{cyclic}} \sqrt{\|x\| \|y\| \|z\| \|\langle y, z \rangle\| \|\langle z, x \rangle\|} \leq \sum_{\text{cyclic}} \|x\| \|y\| \|z\|.
\]

Hence, by dividing this last inequality by \(\sqrt{\|x\| \|y\| \|z\|}\), we obtain the required inequality.

Also solved by ARKADY ALT. San Jose, CA, USA; GEORGE APOTOLOPoulos, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; and the proposer.

Prove the following and generalize these results.

(a) \( \tan^2 36^\circ + \tan^2 72^\circ = 10 \),
(b) \( \tan^4 36^\circ + \tan^4 72^\circ = 90 \),
(c) \( \tan^6 36^\circ + \tan^6 72^\circ = 850 \),
(d) \( \tan^8 36^\circ + \tan^8 72^\circ = 8050 \).

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Set \( a = \tan^2 36^\circ \), \( b = \tan^2 72^\circ \) and \( T_n = a^n + b^n \) for \( n \geq 1 \). It is well-known that

\[
\cos 36^\circ = \frac{\sqrt{5} + 1}{4} \quad \text{and} \quad \cos 72^\circ = 2 \left( \frac{\sqrt{5} + 1}{4} \right)^2 - 1 = \frac{\sqrt{5} - 1}{4}.
\]

Since \( \tan^2 x = \frac{1}{\cos^2 x} - 1 \) and \( \frac{4}{\sqrt{5} + 1} = \sqrt{5} - 1 \), we have

\[
a = (\sqrt{5} - 1)^2 - 1 = 5 - 2\sqrt{5},
\]
\[
b = (\sqrt{5} + 1)^2 - 1 = 5 + 2\sqrt{5},
\]

so that \( T_n = (5 - 2\sqrt{5})^n + (5 + 2\sqrt{5})^n \).

To verify equations (a)-(d), we give a way to calculate the values of \( T_n \) inductively. From \( (a + b)(a^{n+1} + b^{n+1}) = (a^{n+2} + b^{n+2}) + ab(a^n + b^n) \) and \( a + b = 10 \), \( ab = 5 \), we obtain the recurrence \( T_{n+2} = 10T_{n+1} - 5T_n \) for \( n \geq 1 \). Since \( T_1 = 10 \) and \( T_2 = (5 - 2\sqrt{5})^2 + (5 + 2\sqrt{5})^2 = 90 \), we easily obtain \( T_3 = 10 \cdot 90 - 5 \cdot 10 = 850 \), \( T_4 = 10 \cdot 850 - 5 \cdot 90 = 8050 \), \( T_5 = 10 \cdot 8050 - 5 \cdot 850 = 76250 \), and so forth.

Also solved by ARRADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messoudi, Greece; MICHELE ARNOLD, Southeast Missouri State University, Cape Girardeau, MO, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; JESI BAYLESS, Southeast Missouri State University, Cape Girardeau, MO, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DOUGLASS L. GRANT, Cape Breton University, Sydney, NS; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; TOM LEONG, The University of Scranton, Scranton, PA, USA; JOSHUA LONG, Southeast Missouri State University, Cape Girardeau, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; JOHN POSTL, St. Bonaventure University, St. Bonaventure, NY, USA; JOEL SCHLOSBERG, Bay side, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUNDO SWYLAN, Riga, Latvia; VASILE TEO DO BICI, Toronto, ON; PANOS F. TSAOUSSOGLOU, Athens, Greece; HAOHAO WANG and JERZY WOJDILO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biao University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.
Bataille remarked that there exists a general result for the sum
\[
\sum_{k=1}^{(m-1)/2} \tan^2\left(\frac{k\pi}{m}\right),
\]
where \(m\) is an odd positive integer and refers readers to his Problem 11044 in the American Math Monthly with solution in Vol. 112, No. 7, 2005; pp. 657-9. Grant pointed out that the formula \(\tan^n(36^\circ) + \tan^n(72^\circ) = (5 - 2\sqrt{5})^{n/2} + (5 + 2\sqrt{5})^{n/2}\) holds for any positive integer \(n\).

**3453.** [2009 : 325, 328] Proposed by Scott Brown, Auburn University, Montgomery, AL, USA.

Triangle \(ABC\) has side lengths \(a = BC, b = AC, c = AB\); and altitudes \(h_a, h_b, h_c\) from the vertices \(A, B, C\), respectively. Prove that
\[
8 \left( \sum_{\text{cyclic}} h_a^2(h_b + h_c) \right) + 16h_a h_b h_c \leq 3\sqrt{3} \left( \sum_{\text{cyclic}} a^2(b + c) \right) + 6\sqrt{3}abc.
\]

I. Solution by Joe Howard, Portales, NM, USA.

It suffices to show that
\[
8(h_a + h_b)(h_b + h_c)(h_c + h_a) \leq 3\sqrt{3}(a + b)(b + c)(c + a).
\]
Since \(h_a = c \sin B = b \sin C\) etc., then \(h_a + h_b = (a + b) \sin C\) etc. Multiplying yields
\[
\prod_{\text{cyclic}} (h_a + h_b) = \left( \prod_{\text{cyclic}} (a + b) \right) \left( \prod_{\text{cyclic}} \sin A \right).
\]
The result now follows from
\[
\prod_{\text{cyclic}} \sin A \leq \frac{3\sqrt{3}}{8},
\]
which is item 2.7, page 19 of O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

II. Solution by George Apostolopoulos, Messolonghi, Greece.

Let \(F\) denote the area of triangle \(ABC\). It is known that \(h_a = \frac{2F}{a}\) etc.
Now,
\[
8 \sum_{\text{cyclic}} h_a^2(h_b + h_c) + 16h_a h_b h_c
\]
\[
= 8 \prod_{\text{cyclic}} (h_a + h_b) = 8(2F)^3 \prod_{\text{cyclic}} \left( \frac{1}{a} + \frac{1}{b} \right)
\]
\[
= \frac{64F^3}{a^2b^2c^2} \prod_{\text{cyclic}} (a + b).
\]
Also
\[ 3\sqrt{3} \sum_{\text{cyclic}} a^2(b + c) + 6\sqrt{3}abc = 3\sqrt{3} \prod_{\text{cyclic}} (a + b), \]
so the conclusion is equivalent to
\[ F^3 \leq \frac{3\sqrt{3}}{64} a^2 b^2 c^2. \]

[Ed.: Apostolopoulos then gave a proof of this last inequality. However, it is equivalent to item 4.14, page 46 of O. Bottema et al., Geometric Inequalities, Groningen, 1969, as some other solvers pointed out.]

Also solved by ARKADY ALT, San Jose, CA, USA; SEFRET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brau, NRW, Germany; WALTHER JANOUS, Ursulineum Gymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Most solvers used one of the above two methods.


Let \( a, b, c, \) and \( d \) be positive real numbers such that \( a + b + c + d = 1. \) Prove that
\[ \frac{a^3}{b + c} + \frac{b^3}{c + d} + \frac{c^3}{d + a} + \frac{d^3}{a + b} \geq \frac{1}{8}. \]


By the AM–GM Inequality we have
\[ \frac{a^3}{b + c} + \frac{b + c}{16} + \frac{1}{32} \geq 3 \sqrt[3]{\frac{a^3(b + c)}{(b + c) \cdot 16 \cdot 32}} = \frac{3a}{8}. \]

Similarly,
\[ \frac{b^3}{c + d} + \frac{c + d}{16} + \frac{1}{32} \geq \frac{3b}{8}; \]
\[ \frac{c^3}{d + a} + \frac{d + a}{16} + \frac{1}{32} \geq \frac{3c}{8}; \]
\[ \frac{d^3}{a + b} + \frac{a + b}{16} + \frac{1}{32} \geq \frac{3d}{8}. \]
Adding the four inequalities, and using \( a + b + c + d = 1 \), we have
\[
\frac{a^3}{b + c} + \frac{b^3}{c + d} + \frac{c^3}{d + a} + \frac{d^3}{a + b} \geq \frac{3}{8},
\]
and the inequality follows.

II. Similar solutions by Tom Leong, The University of Scranton, Scranton, PA, USA and Pedro Henrique O. Pantoja, student, UFRN, Brazil.

By the Generalized Hölder Inequality, we have
\[
\left( \frac{a^3}{b + c} + \frac{b^3}{c + d} + \frac{c^3}{d + a} + \frac{d^3}{a + b} \right) \cdot \left[ (b + c) + (c + d) + (d + a) + (a + b) \right] \cdot (1 + 1 + 1 + 1) \geq (a + b + c + d)^3.
\]
Hence,
\[
8 \left( \frac{a^3}{b + c} + \frac{b^3}{c + d} + \frac{c^3}{d + a} + \frac{d^3}{a + b} \right) \geq 1,
\]
and the inequality follows.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece [2nd solution]; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; SALVATORE INGALA, student, Scuola Superiore di Catania, University of Catania, Catania, Italy; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; ALBERT STADLER, Herrliberg, Switzerland; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.


Find the minimum value of \( x^2 + y^2 + z^2 \) over all triples \((x, y, z)\) of real numbers such that
\[
13x^2 + 40y^2 + 45z^2 - 36xy + 12yz + 24xz \geq 2009
\]
and characterize all the triples at which the minimum is attained.

Solution by Kee-Wai Lau, Hong Kong, China.

Since
\[
x^2 + y^2 + z^2 = \frac{(6x + 3y - 2z)^2 + 13x^2 + 40y^2 + 45z^2 - 36xy + 12yz + 24xz}{49} \geq \frac{2009}{49} = 41,
\]
the required minimum is 41.
The minimum is attained if and only if

\[ 6x + 3y - 2z = 0, \]
\[ 13x^2 + 40y^2 + 45z^2 - 36xy + 12yz + 24xz = 2009. \]

Eliminating \( z \) from the last equation yields \( 40x^2 + 36xy + 13y^2 = 164 \), which is an ellipse, \( E \), in the \( xy \)-plane. Hence, the triples at which the minimum is attained are given by \( (x, y, z) = \left(s, t, \frac{6s + 3t}{2}\right) \), where \((s, t)\) is any point on \( E \).

*Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

Curtis’ solution was similar to the featured solution. All five incomplete solutions used Lagrange Multipliers and determined the minimum value of the objective function to be 41, but did not determine the complete set of triples at which the minimum is attained.

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Given a triangle \( ABC \) with circumcircle \( \Gamma \), let circle \( \Gamma' \) centred on the line \( BC \) intersect \( \Gamma \) at \( D \) and \( D' \). Denote by \( Q \) and \( Q' \) the projections of \( D \) and \( D' \) on the line \( AB \), and by \( R \) and \( R' \) their projections on \( AC \); assume that none of these projections coincide with a vertex of the triangle.

Show that if \( \Gamma' \) is orthogonal to \( \Gamma \), then \( \frac{BQ}{BQ'} = \frac{CR}{CR'} \). Does the converse hold?

*Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We shall see that the converse does not hold. Let \( T \) be the point where the tangents to \( \Gamma \) at \( D \) and \( D' \) meet. We prove that the given ratios are equal when \( T \) lies on the line \( BC \).

Because \( ABCD \) is cyclic, the directed angle from line \( BA \) to \( BD \) equals the directed angle from \( CA \) to \( CD \); thus, because \( Q \) is on \( AB \) and \( R \) is on \( AC \), the acute angles \( \angle QBD \) and \( \angle RCD \) must be equal, whence the right triangles \( BQD \) and \( CRD \) are similar. Likewise, \( \triangle BQ'D' \sim \triangle CR'D' \).

From these two pairs of similar triangles we deduce that

\[
\frac{BQ}{CR} = \frac{DB}{DC} \quad \text{and} \quad \frac{BQ'}{CR'} = \frac{D'B}{D'C}. \tag{1}
\]

If we assume that \( T \in BC \) and that the given triangle has been labeled so that \( B \) is between \( T \) and \( C \), then we have \( \angle TDB = \angle DCB = \angle DCT \) (the angle between a chord and tangent equals the angle subtended by the chord). We deduce that \( \triangle TCD \sim \triangle TDB \). Likewise, \( \triangle TCD' \sim \triangle TD'B \), and these two pairs of similar triangles give us

\[
\frac{DB}{CD} = \frac{TD}{TC} \quad \text{and} \quad \frac{D'B}{CD'} = \frac{TD'}{TC}. \tag{2}
\]
But $TD = TD'$, so all four quotients in (2) are equal, and (1) implies that $BQ/CR = BQ'/CR'$, or

$$\frac{BQ}{BQ'} = \frac{CR}{CR'}.$$  \hspace{1cm} (3)

When $\Gamma'$ is orthogonal to $\Gamma$, $T$ becomes the centre of $\Gamma'$ and lies on $BC$ by assumption. Thus, the orthogonality of the two circles implies that (3) holds, as desired. But when $BC$ is the diameter of $\Gamma$, $T$ lies on $BC$ for all circles $\Gamma'$ whose centre lies on $BC$. For each choice of $D$ on $\Gamma$, only one of these circles will be orthogonal to $\Gamma$; the rest serve as counterexamples to the converse.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Both Bataille and Geupe proved that the converse holds exactly when $\angle BAC$ is not a right angle. In other words, equation (3) holds if and only if (a) $\Gamma'$ is orthogonal to $\Gamma$, or (b) $\angle BAC = 90^\circ$, which is the case (in the notation of our featured solution) if and only if $T$ lies on $BC$.


Let $A_n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-3)^j j^n (2j+1)^{k+1}$, where $\alpha_j = \sum_{i=1}^{n} \binom{k}{j} \binom{k+1}{2j+1}$ and $n$ is a positive integer. Prove that $A_1 + A_2 + \cdots + A_n \geq n$ with equality for infinitely many $n$.

Solution by George Apostolopoulos, Messolonghi, Greece, modified by the editor.

By using the Binomial Theorem, the identity $(\binom{k}{j} \binom{k+1}{2j+1} = \binom{k+1}{2j+1}$, and the fact that $(\binom{k+1}{j} = 0$ if $2j > k$, we obtain

$$A_n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-3)^j \sum_{k=2j}^{n} \binom{k}{2j} \binom{k+1}{2j+1} = \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=2j}^{n} (-3)^j \binom{k+1}{2j+1}$$

$$= \sum_{k=2j}^{n} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-3)^j}{2k} \binom{k+1}{2j+1} = \frac{1}{\sqrt{3i}} \sum_{k=0}^{n} \frac{1}{2k} \sum_{j=0}^{k+1} \binom{k+1}{2j+1} (\sqrt{3i})^{2j+1}$$

$$= \frac{1}{\sqrt{3i}} \sum_{k=0}^{n} \frac{1}{2k+1} \left[(1 + \sqrt{3i})^{k+1} - (1 - \sqrt{3i})^{k+1}\right]$$

$$= \frac{1}{\sqrt{3i}} \sum_{k=0}^{n} \left[(1 + \sqrt{3i})^{k+1} - (1 - \sqrt{3i})^{k+1}\right].$$
where $i^2 = -1$. Let $\alpha = \frac{1 + \sqrt{3}i}{2}$ and $\beta = \frac{1 - \sqrt{3}i}{2}$, then $\alpha + \beta = \alpha \beta = 1$ and $\alpha - \beta = \sqrt{3}i$. Set $S_\ell = \frac{\alpha^\ell - \beta^\ell}{\alpha - \beta}$; then $S_0 = 0$, $S_1 = 1$, and for each integer $\ell$ we have $S_{\ell+2} = S_{\ell+1} - S_\ell$. It follows that $S_{6k+1} = S_{6k+2} = 1$, $S_{6k+4} = S_{6k+5} = -1$, and $S_{6k} = 0$ for all integers $k$.

This allows us to write $A_n = \sum_{k=0}^{n} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \sum_{k=1}^{n+1} S_k$, and we deduce that $A_{n+6} = A_n$ for all $n$. The initial values are $A_1 = 2$, $A_2 = 2$, $A_3 = 1$, $A_4 = 0$, $A_5 = 0$, and $A_6 = 1$. Thus, $A_1 + A_2 + \cdots + A_n \geq n$ with equality if and only if $n$ is congruent to 0 or 5 modulo 6.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, The University of Scranton, Scranton, PA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3458. [2009: 326, 329] Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Determine the central angle of a sector, such that the square drawn with one vertex on each radius of the sector and two vertices on the circumference, has area equal to the square of the radius of the sector.

Similar solutions by Richard I. Hess, Rancho Palos Verdes, CA, USA; Tom Leong, The University of Scranton, Scranton, PA, USA; and Albert Stadler, Herrliberg, Switzerland.

Call the square $ABCD$ where $A$ and $B$ lie on the the given circle with centre $O$. The given conditions imply that the side length of the square equals the radius of the sector; hence, $\triangle ABO$ is equilateral and $\angle AOB = 60^\circ$. Since $\angle OAD = \angle BAD - \angle BAO = 90^\circ - 60^\circ = 30^\circ$, and $\triangle OAD$ is isosceles, we find that $\angle DOA = 75^\circ$. Similarly, $\angle BOC = 75^\circ$. Thus, the central angle of the sector is

$$\angle DOC = \angle DOA + \angle AOB + \angle BOC$$

$$= 75^\circ + 60^\circ + 75^\circ = 210^\circ.$$

Also solved by GEOFRE APSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; EDM UND SWYLAN, Riga, Latvia; and the proposer.
Proposed by Zafar Ahmed, BARC, Mumbai, India. Let \(a, b, c\) and \(p, q, r\) be positive real numbers. Prove that if \(q^2 \leq pr\) and \(r^2 \leq pq\), then

\[
\frac{a}{pa + qb + rc} + \frac{b}{pb + qc + ra} + \frac{c}{pc + qa + rb} \leq \frac{3}{p + q + r}.
\]

When does equality hold?

Solution by Arkady Alt, San Jose, CA, USA, expanded by the editor.

Consider the system of linear equations below:

\[
\begin{align*}
pa + qb + rc &= x, \\
ra + pb + qc &= y, \\
qa + rb + pc &= z,
\end{align*}
\]

The coefficient matrix has determinant \(\Delta = p^3 + q^3 + r^3 - 3pqr\). If \(\Delta = 0\), then by the AM–GM Inequality \(p = q = r\), in which case equality holds in the required inequality.

Otherwise, by Cramer’s Rule, we obtain that \(a = \frac{1}{\Delta}(x\alpha + y\gamma + z\beta)\), \(b = \frac{1}{\Delta}(x\beta + y\alpha + z\gamma)\), and \(c = \frac{1}{\Delta}(x\gamma + y\beta + z\alpha)\), where \(\alpha = p^2 - qr\), \(\beta = q^2 - rp\), and \(\gamma = r^2 - pq\).

Thus,

\[
\frac{a}{pa + qb + rc} + \frac{b}{pb + qc + ra} + \frac{c}{pc + qa + rb} = \frac{1}{\Delta} \left( \frac{x\alpha + y\gamma + z\beta}{x} + \frac{x\beta + y\alpha + z\gamma}{y} + \frac{x\gamma + y\beta + z\alpha}{z} \right)
\]

\[
= \frac{3\alpha}{\Delta} + \frac{\beta}{\Delta} \left( \frac{z}{x} + \frac{x}{y} + \frac{y}{z} \right) + \frac{\gamma}{\Delta} \left( \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \right).
\]

(1)

Since \(x, y,\) and \(z\) are positive, the AM–GM Inequality yields

\[
\frac{z}{x} + \frac{x}{y} + \frac{y}{z} \geq 3 \quad \text{and} \quad \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \geq 3.
\]

(2)

Since \(\beta \leq 0\) and \(\gamma \leq 0\) by assumption, we obtain from (1) that

\[
\frac{a}{pa + qb + rc} + \frac{b}{pb + qc + ra} + \frac{c}{pc + qa + rb} \leq \frac{3\alpha}{\Delta} + \frac{3\beta}{\Delta} + \frac{3\gamma}{\Delta} = \frac{3(\alpha + \beta + \gamma)}{\Delta} = \frac{3}{p + q + r}.
\]

Equality holds if \(p = q = r\) or if \(\Delta \neq 0\) and \(x = y = z\) (necessary for equality to hold in (2)), and in the latter case \(a = b = c = \frac{x}{\Delta}(\alpha + \beta + \gamma)\).
Conversely, if $a = b = c$ then equality holds; hence, equality holds if and only if $p = q = r$ or $a = b = c$.

Also solved by MOHAMED AASSILA, Strasbourg, France; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comanesti, Romania; and the proposer. There was one incorrect solution submitted.


The triangle $ABC$ has circumcentre $O$, orthocentre $H$, and circumradius $R$. Prove that

$$3R - 2OH \leq HA + HB + HC \leq 3R + OH.$$  

Solution by George Apostolopoulos, Messolonghi, Greece.

We need two known results:

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC},$$  

and, for any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ in Euclidean space,

$$|\vec{b} + \vec{c}| + |\vec{c} + \vec{a}| + |\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}| + |\vec{c}| + |\vec{a} + \vec{b} + \vec{c}|.$$  

The first is the observation (in vector notation) that the segment joining the orthocentre of a triangle to any vertex (represented, for example, by the vector $\overrightarrow{HA} = \overrightarrow{HO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OH}$) is parallel to and twice as long as the segment joining the midpoint of the opposite side to the circumcentre (namely $-\frac{1}{2} (\overrightarrow{OB} + \overrightarrow{OC})$). The second, follows from the triangle inequality applied to Hlawka’s identity (see D. S. Mitrinovic, *Analytic Inequalities*, Springer, Berlin, 1970, page 171 item 2.25.2, for the proof and for further references).

For the rightmost inequality use (1) together with $HA = |\overrightarrow{OB} + \overrightarrow{OC}|$, $R = |\overrightarrow{OA}|$, and analogous expressions for the other two vertices to rewrite the inequality $HA + HB + HC \leq 3R + OH$ as

$$|\overrightarrow{OB} + \overrightarrow{OC}| + |\overrightarrow{OC} + \overrightarrow{OA}| + |\overrightarrow{OA} + \overrightarrow{OB}| \leq |\overrightarrow{OA}| + |\overrightarrow{OB}| + |\overrightarrow{OC}| + |\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}|,$$

which holds by (2).

For the inequality on the left, without loss of generality we can assume that $\angle A \leq 60^\circ$. Then $|\overrightarrow{HA}| = 2R \cos A \geq 2R \cos 60^\circ = R$, while

$$|\overrightarrow{HB}| + |\overrightarrow{OH}| \geq |\overrightarrow{OB}| = R, \quad \text{and} \quad |\overrightarrow{HC}| + |\overrightarrow{OH}| \geq |\overrightarrow{OC}| = R.$$
Add these three inequalities to obtain
\[ 3R - 2|\overline{OH}| \leq |\overline{HA}| + |\overline{HB}| + |\overline{HC}|, \]
and the solution is complete.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

In addition to Apostolopoulos and Tran, only Geupel and Swylan established both inequalities correctly for all triangles; the other submissions relied on an identity that holds only for those triangles having no obtuse angles, namely \( HA + HB + HC = 2R + 2r \). This formula can be found in O. Bottema et al., Geometric Inequalities, Wolters-Noordhoff Publ., Groningen, 1969, page 103 item 2.2, where unfortunately the authors have omitted the necessary condition on the angles; should the triangle have \( \angle A > 90^\circ \), for example, the identity would become \( -HA + HB + HC = 2R + 2r \). The proposer observed that because \( R \geq 2r \), the identity implies that for acute triangles his inequality on the right can be strengthened to \( HA + HB + HC \leq 3R \), an inequality that fails for obtuse triangles.


Let \( I \) be the incentre of triangle \( ABC \) and let \( A', B', \) and \( C' \) be the intersections of the rays \( AI, BI, \) and \( CI \) with the respective sides of the triangle. Prove that
\[ IA + IB + IC \geq 2(I'A' + IB' + IC'). \]

Comment by Oliver Geupel, Brühl, NRW, Germany.


[Editor's note. Geupel then gave a solution from this website, which he attributes to C. Pohoță. As all of the solutions received have some similarities to this one, no solution will be presented here.

Readers are reminded once again that CRUX is not interested in receiving problems that have appeared recently elsewhere, and certainly no problems taken from another source should ever be submitted to CRUX unless that source is identified.]

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE POSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. One incorrect solution was received.

Let \( x, y, \) and \( z \) be positive real numbers such that
\[
(x^3 + z^3 - y^3) (y^3 + x^3 - z^3) (z^3 + y^3 - x^3) > 0.
\]
Prove that
\[
(x^3 + y^3 + z^3 + 3xyz) \prod_{\text{cyclic}} (x^3 + y^3 - z^3 + xyz) \leq 3 \prod_{\text{cyclic}} \sqrt[3]{x^4 (x^2 + yz)^4}.
\]

Composite of similar solutions by Arkady Alt, San Jose, CA, USA, and Thanos Magkos, 3rd High School of Kozani, Kozani, Greece, modified by the editor.

First note that the hypotheses imply that each of the terms \( x^3 + y^3 - z^3, \) \( y^3 + z^3 - x^3, \) and \( z^3 + x^3 - y^3 \) is positive, since if two of them are negative, say \( x^3 + y^3 - z^3 < 0 \) and \( y^3 + z^3 - x^3 < 0, \) then we would have \( 2y^3 < 0, \) or \( y < 0, \) a contradiction. Hence, if we set \( a = x^3 + 3xyz, b = y^3 + 3xyz, \) and \( c = z^3 + 3xyz, \) then \( a + b - c = x^3 + y^3 - z^3 + 3xyz > 0, \) which implies that \( a + b > c. \) Similarly, \( b + c > a \) and \( c + a > b. \) Therefore, since \( a, b, \) and \( c \) are positive, they are the side lengths of a triangle \( ABC. \) In this context, the inequality to be proved is now rewritten as
\[
(a + b + c)(a + b - c)(b + c - a)(c + a - b) \leq 3 \sqrt[3]{a^4b^4c^4}.
\] (1)

Let \( s, R, \) and \( F \) denote the semiperimeter, the circumradius, and the area of triangle \( ABC. \) The following formulas are well known:
\[
\begin{align*}
F &= \sqrt{s(s-a)(s-b)(s-c)}; \\
\frac{a}{\sin \ A} &= \frac{b}{\sin \ B} = \frac{c}{\sin \ C} = 2R; \\
abc &= 4RF.
\end{align*}
\]

Hence, inequality (1) is equivalent to each of the following:
\[
\begin{align*}
16s(s-a)(s-b)(s-c) &\leq 3 \sqrt[3]{(4RF)^4}, \\
16^3F^6 &\leq 3^3 \cdot 4^4 \cdot R^4 \cdot F^4, \\
abc &= 4RF \leq 3\sqrt{3}R^3, \\
8(\sin \ A \, \sin \ B \, \sin \ C)R^3 &\leq 3\sqrt{3}R^3, \\
\sin \ A \, \sin \ B \, \sin \ C &\leq \frac{3\sqrt{3}}{8},
\end{align*}
\]
and it is well known that the last inequality is true [Ed: c.f. Formula 2.8 on p. 20 of O. Bottema et al., Geometric Inequalities, Wolters-Noordhoff Publ., Groningen, 1969.]
Thus, inequality (1) is established, and the problem is solved.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There was one incorrect solution submitted.

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