TOTTEN SOLUTIONS

These are the solutions to the special section of problems appearing in the September 2009 issue and dedicated to the memory of Jim Totten.


Let \( H \) be the orthocentre of triangle \( ABC \) and let \( P \) be the second intersection of the circumcircle of triangle \( AHC \) with the internal bisector of \( \angle BAC \). If \( X \) is the circumcentre of triangle \( APB \) and if \( Y \) is the orthocentre of triangle \( APC \), prove that the length of \( XY \) is equal to the circumradius of triangle \( ABC \).

Solution by John G. Heuver, Grande Prairie, AB.

The points \( A, B, C, H \) form an orthocentric set, so the circumcircles of the four triangles formed have congruent radii. In particular, triangles \( ABC \) and \( AHC \) have equal circumradii with \( AC \) as axis of symmetry. Now \( A, P, C, Y \) also form an orthocentric set with the circumcircles of triangles \( APC \) and \( ACY \) having \( AC \) as axis of symmetry, and \( P \) lies on the circumcircle of triangle \( AHC \), hence \( Y \) lies on the circumcircle of triangle \( ABC \).

Let \( O, O' \) be the circumcentres of triangles \( ABC, AHC \).

Line segments \( YC \) and \( O'X \) are parallel as both are perpendicular to \( AP \). Consider the perpendicular bisector of \( YC \) passing through \( O \) and making a right angle with \( O'X \). The rays forming \( \angle O'OX \) make right angles with the rays of \( \angle PAC \), hence \( \angle O'OX = \frac{1}{2} \angle A \), and similarly \( \angle OXO' = \frac{1}{2} \angle A \). It follows that the perpendicular bisector of \( YC \) also bisects \( O'X \), since triangle \( ABC \) is an orthocentric set.
$OXO'$ is isosceles. Thus, $O'C$ and $XY$ are symmetric in the perpendicular bisector of $YC$, and hence $O'C = XY$, which solves the problem since triangles $ABC$ and $APC$ have the same circumradius.

Also solved by MICHEL BATAILLE, Rouen, France; WALther JANOUS, Ursulinengymnasium, Innsbruck, Austria; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Bola University, La Mirada, CA, USA; and the proposer.


Let $k \geq 2$ be an integer and let $f : [0, \infty) \to \mathbb{R}$ be a bounded continuous function. If $x$ is a positive real number, find the value of

$$
\lim_{n \to \infty} \sqrt{n} \int_0^x \frac{f(t)}{(1 + t^k)^n} \, dt .
$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The substitution $u = t \cdot \sqrt[n]{n}$ transforms

$$
I(n, k, x) = \sqrt{n} \int_0^x \frac{f(t)}{(1 + t^k)^n} \, dt
$$

into

$$
I(n, k, x) = \int_0^{x \cdot \sqrt[n]{n}} \frac{f \left( \frac{u}{\sqrt[n]{n}} \right)}{(1 + u^k)^n} \, du = \int_0^{\infty} \frac{f \left( \frac{u}{\sqrt[n]{n}} \right)}{(1 + u^k)^n} \cdot \chi(u) \, du ,
$$

where $\chi(u) = 1$ if $u \in [0, x \cdot \sqrt[n]{n}]$ and $\chi(u) = 0$ otherwise. Define

$$
g_n(u) = \frac{f \left( \frac{u}{\sqrt[n]{n}} \right)}{(1 + u^k)^n} \cdot \chi(u) .
$$

For fixed $u \geq 0$ and $k$, the denominator is an increasing function of $n$, so that with $|f| \leq M$ we have

$$
|g_n(u)| \leq \frac{M}{1 + u^k} ,
$$

that is, an integrable function dominates $g_n(u)$. By Lebesgue's Dominated Convergence Theorem,

$$
\lim_{n \to \infty} I(n, k, x) = \int_0^{\infty} \left[ \lim_{n \to \infty} g_n(u) \right] \, du = \int_0^{\infty} \frac{f(0)}{\exp(u^k)} \, du = \frac{f(0)}{k} \Gamma \left( \frac{1}{k} \right) .
$$
Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Hertliëg, Switzerland; and the proposer. There was one incorrect solution and one incomplete solution submitted.


Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Find the limit

$$
\lim_{n \to \infty} \int_0^1 \cdots \int_0^1 f \left( \frac{n}{x_1 + \cdots + x_n} \right) \, dx_1 \cdots dx_n.
$$

**Solution by the proposer, modified by the editor.**

Let $k$ be a given positive integer. For any positive integer $n$, let

$$
I_n = \int_0^1 \cdots \int_0^1 \left( \frac{n}{x_1 + \cdots + x_n} \right)^k \, dx_1 \cdots dx_n.
$$

Put $C_k = \int_0^\infty t^{k-1} e^{-t} \, dt < \infty$. We will first show that

$$
\lim_{n \to \infty} I_n = 0.
$$

Making the substitutions $y_i = \frac{1}{x_i}, i = 1, \ldots, n$ in $I_n$, we obtain

$$
I_n = n^k \int_1^\infty \cdots \int_1^\infty \frac{dy_1 \cdots dy_n}{(y_1 + \cdots + y_n)^k y_1^2 \cdots y_n^2}
$$

and also we have

$$
\frac{1}{C_k} \int_0^\infty e^{-t(y_1 + \cdots + y_n)} t^{k-1} \, dt
= \frac{1}{C_k} \int_0^\infty e^{-u} \left( \frac{u}{y_1 + \cdots + y_n} \right)^{k-1} \frac{dy}{y_1 + \cdots + y_n}
= \frac{1}{C_k} \cdot \frac{1}{(y_1 + \cdots + y_n)^k} \int_0^\infty e^{-u} u^{k-1} \, du
= \frac{1}{(y_1 + \cdots + y_n)^k}.
$$

Hence,

$$
I_n = n^k \int_1^\infty \cdots \int_1^\infty \left( \frac{1}{C_k} \int_0^\infty e^{-t(y_1 + \cdots + y_n)} \, dt \right) \frac{dy_1 \cdots dy_n}{y_1^2 \cdots y_n^2}
= \frac{n^k}{C_k} \int_0^\infty t^{k-1} \left( \int_1^\infty \frac{e^{-ty}}{y^2} \, dy \right) \, dt
= \frac{1}{C_k} \int_0^\infty s^{k-1} \left( \int_1^\infty \frac{e^{-sy/n}}{y^2} \, dy \right) \, ds, \quad \text{where } t = \frac{s}{n}.
$$
Let
\[ f_n(s) = s^{k-1} \left( \int_1^\infty \frac{e^{-sy/n}}{y^2} \, dy \right)^n \]
so that
\[ I_n = \frac{1}{C_k} \int_0^\infty f_n(s) \, ds. \tag{2} \]

Now, for all \( n \geq 1 \),
\[ f_n(s) \leq s^{k-1} \left( e^{-s} \int_1^\infty \frac{1}{y^2} \, dy \right)^n = s^{k-1}e^{-s}, \]
and thus, \( f_n \) is integrable for all \( n \geq 1 \). So, if we can show \( \lim_{n \to \infty} f_n(s) = 0 \), then (1) would follow from the Lebesgue Dominated Convergence Theorem and (2). To show that \( \lim_{n \to \infty} f_n(s) = 0 \), put
\[ X_n = \int_1^\infty \frac{e^{-sy/n}}{y^2} \, dy. \]

Then \( 0 < X_n < 1 \) for all \( n \) and by the Monotone Convergence Theorem,
\[ \lim_{n \to \infty} X_n = \int_1^\infty \left( \lim_{n \to \infty} \frac{e^{-sy/n}}{y^2} \right) \, dy = \int_1^\infty \frac{1}{y^2} \, dy = 1. \]

A power series expansion for \( \ln t \) is
\[ \ln t = (t - 1) - \frac{(t - 1)^2}{2} + \frac{(t - 1)^3}{3} - \frac{(t - 1)^4}{4} + \cdots, \quad 0 < t < 2, \]
so that
\[ |n \ln X_n - n(X_n - 1)| = n(X_n - 1)^2 \left| \frac{1}{2} - \frac{1}{3}(X_n - 1) + \frac{1}{4}(X_n - 1)^2 - \cdots \right|. \]

Since \( \lim_{n \to \infty} X_n = 1 \), the sum inside the absolute signs in the above equation approaches \( \frac{1}{2} \) in the limit as \( n \to \infty \). Also,
\[ X_n - 1 = \int_1^n \frac{e^{-sy/n}}{y^2} \, dy + \int_n^\infty \frac{e^{-sy/n}}{y^2} \, dy - 1 \]
\[ = \left( \frac{e^{-sy/n}}{y} \right)_1^n - 1 - \frac{s}{n} \int_1^n \frac{e^{-sy/n}}{y} \, dy + \int_n^\infty \frac{e^{-sy/n}}{y^2} \, dy \]
\[ = \frac{-e^{-s}}{n} + (e^{-s/n} - 1) - \frac{s}{n} \int_1^n \frac{e^{-sy/n}}{y} \, dy + \int_n^\infty \frac{e^{-sy/n}}{y^2} \, dy. \]

Because
\[ \int_1^n \frac{e^{-sy/n}}{y} \, dy < \int_1^n \frac{1}{y} \, dy = \ln n \]
and
\[
\int_{n}^{\infty} \frac{e^{-sy/n}}{y^2} \, dy \leq \int_{n}^{\infty} \frac{1}{y^2} \, dy = \frac{1}{n},
\]
it follows that there is some constant \( C \) such that for sufficiently large \( n \)
\[
X_n - 1 \leq C \left( \frac{\ln n}{n} \right).
\]
Thus, \( \lim_{n \to \infty} n(X_n - 1)^2 = 0 \) and
\[
\lim_{n \to \infty} n \ln X_n = \lim_{n \to \infty} n(X_n - 1),
\]
in the sense that both limits equal the same extended real number or both limits do not exist. In particular
\[
\lim_{n \to \infty} X_n^n = \lim_{n \to \infty} e^{n \ln X_n} = \lim_{n \to \infty} e^{n(X_n - 1)}.
\]  
(3)
Recalling that \( s \) is positive, we have the estimate
\[
n(X_n - 1) = n \int_{1}^{\infty} \frac{e^{-sy/n} - 1}{y^2} \, dy \leq n \int_{1}^{\infty} \frac{e^{-sy/n} - 1}{y^2} \, dy = n \int_{1}^{\infty} \frac{1 - e^{-sy/n}}{y^2} \, dy
\]
\[
= n \left( 1 - e^{-sy/n} \right)^n - s \int_{1}^{\infty} \frac{e^{-sy/n}}{y} \, dy
\]
\[
\leq 1 - s \int_{1}^{\infty} \frac{e^{-s}}{y} \, dy = 1 - se^{-s} \ln n,
\]
hence, \( \lim_{n \to \infty} n(X_n - 1) = -\infty \). Therefore, from (3), \( \lim_{n \to \infty} X_n^n = 0 \), and so
\[
\lim_{n \to \infty} f_n(s) = s^{k-1} \lim_{n \to \infty} X_n^n = 0,
\]
which establishes (1). In particular, this implies that for any polynomial
\[
p(t) = a_0 + a_1 t + \cdots + a_m t^m,
\]
we have
\[
\lim_{n \to \infty} \int_{0}^{1} \cdots \int_{0}^{1} p \left( \frac{n}{x_1 + \cdots + x_n} \right) \, dx_1 \cdots dx_n = a_0 = p(0).
\]
Finally, let \( f \) be any continuous function on \([0, 1]\) and let \( \epsilon > 0 \) be arbitrary.
By the Weierstrass Approximation Theorem, there is a polynomial \( p_{\epsilon} \) such that
\[|f(x) - p_{\epsilon}(x)| \leq \epsilon \text{ for all } x \in [0, 1].\]
Hence,
\[
L = \lim_{n \to \infty} \int_{0}^{1} \cdots \int_{0}^{1} (f - p_{\epsilon}) \left( \frac{n}{x_1 + \cdots + x_n} \right) \, dx_1 \cdots dx_n
\]
\[+ \lim_{n \to \infty} \int_{0}^{1} \cdots \int_{0}^{1} p_{\epsilon} \left( \frac{n}{x_1 + \cdots + x_n} \right) \, dx_1 \cdots dx_n
\]
\[= \lim_{n \to \infty} \left( \int_{0}^{1} \cdots \int_{0}^{1} (f - p_{\epsilon}) \left( \frac{n}{x_1 + \cdots + x_n} \right) \, dx_1 \cdots dx_n \right) + p_{\epsilon}(0).
\]
As $0 < n \left( \frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)^{-1} \leq 1$ for all $x_1, \ldots, x_n$ in $(0, 1]$, we have

$$-\epsilon \leq (f - p_c) \left( \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \right) \leq \epsilon$$

for all $x_1, \ldots, x_n$ in $(0, 1]$. It follows that $|L - p_c(0)| \leq \epsilon$, which together with the fact that $p_c(0) \to f(0)$ as $\epsilon \to 0^+$ implies that $L = f(0)$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and ALBERT STADLER, Herrliberg, Switzerland.


Suppose that $0 < a < b$, $m_0 = \sqrt{ab}$, and $m_1 = \frac{a + b}{2}$. If $x \geq 0$, prove that

$$\frac{x}{m_1(x + m_1)} \leq \frac{1}{b - a} \log \frac{b(x + a)}{a(x + b)} \leq \frac{x}{m_0(x + m_0)}.$$  

**Solution by Oliver Geupel, Brühl, NRW, Germany.**

For $x \geq 0$, let

$$f(x) = \frac{x}{m_1(x + m_1)},$$
$$g(x) = \frac{1}{b - a} \log \frac{b(x + a)}{a(x + b)},$$
$$h(x) = \frac{x}{m_0(x + m_0)}.$$

Since $f$, $g$, and $h$ are differentiable for $x \geq 0$ and $f(0) = g(0) = h(0)$, it suffices to show that

$$f'(x) \leq g'(x) \leq h'(x) \quad (1)$$

for all $x \geq 0$. Since $m_0 \leq m_1$, we have

$$2m_0 x + m_0^2 \leq (a + b) x + ab \leq 2m_1 x + m_1^2.$$

Hence,

$$(x + m_0)^2 \leq (x + a)(x + b) \leq (x + m_1)^2,$$

so that

$$\frac{1}{(x + m_0)^2} \leq \frac{1}{b - a} \left( \frac{1}{x + a} - \frac{1}{x + b} \right) \leq \frac{1}{(x + m_0)^2},$$

and (1) is established.

Let I be the incentre of triangle ABC. Let the point A' be such that \( \overrightarrow{AA'} = (\cos A)\overrightarrow{AI} \), and let points B’ and C’ be defined similarly. Find the radius of the circle passing through A’, B’, and C’ and locate its centre.

I. Solution by J. Chris Fisher, University of Regina, Regina, SK.

We shall see that the circumradius of \( \triangle A'B'C' \) equals the inradius of the original triangle, while its circumcentre is the orthocentre of the triangle whose vertices are the points where the incircle of \( \triangle ABC \) touches its sides.

Denote by \( I_a, I_b, \) and \( I_c \) the images of the incentre I of \( \triangle ABC \) under reflections in the sides \( BC, CA, \) and \( AB, \) respectively. Because \( AI \) bisects \( \angle A, \) we have \( \angle IAI_c = \angle IAI_b = \angle A. \) Consequently, setting \( X = I_bI_c \cap AI \) we see that \( \angle I_cXA = 90^\circ \) implies that

\[
\cos A = \frac{AX}{AI_c} = \frac{AX}{AI},
\]

whence \( X = A'. \) Similarly,

\[
B' = I_cI_a \cap BI, \quad \text{and} \quad C' = I_aI_b \cap CI.
\]

Because \( A', B', \) and \( C' \) are the midpoints of the sides of \( \triangle I_aI_bI_c, \)

the sides of \( \triangle A'B'C' \) are parallel to and half the length of the sides of \( \triangle I_aI_bI_c. \)

By the definition of reflection, the midpoints \( D \) of \( II_a, \) \( E \) of \( II_b, \) and \( F \) of \( II_c \) are the feet of the perpendiculars from I to the sides of the given triangle \( \triangle ABC; \) in other words, the circumcircle of \( \triangle DEF \) is the incircle of \( \triangle ABC. \) Moreover, the dilatation with centre I and ratio 1/2 takes \( \triangle I_aI_bI_c \) to \( \triangle DEF; \) whence,

the sides of \( \triangle DEF \) are parallel to and half the length of the sides of \( \triangle I_aI_bI_c. \)

We conclude that the triangles \( A'B'C' \) and \( DEF \) are congruent, so that the circumradius of \( \triangle A'B'C' \) equals the inradius of \( \triangle ABC, \) as claimed. We now use the parallel corresponding sides of these two triangles to locate the
circumcentre of $\triangle A'B'C'$, call it $M$. Since $D$ is the midpoint of $II_a$ and $\angle IB'I_a = \angle IC'I_a = 90^\circ$, $DB' = DC'$. That is, the perpendicular bisector of $B'C'$ passes through $D$ as well as through $M$. But we saw that $B'C' \parallel EF$, so that $DM$ must be perpendicular also to $EF$. Similarly, $EM \perp FD$, whence $M$ is the orthocentre of $\triangle DEF$.

II. Solution by the proposer.

Let $X(\sigma)$ be the circle with centre $X$ and radius $\sigma$. Since $\sin \frac{A}{2} = \frac{r}{IA}$, where $r$ is the inradius of $ABC$, and $IA_1 = (1 - \cos A)IA = (2 \sin^2 \frac{A}{2})IA$, we see that $IA \cdot IA' = 2r^2$. Thus, $A'$, $B'$, and $C'$ are the inverses of $A, B,$ and $C$ in the circle $I(\sqrt{2r})$, and the circumcircle of $\triangle A'B'C'$, say $M(\rho)$, is the inverse of the circumcircle $O(R)$ of $\triangle ABC$. It follows that $M(\rho)$ is the image of $O(R)$ under the homothety with centre $I$ and factor $\frac{2r^2}{p}$, where $p$ is the power of $I$ with respect to $O(R)$. [Ed.: Details can be found in references dealing with inversive geometry, such as H.S.M. Coxeter, Introduction to Geometry, Section 6.3.] Since $p = IO^2 - R^2 = -2rR$, this factor is $-\frac{r}{2R}$, whence $M$ is defined by $\overrightarrow{MI} = \frac{r}{R} \overrightarrow{IO}$ and $\rho = \frac{r}{R} \cdot R = r$.

Comment. Using classical expressions for $R$ and $r$, one easily obtains trilinear coordinates of $M$ relative to $ABC$: $M(\cos B + \cos C, \cos C + \cos A, \cos A + \cos B)$. Thus, $M$ is point $X_{65}$ of Clark Kimberling’s Encyclopedia of Triangle Centers. (See Math. Magazine, 67:3 (June 1994), p. 179, or the web page, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html). In particular, $M$ is also on the line through the Nagel and Gergonne points of the original triangle.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; and PETER Y. WOO, Bida University, La Mirada, CA, USA.

Bataille further commented that under the inversion of Solution II, the inverses of the sides of triangle $ABC$ are the circles $(IB'C')$, $(IC'A')$, and $(IA'B')$. Since the distance to each side from $I$ is $r$, we see that each of these circles has radius $r$ and contains $I$. We recognize this configuration of three congruent circles containing a common point from earlier problems 2455 [1999 : 307; 2000 : 314] and 3337 [2008 : 173, 175; 2009 : 191-192], where further references are provided. From results established there, we see that $I$ is the orthocentre of $\triangle A'B'C'$, because that triangle is interchanged with $\triangle DEF$ by a halfturn about the common midpoint of $A'$ and $B'$, and $E$, $C'$ and $F$, and $M$ and $I$, (according to Solution I above), we again see that $\rho = r$ and $M$ is the orthocentre of $\triangle DEF$.

TOTTEN–06. [2009 : 321, 323] Proposed by Bill Sands, University of Calgary, Calgary, AB.

Jim and three of his buddies played a round of golf. As usual, Jim won the game. In fact, he beat every two of his three buddies, in the following sense. Let his three buddies be $A$, $B$, and $C$ and let $a_i$ be $A$’s score on hole $i$, for all $1 \leq i \leq 18$, and similarly define $b_i$ and $c_i$. Set $S_{ab} = \sum_{i=1}^{18} \min(a_i, b_i)$,
and similarly define $S_{ac}$ and $S_{bc}$. Then Jim's total score was less than $S_{ab}$, $S_{ac}$, and $S_{bc}$. However, Jim's score was more than $S_{abc} = \sum_{i=1}^{18} \min(a_i, b_i, c_i)$. Jim's score was 72. What was the minimum possible score of any of his buddies?

**Solution by Oliver Geupel, Brühl, NRW, Germany.**

The following example shows that Jim's buddies can have score 75 each:

$$
\begin{align*}
\begin{bmatrix}
    a_1 & a_2 & \ldots & a_{18} \\
    b_1 & b_2 & \ldots & b_{18} \\
    c_1 & c_2 & \ldots & c_{18}
\end{bmatrix} &=
\begin{bmatrix}
    3 & 5 & 5 & 5 & 4 & 4 & \ldots & 4 \\
    5 & 3 & 5 & 5 & 4 & 4 & \ldots & 4 \\
    5 & 5 & 3 & 5 & 4 & 4 & \ldots & 4
\end{bmatrix}.
\end{align*}
$$

We will prove that 75 is the minimum possible score of any of Jim's buddies.

Note that $S_{abc} \leq 71$, whereas each of $S_{ab}$, $S_{bc}$, and $S_{ac}$ is at least 73.

Let $S_a = \sum_{i=1}^{18} a_i$. We show that $S_{ab} < S_a$, the proof being by contradiction.

Assume that $S_a = S_{ab}$. Then for each $1 \leq i \leq 18$ we have that $a_i = \min(a_i, b_i) \leq b_i$; hence $73 \leq S_{ac} = S_{abc} \leq 71$, a contradiction.

Thus, $74 \leq S_{ab} + 1 \leq S_a$. It remains to show that $S_a = 74$ is impossible. The proof is again by contradiction.

Assume that $S_a = 74$. Let us define the quantities

$$
\begin{align*}
m_i &= \min(a_i, b_i, c_i), \\
a'_i &= a_i - m_i, \quad S_{a'} = S_a - S_{abc}, \\
b'_i &= b_i - m_i, \quad S_{a'b'} = S_{ab} - S_{abc} = \sum_{i=1}^{18} \min(a'_i, b'_i), \\
c'_i &= c_i - m_i, \quad S_{a'c'} = S_{ac} - S_{abc} = \sum_{i=1}^{18} \min(a'_i, c'_i).
\end{align*}
$$

For each $1 \leq i \leq 18$, at least one of the numbers $\min(a'_i, b'_i)$ or $\min(a'_i, c'_i)$ is zero. Hence, $\min(a'_i, b'_i) + \min(a'_i, c'_i) \leq a'_i$. Therefore,

$$
S_{a'b'} + S_{a'c'} \leq S_{a'} = S_a - S_{abc} = S_{ab} + 1 - S_{abc} = S_{a'b'} + 1,
$$

which implies that $S_{a'c'} \leq 1$. On the other hand, $S_{a'c'} = S_{ac} - S_{abc} \geq 73 - 71 = 2$, a contradiction.

This completes the proof.

*Also solved by TOM LEONG, The University of Scranton, Scranton, PA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.*

Note that if $S_a = 74$, then $S_a = S_{ab} + 1$, hence $a_i = \min(a_i, b_i) \leq b_i$ fails for exactly one index $i = j$ and $b_j = a_j - 1$. The contradiction $S_{abc} \geq S_{ac} - 1 \geq 73 - 1 = 72$ then arises by an argument similar to the third paragraph of our featured solution.
TOTTEN--07. [2009 : 321, 323] Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Let \( a, b, \) and \( c \) be nonnegative real numbers such that \( a^2 + b^2 + c^2 = 1 \). Prove or disprove that

\[
(a) \quad 1 \leq \frac{a}{1 - ab} + \frac{b}{1 - bc} + \frac{c}{1 - ca} \leq \frac{3\sqrt{3}}{2},
\]

\[
(b) \quad 1 \leq \frac{a}{1 + ab} + \frac{b}{1 + bc} + \frac{c}{1 + ca} \leq \frac{3\sqrt{3}}{4}.
\]

Composite of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and the second proposer.

We show that all four inequalities hold. Both inequalities are cyclic symmetric, so without loss of generality we can assume that either \( a \leq b \leq c \) or \( a \leq c \leq b \).

(a) The left hand side inequality follows from (b).

To prove the right hand side inequality, we homogenize it and prove more generally that for all nonnegative real numbers \( a, b, c \) we have:

\[
\sum_{\text{cyclic}} \frac{a}{a^2 + b^2 + c^2 - ab} \leq \frac{3\sqrt{3}}{2} \frac{1}{\sqrt{a^2 + b^2 + c^2}}.
\]

or

\[
\left( \frac{3\sqrt{3}}{2\sqrt{a^2 + b^2 + c^2}} \right)^2 \geq \left( \sum_{\text{cyclic}} \frac{a}{a^2 + b^2 + c^2 - ab} \right)^2.
\]

This is equivalent to

\[
\frac{1}{4(a^2 + b^2 + c^2)(a^2 + b^2 + c^2 - ab)^2(a^2 + b^2 + c^2 - bc)^2(a^2 + b^2 + c^2 - ca)^2}.
\]

\[
\left[ 23a^{12} - 2a^{11}(31b + 27c) + a^{10}(173b^2 + 78bc + 169c^2)
\right.
\]

\[
- 2a^9(151b^3 + 104b^2c + 128bc^2 + 139c^3)
\]

\[
- a^8(481b^4 + 164b^3c + 752b^2c^2 + 196bc^3 + 473c^4)
\]

\[
- 2a^7(294b^5 + 107b^4c + 418b^3c^2 + 370b^2c^3 + 159bc^4 + 286c^5)
\]

\[
+ a^6(658b^6 - 38b^5c + 1399b^4c^2 + 98b^3c^3 + 1395b^2c^4 + 58bc^5 + 658c^6)
\]

\[
- 2a^5(286b^7 - 296b^6c + 556b^5c^2 + 264b^4c^3 + 316b^3c^4 + 556b^2c^5 + 19bc^6
\]

\[
+ 294c^7)
\]

\[
+ a^4(473b^8 - 318b^7c + 1395b^6c^2 - 632b^5c^3 + 1863b^4c^4
\]

\[
- 528b^3c^5 - 1399b^2c^6 - 214bc^7 + 481c^8)
\]

\[
- 2a^3(b^2 + c^2)^2(139b^7 - 98b^6c
\]

\[
+ 231b^5c^2 + 49b^4c^3 + 33b^3c^4 + 267b^2c^5 - 82bc^6 + 151c^7
\]

\[
+ a^2(b^2 + c^2)(169b^5 - 256b^4c + 583b^3c^2 - 580b^2c^3 - 816b^c^4 + 532bc^5
\]

\[
+ 579b^3c^6 - 208bc^7 + 173c^8)
\]

\[
- 2a(b^2 + c^2)^2(27b^7 - 39b^6c + 50bc^2c^2 - 4b^4c^3
\]

\[
- 20b^3c^4 + 66b^2c^5 - 39bc^6 + 31c^7)
\]

\[
+ (b^2 + c^2)^2(23b^8 - 62b^7c + 127b^6c^2
\]

\[
- 178b^5c^3 + 240bc^4 + 170b^3c^5 + 123b^2c^6 - 54bc^7 + 23c^8)
\]
We denote by $g(a, b, c)$ the numerator of the previous fraction. We need to prove that $g(a, b, c) \geq 0$.

In the first case we suppose that $a \leq b \leq c$. Then $b = a + s$ and $c = a + s + t$, where $s$ and $t$ are nonnegative real numbers. Substituting these into the expression for $g$ yields

$$g(a, a + s, a + s + t) = 576a^{10}(s^2 + st + t^2) + 16a^9(224s^3 + 363s^2t + 411st^2 + 136t^3) + 16a^8(691s^4 + 1544s^3t + 2136s^2t^2 + 1283st^3 + 295t^4) + 4a^7(5428s^5 + 15433s^4t + 25730s^3t^2 + 21866s^2t^3 + 9557st^4 + 1708t^5) + a^6(29876s^6 + 102904s^5t + 202027s^4t^2 + 220046s^3t^3 + 139651s^2t^4 + 48604st^5 + 7364t^6) + 2a^5(14998s^7 + 60523s^6t + 136935s^5t^2 + 181046s^4t^3 + 149499s^3t^4 + 76506s^2t^5 + 22655st^6 + 3006t^7) + a^4(22279s^8 + 102756s^7t + 263080s^6t^2 + 407706s^5t^3 + 412688s^4t^4 + 277136s^3t^5 + 121173s^2t^6 + 31622st^7 + 3815t^8) + 2a^3(6060s^9 + 31334s^8t + 89386s^7t^2 + 158387s^6t^3 + 189197s^5t^4 + 156661s^4t^5 + 90153s^3t^6 + 34826s^2t^7 + 8248st^8 + 918t^9) + a^2(4656s^{10} + 26560s^9t + 83312s^8t^2 + 165648s^7t^3 + 227450s^6t^4 + 223346s^5t^5 + 158989s^4t^6 + 81084s^3t^7 + 28487s^2t^8 + 6256st^9 + 662t^{10}) + 2a(576s^{11} + 3576s^{10}t + 12192s^9t^2 + 26796s^8t^3 + 41472s^7t^4 + 47006s^6t^5 + 39796s^5t^6 + 25167s^4t^7 + 11688s^3t^8 + 3810s^2t^9 + 791st^{10} + 80t^{11}) + (4s^4 + 8s^3t + 8s^2t^2 + 4st^3 + t^4)(36s^8 + 168s^7t + 472s^6t^2 + 796s^5t^3 + 919s^4t^4 + 722s^3t^5 + 389s^2t^6 + 130st^7 + 23t^8) \geq 0. $$

In the second case we have $a \leq b \leq c$, so that $b = a + s + t$ and $c = a + s$, where $s$ and $t$ are nonnegative real numbers. As in the first case, substituting these into the expression for $g$ and simplifying (with the help of a computer algebra system) yields a polynomial in $s$ and $t$ all of whose coefficients are nonnegative, which completes the proof of part (a).

(b) The left end side inequality is equivalent to

$$(1 + ab)(1 + bc)(1 + ca) \leq a(1 + bc)(1 + ca) + b(1 + ab)(1 + ca) + c(1 + ab)(1 + bc),$$

which upon expanding becomes

$$1 + ab + ac + bc + abc(a + b + c) + a^2b^2c^2 \leq a + b + c + a^2c + ab^2 + bc^2 + abc(ab + ac + bc) + 3abc.$$

The above inequality follows by adding the three inequalities below, so it remains to prove each of these:

$$abc(a + b + c) \leq 3abc, \quad (1)$$

$$a^2b^2c^2 \leq abc(ab + bc + ca), \quad (2)$$

$$1 + ab + bc + ca \leq a + b + c + ab^2 + bc^2 + ca^2. \quad (3)$$
The inequality (1) follows from
\[(a + b + c)^2 \leq 3(a^2 + b^2 + c^2) = 3.\]

Also, since \(0 \leq c \leq 1\), we have \(abc \leq ab \leq ab + bc + ca\), from which inequality (2) follows.

Lastly, we have
\[a(1 - a)(1 - c) + b(1 - a)(1 - b) + c(1 - b)(1 - c) \geq 0,
\]
which yields (3).

To prove the right hand inequality, we homogenise again and prove more generally that:
\[
\sum_{\text{cydic}} \frac{a}{a^2 + b^2 + c^2 + ab} \leq \frac{3\sqrt{3}}{4} \frac{1}{\sqrt{a^2 + b^2 + c^2}},
\]
or
\[
\left(\frac{3\sqrt{3}}{4\sqrt{a^2 + b^2 + c^2}}\right)^2 \geq \left(\sum_{\text{cydic}} \frac{a}{a^2 + b^2 + c^2 + ab}\right)^2.
\]

This is equivalent to
\[
\frac{1}{16(a^2 + b^2 + c^2)(a^2 + b^2 + c^2 + ab)^2(a^2 + b^2 + c^2 + bc)^2(a^2 + b^2 + c^2 + ca)^2}
\]
\[+ 2a^9(39b^3 - 24b^2c + 8bc^2 - 9c^3)
\]
\[+ a^8(129b^4 - 44b^3c + 208b^2c^2 - 44bc^3 + 97c^4)
\]
\[+ 2a^7(46b^5 - 57b^4c - 10b^3c^2 - 74b^2c^3 + 41bc^4 + 14c^5)
\]
\[+ a^6(142b^6 - 122b^5c + 351b^4c^2 - 242b^3c^3 + 335b^2c^4 - 122bc^5 + 142c^6)
\]
\[+ 2a^5(14b^7 - 61b^6c - 68b^5c^2 - 164b^4c^3 - 148b^3c^4 - 68b^2c^5 - 61bc^6 + 46c^7)
\]
\[+ a^4(97b^8 - 82b^7c + 335b^6c^2 - 296b^5c^3 + 471b^4c^4
\]
\[+ 328b^3c^5 + 351b^2c^6 - 114bc^7 + 129c^8) - 2a^3(b^2 + c^2)(9b^2 + 22b^2c +
\]
\[+ 65b^2c^2 + 99b^2c^3 + 99b^3c^3 + 49b^4c^5 + 22bc^6 - 39c^7)
\]
\[+ a^2(b^2 + c^2)(45b^2 + 16b^2c + 163b^2c^2 - 36b^2c^3 + 188b^4c^4 - 100b^4c^5
\]
\[+ 147b^4c^6 - 48bc^7 + 61c^8) - 2a(b^2 + c^2)^2(5b^2 - b^2c + 14b^2c + 24b^4c^3
\]
\[+ 24b^3c^2 + 24b^2c^3 - bc^6 - 11c^7) + (b^2 + c^2)(11b^6 + 22b^7c + 39b^6c^2
\]
\[+ 34b^5c^3 + 40b^4c^4 + 2b^3c^5 + 23b^2c^6 - 10bc^7 + 11c^8)] \geq 0.
\]

We denote by \(f(a, b, c)\) the numerator of the previous fraction. We need to prove that \(f(a, b, c) \geq 0\).

In the first case, \(a \leq b \leq c\). Then \(b = a + s\) and \(c = a + s + t\), where \(s\) and \(t\) are nonnegative real numbers. Substituting these into \(f\) we obtain
\[ f(a, a + s, a + s + t) = 9216a^6(a^2 + st + t^2) + 128a^6(478s^3 + 663s^2t + 669st^2 + 242t^3) + 128a^6(1484s^4 + 2644s^3t + 2931s^2t^2 + 1771st^3 + 422t^4) + 8a^7(45222s^5 + 98637s^4t + 123266s^3t^2 + 96630s^2t^3 + 42063st^4 + 7642t^5) + a^8(467076s^6 + 1207100s^5t + 1705636s^4t^2 + 1593386s^3t^3 + 956463s^2t^4 + 328916st^5 + 49204t^6) + 2a^6(213278s^7 + 637841s^6t + 1014189s^5t^2 + 1085926s^4t^3 + 805617s^3t^4 + 393396s^2t^5 + 113243st^6 + 14582t^7) + a^4(278867s^8 + 948076s^7t + 1683432s^6t^2 + 2022590s^5t^3 + 1753596s^4t^4 + 1082132s^3t^5 + 449425s^2t^6 + 112522st^7 + 12883t^8) + 2a^3(64428s^9 + 245574s^8t + 483174s^7t^2 + 642585s^6t^3 + 630183s^5t^4 + 461513s^4t^5 + 245631s^3t^6 + 89646s^2t^7 + 20990st^8 + 2094t^9) + a^2(40296s^{10} + 170312s^9t + 368644s^8t^2 + 537504s^7t^3 + 584030s^6t^4 + 488290s^5t^5 + 311941s^4t^6 + 147356s^3t^7 + 48511s^2t^8 + 9660st^9 + 966t^{10}) + 2a(3856s^{11} + 17912s^{10}t + 42384s^9t^2 + 67268s^8t^3 + 79860s^7t^4 + 74270s^6t^5 + 54608s^5t^6 + 31241s^4t^7 + 13420s^3t^8 + 4076s^2t^9 + 78st^{10} + 72t^{11}) + (4s^4 + 8s^3t + 8s^2t^2 + 4st^3 + t^4)(172s^8 + 528s^7t + 844s^6t^2 + 940s^5t^3 + 815s^4t^4 + 546s^3t^5 + 261s^2t^6 + 78st^7 + 11t^8) \geq 0. \]

In the second case we have \( a \leq c \leq b \), so that \( b = a + s + t \) and \( c = a + s \), where \( s \) and \( t \) are nonnegative real numbers. Again, substituting these into \( f \) and simplifying (with the help of a computer) yields a polynomial in \( a, s, \) and \( t \) all of whose coefficients are nonnegative, which completes the proof of part (b).

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA.

Bracken’s solution also used a computer, and involved expressions even longer than those of our featured solution. The editor has no other solutions to offer, and would be interested in receiving a more “human” solution to this problem.


In triangle \( ABC \) suppose that \( AB < AC \). Let \( D \) and \( M \) be the points on side \( BC \) for which \( AD \) is the angle bisector and \( AM \) is the median. Let \( F \) be on side \( AC \) so that \( AD \) is perpendicular to \( DF \). Finally, let \( E \) be the intersection of \( AM \) and \( DF \). Prove that \( AB \cdot DE + AB \cdot DF = AC \cdot EF \).

Solution by Edmund Swylan, Riga, Latvia.

Define the points \( C', F', G' \) on \( AB \) and \( B', G \) on \( AC \) so that \( BB', CC', FF', \) and \( GG' \) are all perpendicular to \( AD \), with \( M \in GG' \). Because \( AB \cdot DE + AB \cdot DF = AB(DE + DF) = AB(F'D + DE) = AB \cdot F'E \),
the problem reduces to proving that

\[
\frac{F'E}{EF} = \frac{AC}{AB}.
\]  

(1)

Because the dilatation with centre A that takes E to M takes \(F'E\) to \(G'M\) and \(EF\) to \(MG\), the left-hand side of (1) satisfies

\[
\frac{F'E}{EF} = \frac{G'M}{MG}.
\]

Because \(ACC'\) and \(ABB'\) are similar isosceles triangles, the right-hand side of (1) satisfies

\[
\frac{AC}{AB} = \frac{CC'}{BB'}.
\]

The proof concludes by noting that in triangles \(BC'C\) and \(CB'B\), the midline \(G'M\) is half the base \(CC'\) while the midline \(MG\) is half \(BB'\), so that

\[
\frac{CC'}{BB'} = \frac{G'M}{MG}.
\]

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; TITU ZVONARU, Comănești, Romania; and the proposer.

There was one incorrect submission.


Let \(n\) and \(k\) be integers with \(n \geq 2\) and \(k \geq 0\). Consider \(n\) dinner guests sitting around a circular table. Let \(g_n(k)\) be the number of ways that \(k\) of these \(n\) guests can be chosen so that no two chosen guests are sitting next to one another. To illustrate, \(g_6(0) = 1\), \(g_6(1) = 6\), \(g_6(2) = 9\), \(g_6(3) = 2\), and \(g_6(4) = 0\) for all \(k \geq 4\). For each \(n \geq 2\), let

\[
f_n(x) = \sum_{k \geq 0} g_n(k)x^k.
\]

For example, \(f_6(x) = 1 + 6x + 9x^2 + 2x^3 = (1 + 2x)(1 + 4x + x^2)\). Determine all \(n\) for which \((1 + 2x)\) is a factor of \(f_n(x)\).

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The number of strings consisting of \(k\) ones and \(n - k\) zeros such that no two ones are adjacent is \(\binom{n-k-1}{k}\). Treating the \(k\) chosen guests as ones and
the others as zeros, and arbitrarily assigning a guest as the start of a string of length \( n \), we have

\[
g_n(k) = \binom{n - k + 1}{k} - \binom{n - k - 1}{k - 2},
\]

because we must eliminate strings in which the first and last elements are both ones. Thus,

\[
g_n(k) = \frac{(n - k - 1)!}{k!(n - 2k + 1)!} \left[ (n - k)(n - k + 1) - k(k - 1) \right]
= \frac{n(n - k - 1)!}{k!(n - 2k)!}.
\]

so that

\[
f_n(x) = \sum_{k \geq 0} \frac{n(n - k - 1)!}{k!(n - 2k)!} x^k
= n \sum_{k \geq 0} \frac{1}{n - k} \binom{n - k}{k} x^k.
\]

By equation (5.75) of Concrete Mathematics (Graham, Knuth, and Patashnik, 2nd ed.),

\[
f_n(x) = \left( \frac{1 + \sqrt{1 + 4x}}{2} \right)^n + \left( \frac{1 - \sqrt{1 + 4x}}{2} \right)^n.
\]

Noting that \( 1 + 2x \) is a factor of \( f_n(x) \) if and only if \( f_n(-\frac{1}{2}) = 0 \), we calculate

\[
f_n\left(-\frac{1}{2}\right) = \left( \frac{1 + i}{2} \right)^n + \left( \frac{1 - i}{2} \right)^n
= \left( \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}} \right)^n + \left( \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}} \right)^n
= 2 \left( \frac{\sqrt{2}}{2} \right)^n \cos \frac{n\pi}{4}.
\]

Hence, \( 1 + 2x \) is a factor of \( f_n(x) \) if and only if \( \cos \frac{n\pi}{4} = 0 \), that is, if and only if \( n = 2(2\ell + 1) \), where \( \ell \) is a nonnegative integer.

Also solved by OLIVER GEUPEL, BrüN, NRW, Germany; WALther JANous, Ursulinen-gymnasium, Innsbruck, Austria; Tom LeOng, The University of Scranton, Scranton, PA, USA; JoEL SchiloSBERG, Bayside, NY, USA; Albert StADler, Herrliberg, Switzerland; and the proposer.

GeuPeL and the proposer both solved the problem by using the recursive relation \( f_n(x) = f_{n-1}(x) + xf_{n-2}(x) \).

Determine all triangles $ABC$ whose side lengths are positive integers and such that $\cos C = \frac{4}{5}$.

Solution by Oliver Geipel, Brühl, NRW, Germany, modified by the editor.

Let $ABC$ be a triangle with $a = BC$, $b = CA$, $c = AB$ and $\cos C = \frac{4}{5}$. By the Law of Cosines, we have

$$c^2 = a^2 + b^2 - \frac{8}{5}ab. \quad (1)$$

Conversely, if $a$, $b$, $c$ are positive integers satisfying (1), then

$$(a - b)^2 < c^2 < (a + b)^2,$$

hence $a$, $b$, $c$ are the side lengths of a triangle, and again by the Law of Cosines, $\cos C = \frac{4}{5}$. Thus, we seek all positive integer solutions of (1).

Either $a$ or $b$ is divisible by 5. Observe that if $(a, b, c)$ is a solution, then so is $(b, a, c)$, hence it suffices to find the solutions for which 5 divides $a$. Let $a = 5d$.

Then (1) becomes

$$c^2 = 25d^2 + b^2 - 8bd,$$

or

$$c^2 = (b - 4d)^2 + (3d)^2.$$ 

This is the well-known Pythagorean Equation. We distinguish two cases:

Case 1: $b - 4d = 2lmn$, $3d = l(m^2 - n^2)$, $c = l(m^2 + n^2)$.

In this case we solve for $a$, $b$, and $c$ to obtain

$$a = \frac{5}{3}l(m^2 - n^2), \quad b = \frac{4}{3}l(m^2 - n^2) + 2lmn, \quad c = l(m^2 + n^2).$$

Here $a$, $b$, $c$ are positive integers if and only if $3|l(m - n)(m + n)$, $l > 0$, and either $m > n \geq 0$ or $m > -2n \geq 0$.

We distinguish three subcases:

Case 1a: $3|l$ and $l = 3k$. Then

$$a = 5k(m^2 - n^2), \quad b = 4k(m^2 - n^2) + 6kmn, \quad c = 3k(m^2 + n^2),$$

where $k > 0$ and either $m > n \geq 0$ or $m > -2n \geq 0$.

Case 1b: $3|(m - n)$ with $m - n = 3k$. Then $m^2 - n^2 = 3k(2n + 3k)$ and thus

$$a = 5k(2n+3k), \quad b = 4k(2n+3k)+2ln(n+3k), \quad c = l(2n^2+6nk+9k^2),$$
where \( l > 0 \) and either \( k > 0 \) and \( n \geq 0 \), or \( k > -n \geq 0 \).

**Case 1c:** \( 3(m + n) \) with \( m + n = 3k \). Then \( m^2 - n^2 = 3k(3k - 2n) \), and thus
\[
a = 5lk(3k - 2n) , \quad b = 4lk(3k - 2n) + 2ln(3k - n) , \quad c = l(2n^2 - 6nk + 9k^2) ,
\]
where \( l > 0 \) and either \( 3k > 2n \geq 0 \) or \( 3k > -n \geq 0 \).

This completes the first case.

**Case 2:** \( b - 4d = l(m^2 - n^2) \), \( 3d = 2lmn \), \( c = l(m^2 + n^2) \).

In this case we obtain
\[
a = l(m^2 - n^2) + \frac{8}{3} lmn , \quad b = \frac{10}{3} lmn , \quad c = l(m^2 + n^2) .
\]

Now \( a, b, c \) are positive integers if and only if \( 3|lmn \), \( l > 0 \), and \( 3m > n \geq 0 \).

In this case three subcases also arise:

**Case 2a:** \( 3|l \) with \( l = 3k \). Then
\[
a = 3k(m^2 - n^2) + 8kmn , \quad b = 10kmn , \quad c = 3k(m^2 + n^2) ,
\]
where \( k > 0 \) and \( 3m > n \geq 0 \).

**Case 2b:** \( 3|m \) with \( m = 3k \). Then
\[
a = l(9k^2 - n^2) + 8kln , \quad b = 10lnk , \quad c = l(9k^2 + n^2) ,
\]
where \( l > 0 \) and \( 9k > n \geq 0 \).

**Case 2c:** \( 3|n \) with \( n = 3k \). Then
\[
a = l(m^2 - 9k^2) + 8lmk , \quad b = 10lmk , \quad c = l(m^2 + 9k^2) ,
\]
where \( l > 0 \) and \( m > n \geq 0 \).

This completes the second and last case.

The above parametrizations explicitly yield all required triples \((a, b, c)\).

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinegymnasi-ium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Two incomplete solutions were submitted. Janous gave the reference http://mathworld.wolfram.com/PythagoreanTriple.html for the well-known fact that all Pythagorean triangles have a leg divisible by 3, and he used this to express his solution in terms of Pythagorean triples.

(a) Let \( x, y, \) and \( z \) be positive real numbers such that \( x + y + z = 1 \). Prove that
\[
\frac{8\sqrt{3}}{9} \leq \left( \frac{1}{\sqrt{x}} - \sqrt{y} \right) \left( \frac{1}{\sqrt{y}} - \sqrt{z} \right) \left( \frac{1}{\sqrt{z}} - \sqrt{x} \right).
\]

(b) \( \star \). Let \( n \geq 2 \) and let \( x_1, x_2, \ldots, x_n \) be positive real numbers such that \( x_1 + x_2 + \cdots + x_n = 1 \). Prove or disprove that
\[
\left( \frac{n-1}{\sqrt{n}} \right)^n \leq \prod_{k=1}^{n} \left( \frac{1}{\sqrt{x_k}} - \sqrt{x_k} \right).
\]

Solution by Albert stadler, Herrliberg, Switzerland.

(a) This is a special case of (b) when \( n = 3 \).

(b) The inequality fails for \( n = 2 \). [Ed.: Both Bataille and Geupel provided the counterexample \( x_1 = \frac{1}{4}, x_2 = \frac{3}{4} \).]

We assume \( n \geq 3 \) and apply the following inequality proved in the Right–Left Convex Function Theorem (RLCF-Theorem) [Ed.: See V. Cirtoaje, Algebraic Inequalities - Old and New Methods, GIL Publishing House, Romania, 2006.]

RLCF Theorem Let \( f(u) \) be a function defined on an interval \( I \). Suppose \( f \) is convex for either \( u \leq s \) or \( u \geq s \) for some \( s \in I \) and for some fixed positive integer \( n \), \( f(x) + (n-1) f(y) \geq n f(s) \) for all \( x, y \in I \) such that \( x + (n-1)y = ns \). Then \( \sum_{k=1}^{n} f(x_k) \geq n f(s) \) for all \( x_1, x_2, \ldots, x_n \in I \) satisfying \( \sum_{k=1}^{n} x_k = ns \).

If we consider the function \( f(x) = \ln \left( \frac{1}{\sqrt{x}} - \sqrt{x} \right) \) on \( I = (0, 1) \), and let \( s = \frac{1}{n} \), then the given inequality is the same as \( \sum_{k=1}^{n} f(x_k) \geq n f(s) \).

We have that \( f'(x) = \frac{x+1}{2x(x-1)} \) and \( f''(x) = \frac{1-2x-x^2}{2x^2(1-x)^2} \). It is easily verified that for \( x \in (0, 1) \), \( 1 - 2x - x^2 \geq 0 \) if and only if \( x \leq \sqrt{2} - 1 \), so \( f''(x) \geq 0 \) for \( x \leq \sqrt{2} - 1 \). Hence, to apply the RLCF Theorem, it suffices to prove that
\[
f(x) + (n-1)f(y) \geq nf\left( \frac{1}{n} \right)
\]
for all \( x, y \in (0, 1) \) such that \( x + (n-1)y = 1 \).
Since \( y = \frac{1 - x}{n - 1} \), the inequality (1) is successively equivalent to

\[
\ln \left( \frac{1}{\sqrt{x}} - \sqrt{x} \right) + (n - 1) \ln \left( \frac{1}{\sqrt{\frac{1-x}{n-1}}} - \sqrt{\frac{1-x}{n-1}} \right) \geq n \ln \left( \sqrt{n} - \frac{1}{\sqrt{n}} \right);
\]

\[
\left( \frac{1}{\sqrt{x}} - \sqrt{x} \right) \left( \frac{1}{\sqrt{\frac{1-x}{n-1}}} - \sqrt{\frac{1-x}{n-1}} \right)^{n-1} \geq \left( \frac{n-1}{\sqrt{n}} \right)^n;
\]

\[
\left( \frac{1}{\sqrt{x}} - \sqrt{x} \right) \left( \frac{n-1}{1-x} \right)^{(n-1)} \left( 1 - \frac{1-x}{n-1} \right)^{(n-1)} \geq \left( \frac{n-1}{\sqrt{n}} \right)^n;
\]

\[
(1-x)^{1-\left(\frac{n-1}{n}\right)}(n-2+x)^{n-1} \geq n^x(n-1)^{2n-1-\left(\frac{n-1}{2}\right)}\sqrt{x};
\]

\[
(1-x)^{\frac{3-n}{2}}(n-2+x)^{n-1} \geq n^x(n-1)^{\left(\frac{3n-1}{2}\right)}\sqrt{x};
\]

\[
n^n(n-2+x)^{2n-2} \geq (n-1)^{3n-1}(1-x)^{n-3}x.
\]

Let \( f_n(x) = n^n(n-2+x)^{2n-2} - (n-1)^{3n-1}(1-x)^{n-3}x \). We will show that \( f_n(x) \geq 0 \) for all \( x \in (0, 1) \). We have

\[
f'_n(x) = n^n(2n-2)(n-2+x)^{2n-3} - (n-1)^{3n-1}(1-x)^{n-3}x
\]

\[
+ (n-3)(n-1)^{3n-1}(1-x)^{n-4}x
\]

\[
f''_n(x) = n^n(2n-2)(2n-3)(n-2+x)^{2n-4}
\]

\[
+ 2(n-3)(n-1)^{3n-1}(1-x)^{n-4}x
\]

\[
- (n-3)(n-4)(n-1)^{3n-1}(1-x)^{n-5}x
\]

\[
= 2n^n(n-1)(2n-3)(n-2+x)^{2n-4}
\]

\[
+ (n-3)(n-1)^{3n-1}(1-x)^{n-5}[2 - (n-2)x]
\]

(2)

Note that

\[
f_n \left( \frac{1}{n} \right) = n^n \left( n - 2 + \frac{1}{n} \right)^{2n-2} - (n-1)^{3n-1} \left( 1 - \frac{1}{n} \right)^{n-3} \left( \frac{1}{n} \right)
\]

\[
= n^{2-n}(n-1)^{4n-4} - n^{2-n}(n-1)^{4n-4} = 0
\]

(3)

and

\[
f'_n \left( \frac{1}{n} \right) = n^n(2n-2) \left( n - 2 + \frac{1}{n} \right)^{2n-3} - (n-1)^{3n-1} \left( 1 - \frac{1}{n} \right)^{n-3}
\]

\[
+ (n-3)(n-1)^{3n-1} \left( 1 - \frac{1}{n} \right)^{n-4} \left( \frac{1}{n} \right)
\]
\[ = 2n^{3-n}(n-1)^{4n-5} - n^{3-n}(n-1)^{4n-4} \\
+ n^{3-n}(n-3)(n-1)^{4n-5} \\
= n^{3-n}(n-1)^{4n-5}[2 - (n-1) + (n-3)] = 0. \tag{4} \]

Also, we see from (2) that \( f''(x) \geq 0 \) for \( 0 < x \leq \frac{2}{n-2} \). Combining this with (3) and (4) we conclude that

\[ f_n(x) \geq 0 \quad \text{for} \quad 0 < x \leq \frac{2}{n-2}. \tag{5} \]

Hence, (1) is true for \( n = 3 \) and \( n = 4 \). We now assume that \( n \geq 5 \).

Let \( g(x) = x(1-x)^{n-3} \). We find that \( g \) decreases on \( \left[ \frac{1}{n-2}, 1 \right] \), hence \( f_n \) increases on \( \left[ \frac{1}{n-2}, 1 \right] \) since \( n^n(n-2+x)^{2n-2} \) increases on \( (0, 1) \). Now, \( f_n\left(\frac{1}{n-2}\right) \geq 0 \) by (5), so we conclude that \( f_n(x) \geq 0 \) for all \( x \in (0, 1) \).

This completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and ROY BARBARA, Lebanese University, Fanar, Lebanon. Part (a) only was solved by ARKADY ALT, San Jose, CA, USA (2 solutions); SEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHITEY, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; and the proposer. Two incorrect solutions we were submitted.


Let \( w, x, y, \) and \( z \) be positive real numbers with \( w + x + y + z = wxyz \), and let

\[ f(x) = \sqrt[3]{\frac{1}{2} + \frac{1}{4} - \frac{1}{x^3}} + \sqrt[3]{\frac{1}{2} - \frac{1}{4} - \frac{1}{x^3}}. \]

Prove that \( \sqrt[3]{wxyz} + \sqrt[3]{xyz} + \sqrt[3]{yzw} + \sqrt[3]{zwx} \geq f(w) + f(x) + f(y) + f(z) \).

Solution by Oliver Geupel, Brühl, NRW, Germany.

By the AM-GM Inequality, we have

\[ wxy = \frac{w}{z} + \frac{x}{z} + \frac{y}{z} + 1 \]
\[ \geq 4 \left( \frac{wxyz}{z^3} \right)^{\frac{1}{4}}, \]

and by symmetry the cyclic variants of this inequality also hold.
Using the above inequality and the AM–GM Inequality once more, we deduce that

\[ \sum_{\text{cyclic}} \sqrt[3]{wxy} \geq 4^{\frac{1}{2}} \sum_{\text{cyclic}} \frac{(wxy)^{\frac{1}{3}}}{z^{\frac{1}{4}}} \geq 4^{\frac{1}{2}} \cdot 4. \] (1)

The function \( y = g(x) = x^{\frac{1}{3}} \) is concave for \( x > 0 \). Hence, by Jensen’s inequality, we have

\[ f(w) = g \left( \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{w^3}} \right) + g \left( \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{w^3}} \right) \leq 2g \left( \frac{1}{2} \right) = 2 \left( \frac{1}{2} \right)^{\frac{1}{3}} = 4^{\frac{1}{2}}. \]

Similarly, \( f(x) \leq 4^{\frac{1}{2}} \), \( f(y) \leq 4^{\frac{1}{2}} \), and \( f(z) \leq 4^{\frac{1}{2}} \); and hence

\[ f(w) + f(x) + f(y) + f(z) \leq 4 \cdot 4^{\frac{1}{2}}. \] (2)

The desired inequality follows immediately from (1) and (2).

By the conditions for equality to hold in the AM–GM Inequality and Jensen’s inequality, we see that equality holds in the given inequality if and only if \( w = x = y = z = 4^{\frac{1}{2}} \).

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; WALTHE JANOUS, Ursuline gymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

The statement of the problem should have included the extra conditions that \( x^3 \geq 4, y^3 \geq 4, z^3 \geq 4, \) and \( w^3 \geq 4 \). Otherwise, the function \( f \) may not be real valued; for instance, if \( x = y = 1, z = 2, \) and \( w = 4, \) then \( w + x + y + z = wxyz = 8 \) but \( f(x) \) and \( f(y) \) are not real numbers.