THE OLYMPIAD CORNER

No. 287

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We start this issue with problems from the Bulgarian National Olympiad, National Round, 2007. My thanks to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

2007 BULGARIAN NATIONAL OLYMPIAD
National Round
May 12–13, 2007

1. (Emil Kolev, Alexandar Ivanov) The quadrilateral $ABCD$ is such that $\angle BAD + \angle ADC > 180^\circ$ and is circumscribed around a circle of centre $I$. A line through $I$ meets $AB$ and $CD$ at points $X$ and $Y$, respectively. Prove that if $IX = IY$, then $AX \cdot DY = BX \cdot CY$.

2. (Alexandar Ivanov, Emil Kolev) Find the largest positive integer $n$ such that one can choose 2007 distinct integers from the interval $[2 \cdot 10^{n-1}, 10^n)$ with the property that whenever $1 \leq i \leq j \leq n$, then there exists a chosen number with decimal representation $a_1a_2\ldots a_n$ and $a_j \geq a_i + 2$.

3. (Nikolai Nikolov, Oleg Mushkarov) Find the least positive integer $n$ for which $\cos \frac{\pi}{n}$ cannot be expressed in the form $p + \sqrt{q} + \sqrt{r}$, where $p, q, r$ are rational numbers.

4. (Emil Kolev, Alexandar Ivanov) Let $k > 1$ be a fixed integer. A set of positive integers $S$ is called good if all positive integers can be painted in $k$ colours such that no element of $S$ is a sum of two distinct numbers of the same colour. Find the largest positive integer $t$ for which the set

$$S = \{a + 1, a + 2, a + 3, \ldots, a + t\}$$

is good for all positive integers $a$.

5. (Oleg Mushkarov, Nikolai Nikolov) Find the least number $m$ for which any five equilateral triangles with combined area $m$ can cover an equilateral triangle of area 1.

6. (Alexandar Ivanov, Emil Kolev) Let $f(x)$ be a monic polynomial of even degree with integer coefficients. Prove that if there are infinitely many integers $x$ for which $f(x)$ is a perfect square, then there is a polynomial $g(x)$ with integer coefficients such that $f(x) = g^2(x)$. 
Next we continue with problems from the IMO Team Selection Tests for the Bulgarian Team. Thanks again to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

48th IMO
Bulgarian Team First Selection Test
May 16-17, 2007

1. The sequence $\{a_i\}_{i=1}^{\infty}$ is such that $a_1 > 0$ and $a_{n+1} = \frac{a_n}{1 + a_n^2}$ for $n \geq 1$.

   (a) Prove that $a_n \leq \frac{1}{\sqrt{2n}}$ for $n \geq 2$.

   (b) Prove that there exists an $n$ such that $a_n > \frac{7}{10 \sqrt{n}}$.

2. Let $A_1 A_2 A_3 A_4 A_5$ be a convex pentagon such that the triangles $A_1 A_2 A_3$, $A_2 A_3 A_4$, $A_3 A_4 A_5$, $A_4 A_5 A_1$, $A_5 A_1 A_2$ have the same area. Prove that there exists a point $M$ such that the triangles $A_1 MA_2$, $A_2 MA_3$, $A_3 MA_4$, $A_4 MA_5$, $A_5 MA_1$ have the same area.

3. Prove that there are no distinct positive integers $x$ and $y$ such that $x^{2007} + y! = y^{2007} + x!$.

4. Given a point $P$ on the side $AB$ of a triangle $ABC$, consider all pairs of points $(X, Y)$ such that $X \in BC$, $Y \in AC$ and such that $\angle PXB = \angle PYA$. Prove that the midpoints of the segments $XY$ lie on a straight line.

5. The real numbers $a_i, b_i$ for $1 \leq i \leq n$ are such that

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{n} a_i b_i = 0.$$ 

Prove that

$$\left( \sum_{i=1}^{n} a_i \right)^2 + \left( \sum_{i=1}^{n} b_i \right)^2 \leq n.$$ 

6. For a finite set $S$, denote by $\mathcal{P}(S)$ the set of all subsets of $S$. The function $f : \mathcal{P}(S) \to \mathbb{R}$ is such that

$$f(X \cap Y) = \min(f(X), f(Y))$$

for any two subsets $X, Y \in \mathcal{P}(S)$. Find the largest number of distinct values that $f$ can take.
1. Two circles $\Gamma_1$ and $\Gamma_2$ with centres $O_1$ and $O_2$, respectively are externally
tangent at point $P$. A circle $\Gamma_3$ is externally tangent to $\Gamma_1$ at $Q$ and to $\Gamma_2$
at $R$. The lines $PQ$ and $PR$ meet $\Gamma_3$ again at points $A$ and $B$, respectively.
If $AO_2$ meets $BO_1$ at a point $S$, prove that

$$SP \perp O_1O_2.$$ 

2. Find all positive integers $m$ such that

$$\frac{2^m \alpha^m - (\alpha + \beta)^m - (\alpha - \beta)^m}{3\alpha^2 + \beta^2}$$

is an integer for all integers $\alpha$ and $\beta$ with $\alpha\beta \neq 0$.

3. Find all integers $n \geq 3$ such that for any two positive integers $m < n - 1$ and $r < n - 1$ there exist $m$ distinct elements of the set $\{1, 2, \ldots, n - 1\}$ whose sum is congruent to $r$ modulo $n$.

4. Solve the system

$$x^2 + yu = (x + u)^n,$$
$$x^2 + yz = u^4,$$

where $x, y,$ and $z$ are prime numbers and $u$ is a positive integer.

5. Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

(a) $f(xg(y + 1)) + y = xf(y) + f(x + g(y))$ for any $x, y \in \mathbb{R}$, and

(b) $f(0) + g(0) = 0$.

6. Prove that $n = 11$ is the least positive integer such that for any colouring of the edges of a complete graph of $n$ vertices with three colours there exists a monochromatic cycle of length 4.

Next we turn to the problems from Hellenic competitions and the problems of the Mediterranean Mathematical Competition 2007. Thanks again are due to Bill Sands for collecting them for our use.
10th MEDITERRANEAN MATHEMATICAL COMPETITION 2007

1. Let $x \leq y \leq z$ be real numbers satisfying the relation $xy + yz + zx = 1$. Prove that $xz < \frac{1}{2}$. Is it possible to improve the value of the constant $\frac{1}{2}$?

2. The quadrilateral $ABCD$ is convex and cyclic, and the diagonals $AC$ and $BD$ intersect at the point $E$. Given that $AB = 39$, $AE = 45$, $AD = 60$ and $BC = 56$, determine the length of $CD$.

3. In the triangle $ABC$ the angle $\alpha = \angle A$ and the side $a = BC$ are given. It is known that $a = \sqrt{rR}$, where $r$ is the inradius and $R$ is the circumradius of $ABC$. Determine all such triangles, that is, compute the sides $b$ and $c$ of all such triangles.

4. Let $x > 1$ be a real number that is not an integer. Prove that

$$\left(\frac{x + \{x\}}{|x|} - \frac{|x|}{x + \{x\}}\right) + \left(\frac{x + |x|}{\{x\}} - \frac{\{x\}}{x + |x|}\right) > \frac{9}{2},$$

where $\lfloor x \rfloor$ and $\{x\}$ are the integer and the fractional part of $x$, respectively.

Continuing with the Hellenic theme we give the problems of the 24th Balkan Mathematical Olympiad written in Rhodes, Greece, April 2007. Thanks again to Bill Sands for obtaining them for our use.

24th BALKAN MATHEMATICAL OLYMPIAD
April 26, 2007

1. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$ and let $E$ be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^\circ$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y.$$

3. Find all positive integers $n$ for which there exists a permutation $\sigma$ of the set $\{1, 2, \ldots, n\}$ such that the number below is a rational number

$$\sqrt{\sigma(1)} + \sqrt{\sigma(2)} + \sqrt{\cdots} + \sqrt{\sigma(n)}.$$

[Ed.: a permutation of the set $\{1, 2, \ldots, n\}$ is a one-to-one function of this set to itself.]
4. For a given positive integer $n > 2$, let $C_1$, $C_2$, $C_3$ be the boundaries of three convex $n$-gons in the plane such that the sets $C_1 \cap C_2$, $C_2 \cap C_3$, $C_3 \cap C_1$ are finite. Find the maximum cardinality the set $C_1 \cap C_2 \cap C_3$ may have.

To complete the collection of problems for this number we give the Indian Team Selection Test problems, 2007. Thanks again to Bill Sands for obtaining them for the Corner.

INDIAN TEAM SELECTION TEST 2007

1. Let $ABC$ be a triangle with $AB = AC$, and let $\Gamma$ be its circumcircle. The incircle $\gamma$ of $ABC$ moves (slides) on $BC$ in the direction of $B$. Prove that when $\gamma$ touches $\Gamma$ internally, it also touches the altitude through $A$.

2. Consider the quadratic polynomial $p(x) = x^2 + ax + b$, where $a$, $b$ are in the interval $[-2, 2]$. Determine the range of the real roots of $p(x)$ as $a$ and $b$ vary over $[-2, 2]$.

3. Let triangle $ABC$ have side lengths $a$, $b$, $c$; circumradius $R$, and internal angle bisector lengths $w_a$, $w_b$, $w_c$. Prove that

$$\frac{b^2 + c^2}{w_a} + \frac{c^2 + a^2}{w_b} + \frac{a^2 + b^2}{w_c} > 4R.$$

4. Let $a_1, a_2, \ldots, a_n$ be an ordering of the numbers $1, 2, \ldots, n$. Find

$$\min \sum_{j=1}^{n} |a_j - a_{j+1}| \quad \text{and} \quad \max \sum_{j=1}^{n} |a_j - a_{j+1}|,$$

where $a_{n+1} = a_1$ and the extrema are taken over all such possible orderings.

5. Show that in a non-equilateral triangle, the following are equivalent:
   (a) The angles of the triangle are in arithmetic progression.
   (b) The common tangent to the nine-point circle and the incircle is parallel to the Euler line.

6. Let $X$ be the set of all bijective functions from $S = \{1, 2, 3, \ldots, n\}$ to itself. Let $f^0(x) = x$ and $f^{(k)}(x) = f(f^{(k-1)}(x))$ for $k \geq 1$, and for each $f \in X$ define

$$T_f(j) = \begin{cases} 1, & \text{if } f^{(12)}(j) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Determine

$$\sum_{f \in X} \sum_{j=1}^{n} T_f(j).$$
7. Let \( a, b, \) and \( c \) be nonnegative real numbers such that \( a + b \leq c + 1 \), \( b + c \leq a + 1 \), and \( c + a \leq b + 1 \). Prove that
\[
a^2 + b^2 + c^2 \leq 2abc + 1.
\]

8. Given a finite string \( S \) of symbols \( a \) and \( b \), we write \( \triangle(S) \) for the number of \( a \)'s in \( S \) minus the number of \( b \)'s. (For example, \( \triangle(abbabba) = -1 \).) We call a string \( S \) balanced if every substring (of consecutive symbols) \( T \) of \( S \) has the property that \( -1 \leq \triangle(T) \leq 2 \). (Thus, \( abbabba \) is not balanced, as it contains the substring \( bbabb \) and \( \triangle(bbabb) = -3 \).) Find, with proof, the number of balanced strings of length \( n \).

9. Define the functions \( f, g, h \) on \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) as follows:
\[
\begin{align*}
f(x, y, z) &= (3x + 2y + 2z, 2x + 2y + z, 2x + y + 2z), \\
g(x, y, z) &= (3x + 2y - 2z, 2x + 2y - z, 2x + y - 2z), \\
h(x, y, z) &= (3x - 2y + 2z, 2x - y + 2z, 2x - 2y + z).
\end{align*}
\]
Given a primitive Pythagorean triple \((x, y, z)\), with \( x > y > z \), prove that \((x, y, z)\) can be uniquely obtained by repeated application of \( f, g, h \) to the triple \((5, 4, 3)\). For example, \((697, 528, 455) = (f \circ h \circ g \circ h)(5, 4, 3)\).

10. (Short-List, IMO-2007) Circles \( \Gamma_1 \) and \( \Gamma_2 \), with centres \( O_1 \) and \( O_2 \) are externally tangent at the point \( D \) and internally tangent to a circle \( \Gamma \) at points \( E \) and \( F \), respectively. Line \( \ell \) is the common tangent to \( \Gamma_1 \) and \( \Gamma_2 \) at \( D \). Let \( AB \) be the diameter of \( \Gamma \) perpendicular to \( \ell \) so that \( A, E, O_1 \) are on the same side of the line \( \ell \). Prove that \( AO_1, BO_2 \) and \( EF \) are concurrent.

11. Find all pairs of integers \((x, y)\) such that \( y^2 = x^3 - p^2x \), where \( p \) is a prime such that \( p \equiv 3 \pmod{4} \).

12. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying the equation
\[
f(x + y) + f(x)f(y) = (1 + y)f(x) + (1 + x)f(y) + f(xy),
\]
for all \( x, y \in \mathbb{R} \).

Next is an apology for having misfiled some solutions from Mohammed Aassila with a group of solutions to problems for a later number of the Corner. He should also appear as a solver for two problems discussed in the April number of the Corner. These are Problem 3, Thai Mathematical Olympiad [2010 : 155; 2009 : 22] and Problem 2, The Italian Mathematical Olympiad [2010 : 16a; 2009 : 25-26].

1. Find the positive values of $x$ that satisfy

$$x(2\sin x - \cos 2x) < \frac{1}{x}. $$

**Solution by Oliver Geupel, Brühl, NRW, Germany.**

Since $\cos 2x = 1 - 2\sin^2 x$ and $x > 0$, the given inequality is equivalent to $x^2 \sin x (1 + \sin x) < 1$. Write $a = 2\sin x(1 + \sin x)$. For $x > 0$, the inequality $a^2 < 1$ holds if and only if either

1. $0 < x < 1$ and $a > 0$, or
2. $x > 1$ and $a < 0$.

If $0 < x < 1$, then we have $\sin x > 0$; hence the condition (1) is satisfied.

If $x > 1$, then $a < 0$ holds if and only if $-1 < \sin x < 0$. Thus, (2) is equivalent to $(2k + 1)\pi < x < \left(2k + \frac{3}{2}\right)\pi$ or $\left(2k + \frac{3}{2}\right)\pi < x < (2k + 2)\pi$ for some nonnegative integer $k$.

Consequently, the solution set is the union of these open intervals:

$$
(0, 1) \cup \bigcup_{k=0}^{\infty} \left((2k + 1)\pi, \left(2k + \frac{3}{2}\right)\pi\right) \cup \bigcup_{k=0}^{\infty} \left(\left(2k + \frac{3}{2}\right)\pi, (2k + 2)\pi\right).
$$

2. For $f(x) = ax^2 - bx + c$ we know that $0 < |a| < 1$, $f(a) = -b$, and $f(b) = -c$. Prove that $|c| < 3$.

**Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messo-
longhi, Greece; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Ham-
den, CT, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON;
Konstantin Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu
Zvonaru, Comănești, Romania. We give Kandall’s write-up.**

First, suppose $a = b$. Then $f(x) = ax^2 - ax + c$ and $-a = f(a) = a^2 - a^2 + c$, that is, $c = a^2 - a^3$. Then $|c| \leq |a|^2 + |a|^3 + |a| < 3$.

Next, suppose $a \neq b$. Then

$$
-b = f(a) = a^3 - ab + c, \quad (1)
$$

$$
-a = f(b) = ab^2 - b^2 + c. \quad (2)
$$

Subtracting (2) from (1), we obtain successively

$$
a - b = a(a^2 - b^2) - b(a - b),
$$

$$
1 = a(a + b) - b,
$$

$$
b(1 - a) = (a + 1)(a - 1),
$$

$$
b = -(a + 1).$$

From (2) we now successively deduce
\[
-a = a(a + 1)^2 - (a + 1)^2 + c,
\]
\[
c = 1 - a^2 - a^3,
\]
\[
|c| \leq 1 + |a|^2 + |a|^3 < 3,
\]
as desired.


1. Define the function \( t(n) \) on the nonnegative integers by \( t(0) = t(1) = 0, \)
\( t(2) = 1, \) and for \( n > 2 \) let \( t(n) \) be the smallest positive integer which does not divide \( n. \) Let \( T(n) = t(t(t(n))). \) Find the value of \( S \) if
\[
S = T(1) + T(2) + T(3) + \cdots + T(2006).
\]

Solved by Michel Bataille, Rouen, France; Oliver Geupert, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Bataille’s solution, modified by the editor.

We observe that \( T(n) = 0 \) if \( n = 2 \) or if \( n \) is an odd integer, hence
\[
\]
If \( n \in \mathcal{P} = \{4, 6, 8, \ldots, 2006\} \) and \( n \) is not a multiple of 3, then \( t(n) = 3 \) and \( T(n) = 1. \)

Let \( \mathcal{P}_1 = \{6, 18, 30, \ldots, 6 \cdot 333\} \) be the set of elements of \( \mathcal{P} \) of the form \( 6(2m - 1), \) and let \( \mathcal{P}_2 = \{12, 24, 36 \ldots, 6 \cdot 334\} \) be the set of elements of \( \mathcal{P} \) of the form \( 12m. \)

For each \( n \in \mathcal{P}_1, \) we have \( t(n) = 4, \) hence \( T(n) = 2, \) and there are 167 numbers in \( \mathcal{P}_1. \)

If \( n \in \mathcal{P}_2 \) and \( n \) is not a multiple of 5 or \( n \) is not a multiple of 7, then \( t(n) = 5 \) or \( t(n) = 7, \) and then \( T(n) = 1. \) Otherwise \( T(n) \) is one of the following:
\( T(420) = 2, T(840) = 1, T(1260) = 2, \) or \( T(1680) = 1. \)

Thus, \( T(n) = 1 \) for each \( n \in \mathcal{P}_2 \) except for two numbers where \( T \) takes the value 2.

In summary, \( T(n) = 2 \) for \( 167 + 2 = 169 \) numbers and \( T(n) = 1 \) for the \( 1002 - 169 = 833 \) remaining numbers. Therefore, \( S = 833 + 2 \cdot 169 = 1171. \)

3. A unit circle \( k \) with centre \( K \) and a line \( e \) are given in the plane. The perpendicular from \( K \) to \( e \) intersects \( e \) in point \( O \) and \( KO = 2. \) Let \( \mathcal{H} \) be the set of all circles centred on \( e \) and externally tangent to \( K. \)

Prove that there is a point \( P \) in the plane and an angle \( \alpha > 0 \) such that \( \angle APB = \alpha \) for any circle in \( \mathcal{H} \) with diameter \( AB \) on \( e. \) Determine \( \alpha \) and the location of \( P. \)
Solution by Titu Zvonaru, Comănești, Romania.

Let $M$ and $N$ be points on $e$ such that $OM = ON$ and the circles with diameter $OM$ and $ON$ are in $\mathcal{H}$. Let $ST$ be the diameter of the circle belonging to $\mathcal{H}$ and centred at $O$.

The $\triangle MPN$ is isosceles since $\angle MPO = \angle NPO = \alpha$ and $O$ is the midpoint of $MN$, hence the point $P$ lies on $KO$.

Let $O_1$ be the midpoint of $OM$, and let $r = MO_1 = O_1O$. In $\triangle KO_1O$ we have $O_1O^2 + OK^2 = O_1K^2$, or $r^2 + 4 = (r + 1)^2$, and hence $r = \frac{3}{2}$.

Let $OP = t$; then in $\triangle PMO$ we have $\tan \alpha = \frac{3}{t}$, and in $\triangle SPO$ we have $\tan \frac{\alpha}{2} = \frac{1}{t}$. Using the formula $\tan \alpha = \frac{2\tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$ we obtain

$$\frac{3}{t} = \frac{\frac{2}{t}}{1 - \frac{1}{t^2}} \iff \frac{3}{t} = \frac{2t}{t^2 - 1} \iff t^2 = 3,$$

so $OP = \sqrt{3}$ and $\alpha = 60^\circ$.

It remains to prove that the point $P$ and the angle $\alpha$ have the property that $\angle APB = \alpha$ for any circle in $\mathcal{H}$ with diameter $AB$ on $e$.

Suppose that $A$ lies between $O$ and $B$. Let $OA = a, AO' = x$, and let the midpoint of $AB$ be $O'$. 
In $\triangle KOO'$ we obtain
\[
OO'^2 + OK^2 = O'K^2 \iff (a + x)^2 + 4 = (x + 1)^2.
\]
hence, $x = \frac{a^2 + 3}{2(1 - a)}$. It follows that $OB = a + 2 \cdot \frac{a^2 + 3}{2(1 - a)} = \frac{a + 3}{1 - a}$.

Thus
\[
\tan \angle APB = \tan(\angle OPB - \angle OPA) = \frac{\tan \angle OPB - \tan \angle OPA}{1 + \tan \angle OPB \cdot \tan \angle OPA} = \frac{\frac{a + 3}{\sqrt{3(1 - a)}} - \frac{a}{\sqrt{3}}}{1 + \frac{a(a + 3)}{3(1 - a)}} = \frac{a^2 + 3}{\sqrt{3}(1 - a)} \cdot 3 + a^2 = \sqrt{3},
\]
hence $\angle APB = 60^\circ$.

For other locations of $AB$, the calculations are similar.

Next we move to the Hungarian Mathematical Olympiad 2005–2006, Specialized Mathematical Classes, First Round given at [2009 : 212].

1. Is it true that there are infinitely many palindromes in the arithmetic progression $7k + 3$, $k = 0, 1, 2, \ldots$? (A number is a palindrome if reversing its digits yields the same number, for example, $12321$ is a palindrome.)

Solution by Titu Zvonaru, Comănești, Romania.

The answer is YES.

To see this take $k = 10^n - 1$ (with $n$ a positive integer), then we have $7k + 3 = 7 \cdot 10^n - 4 = 70 \ldots 00 - 4 = 699 \ldots 96$, which is a palindrome.

3. The interval $[0, 1]$ is divided by 999 red points into 1000 equal parts and by 1110 blue points into 1111 equal parts. Find the minimum distance between a red point and a blue point. How many pairs of blue and red points achieve this minimum distance?

Solution by Titu Zvonaru, Comănești, Romania.

Let $R_1, R_2, \ldots, R_{999}$ be the red points and $B_1, B_2, \ldots, B_{1110}$ be the blue points. The distance between $R_k$ and $B_t$ is
\[
d = \left| \frac{k}{1000} - \frac{t}{1111} \right| = \frac{|1111k - 1000t|}{1000 \cdot 1111}.
\]

Since 1111 and 1000 are coprime, we deduce that $d \neq 0$ (the equation $1111k - 1000t = 0$ has the solution $k = 1000$ and $t = 1111$). So, we search for solutions to $|1111k - 1000t| = 1$ with $1 \leq k \leq 999$ and $1 \leq t \leq 1110$. 
Case 1: $1111k - 1000t = 1$.
To solve this equation, let $k = \overline{abc}$, where $a, b, c$ are the digits of $k$. Since $1111 \cdot \overline{abc} - 1 = 1000t$, we have $c = 1$ and successively

$$
1111(10 \cdot \overline{ab} + 1) - 1 = 10000t, \\
11110 \cdot \overline{ab} + 1110 = 1000t, \\
1111 \cdot \overline{ab} + 111 = 100t.
$$

We deduce that $b = 9$, hence

$$
1111(10a + 9) + 111 = 100t, \\
11110 \cdot a + 10110 = 100t, \\
1111 \cdot a + 1011 = 1001.
$$

We thus obtain the solution $k = 991$ and $t = 1001$.

Case 2: $1111k - 1000t = -1$.
Then $1111 \cdot \overline{abc} + 1 = 1000t$. We have $c = 9$, and successively

$$
1111(10\overline{ab} + 9) + 1 = 1000t, \\
11110 \cdot \overline{ab} + 10000 = 1000t, \\
1111 \cdot \overline{ab} + 1000 = 100t.
$$

Hence, $\overline{ab} = 0$, and $k = 9, t = 10$.

The minimum distance between a red and a blue point is thus achieved for the pairs $(R_{991}, B_{1001})$ and $(R_9, B_{10})$ and no others.

5. Let $k$ be a circle with centre $O$ and let $AB$ be a chord of $k$ whose midpoint, $M$, is distinct from $O$. The ray from $O$ through $M$ meets $k$ at $R$. Let $P$ be a point on the minor arc $AR$ of $k$, let $PM$ meet $k$ again at $Q$, and let $AB$ meet $QR$ at $S$. Which segment is longer, $RS$ or $PM$?

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Titi Zvonaru, Comănești, Romania. Kandall's solution is presented.

Extend $RO$ to meet $k$ at $T$. Note that $\angle RMS, \angle RQT$ are right angles. Let $\angle MRP = \theta, \angle MRS = \varphi$. Then $RM = RS \cos \varphi$ and $\angle RPQ = \angle RTQ = 90^\circ - \varphi$. Consequently,

$$
\frac{PM}{\sin \theta} = \frac{RM}{\sin(90^\circ - \varphi)}, \\
\quad = \frac{RM}{\cos \varphi} = RS,
$$

that is, $PM = RS \sin \theta < RS$. 

\[
\begin{array}{c}
\text{Diagram showing circle with centre} \ O, \ \text{chord} \ AB, \ \text{point} \ P, \ \text{midpoint} \ M, \ \text{intersection} \ T, \ \text{point} \ Q, \ \text{intersection} \ S, \ \text{rays} \ OR, \ OP, \ OT, \ PM, \ \text{incidence}.
\end{array}
\]
We now give a solution to a problem of the 2005 Kürschák Competition, given at [2009: 213].

2. Ann and Bob are playing tennis. The winner of a match is the player who is the first to win at least four games, being at least two games ahead of his or her opponent. Ann wins a game with probability \( p \leq \frac{1}{2} \) independently of the outcome of the previous games. Prove that Ann wins the match with probability at most \( 2p^2 \).

**Solution by Oliver Geupel, Brühl, NRW, Germany.**

Let \( p_{i,j} \) denote the probability that, after \( i + j \) games, Ann has won exactly \( i \) games. Write \( q = p - 1 \) and \( p_{i,j} = c_{i,j} \cdot p^i q^j \). For \( 1 \leq i, j \leq 3 \) as well as for \( i = j \geq 4 \), it holds that \( p_{i,j} = p_{i-1,j}p + p_{i,j-1}q \), hence

\[
c_{i,j} = c_{i-1,j} + c_{i,j-1} \quad (1 \leq i, j \leq 3 \text{ or } i, j \geq 4).
\]

There are no intermediate scores such that one player won four games being two games ahead of his or her opponent. Therefore, for \( j \geq 5 \), we have \( p_{j-2,j} = p_{j-2,j-1}q = p_{j-2,j-2}q^2 \) and \( p_{j,j-2} = p_{j-1,j-2}p = p_{j-2,j-2}p^2 \); thus

\[
c_{j-2,j} = c_{j-2,j-1} = c_{j-2,j-2} = c_{j-1,j-2} = c_{j,j-2} \quad (j \geq 5).
\]

The constants \( c_{i,j} \) for small indices \( i, j \) can now be directly computed from (1) and (2); some are computed at right.

We guess from the examples that the common value of the constants in (2) is \( 2^{j-3} \cdot 5 \). This can easily be proved by Mathematical Induction.

Consequently, Ann wins the match with probability

\[
p_{4,0} + p_{4,1} + \sum_{i=4}^{\infty} p_{i,i-2} \]

\[
= p^4 + 4p^4q + 10p^4q^2 \sum_{i=0}^{\infty} (2pq)^i = p^4(1 + 4q) + \frac{10p^4q^2}{1 - 2pq}
\]

\[
= \frac{15p^4 - 34p^5 + 2p^6 - 8p^7}{1 - 2p + 2p^2}
\]

\[
= 2p^2 + 2p^2 \left( \frac{1}{2} - p \right) (2 - 11p^2 + 12p^3 - 4p^4).
\]
It remains to prove that the function \( f(p) = 2 - 11p^2 + 12p^3 - 4p^4 \) is positive for \( 0 \leq p \leq \frac{1}{2} \). Since \( f'(p) = -2p(11 - 18p + 8p^2) \) is negative, \( f \) is decreasing and \( f(p) \geq f\left(\frac{1}{2}\right) = \frac{1}{2} > 0 \), which completes the proof.

Now we move to solutions to the problems of the 8th Hong Kong (China) Mathematical Olympiad, given at [2009 : 213].

1. On a planet there are \( 3 \cdot 2005! \) aliens and 2005 languages. Each pair of aliens communicate with each other in exactly one language. Show that there are 3 aliens who communicate with each other in one common language.

Solution by Titu Zvonaru, Comănești, Romania.

We will prove by induction the statement of the problem for \( 3 \cdot n! \) aliens and \( n \) languages.

For \( n = 2 \), let \( A_1, A_2, A_3, A_4, A_5, \) and \( A_6 \) be the aliens and \( \ell_1, \ell_2 \) be the languages. By the Pigeonhole Principle, at least three pairs among \((A_1, A_2), (A_1, A_3), (A_1, A_4), (A_1, A_5), (A_1, A_6)\) use the same language. We may assume that the pairs \((A_1, A_2), (A_1, A_3)\) and \((A_1, A_4)\) use the language \( \ell_1 \).

If one of the pairs \((A_2, A_3), (A_2, A_4)\) or \((A_3, A_4)\) use the language \( \ell_1 \), we are done. If not, then \( A_2, A_3, \) and \( A_4 \) communicate with each other in language \( \ell_2 \).

Suppose that the statement is valid for \( 3 \cdot n! \) aliens and \( n \) languages.

Let \( m = 3 \cdot (n + 1)! \), \( A_1, \ldots, A_m \) be the aliens, and \( \ell_1, \ldots, \ell_{n+1} \) be the languages.

By the Pigeonhole Principle, at least \( \left\lceil \frac{m-1}{n+1} \right\rceil = 3 \cdot n! \) pairs among \((A_1, A_2), (A_1, A_3), \ldots, (A_1, A_m)\) use the same language. We assume then that the pairs \((A_1, A_2), (A_1, A_3), \ldots, (A_1, A_{3n+1})\) use the language \( \ell_1 \).

Now, if two aliens among \( A_2, A_3, \ldots, A_{3n+1} \) use the language \( \ell_1 \), then we are done. Otherwise, these \( 3 \cdot n! \) aliens use only the \( n \) languages \( \ell_2, \ell_3, \ldots, \ell_{n+1} \), and then some three among them use a common language by the induction hypothesis.

This completes the proof.

2. Suppose that there are \( 4n \) line segments of unit length inside a circle of radius \( n \). Given a straight line \( \ell \), prove that there exists a straight line \( \ell' \) that is either parallel to or perpendicular to \( \ell \) and such that \( \ell' \) intersects at least two of the given line segments.

Solved by Oliver Geipel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geipel’s solution.

Let \( s_1, s_2, \ldots, s_{4n} \) denote the \( 4n \) line segments inside the circle \( \Gamma \); let \( x_i \) and \( y_i \) denote the projections of \( s_i \) onto \( \ell \) and onto a fixed perpendicular
\( \ell_\perp \) to \( \ell \), respectively. By the Triangle Inequality, we have
\[
\sum_{i=1}^{4n} |x_i| + \sum_{i=1}^{4n} |y_i| \geq \sum_{i=1}^{4n} |s_i| = 4n.
\]

We consider three cases. First, if \( \sum_{i=1}^{4n} |x_i| > 2n \), then there is a point \( P \) on \( \ell \) that belongs to two of the \( x_i \). In this case, the perpendicular to \( \ell \) through \( P \) is a suitable choice for \( \ell' \). Second, if \( \sum_{i=1}^{4n} |y_i| > 2n \), then there is a point \( Q \) on \( \ell_\perp \) that belongs to two of the \( y_i \). The parallel line to \( \ell \) through \( Q \) is a suitable choice for \( \ell' \). It remains to consider the case \( \sum_{i=1}^{4n} |x_i| = \sum_{i=1}^{4n} |y_i| = 2n \). In this situation, the parallel line to \( \ell \) through the midpoint of \( \Gamma \) is adequate for \( \ell' \).

3. Let \( a, b, c, \) and \( d \) be positive real numbers such that \( a + b + c + d = 1 \). Prove that \( 6(a^3 + b^3 + c^3 + d^3) \geq (a^2 + b^2 + c^2 + d^2) + \frac{1}{8} \).

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Comănești, Romania. We give Alt’s presentation.

By the Power Mean Inequality
\[
\frac{a^3 + b^3 + c^3 + d^3}{4} \geq \left( \frac{a + b + c + d}{4} \right)^3;
\]
\[
a^3 + b^3 + c^3 + d^3 \geq \frac{(a + b + c + d)^3}{16} = \frac{1}{16},
\]
and by Chebychev’s inequality
\[
a^3 + b^3 + c^3 + d^3 \geq \frac{a + b + c + d}{4} (a^2 + b^2 + c^2 + d^2) = \frac{a^2 + b^2 + c^2 + d^2}{4}.
\]

This yields
\[
6(a^3 + b^3 + c^3 + d^3)
\]
\[
= 4(a^3 + b^3 + c^3 + d^3) + 2(a^3 + b^3 + c^3 + d^3)
\]
\[
\geq (a^2 + b^2 + c^2 + d^2) + \frac{1}{8},
\]

as desired.

Next we move to solutions to problems of the Hong Kong Team Selection Test 1 given at [2009: 21a].

1. Find the integer solutions of the equation \( 7(x + y) = 3(x^2 - xy + y^2) \).
Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Bataille’s solution.

The pairs \((0, 0), (4, 5)\) and \((5, 4)\) are clearly solutions for \((x, y)\). We show that there are no other solutions.

Let \((x, y) \neq (0, 0)\) be a solution. Since \(x^2 - xy + y^2 > 0\), we must have \(x + y > 0\). Since 3 is coprime to 7, it divides \(x + y\). Let \(x + y = 3k\).

Since \(7(x + y) = 3(x + y)^2 - 3xy\), we have \(7k = 3(3k^2 - xy)\) so that 3 divides \(k\) as well and finally we see that \(x + y\) is a multiple of 9.

Also, \(xy \leq \frac{(x+y)^2}{4}\), hence \((x+y)^2 - 3xy \geq \frac{(x+y)^2}{4}\) and it follows that \(7(x+y) \geq 3\frac{(x+y)^2}{4}\). From this (and \(x + y > 0\), we deduce that \(x + y \leq \frac{28}{3}\).

Since \(x + y\) is a multiple of 9 and \(0 < x + y \leq \frac{28}{3}\), it follows that \(x + y = 9\), so \(7 \cdot 9 = 3(9^2 - 3xy)\) and \(xy = 20\). Thus, \((x, y) = (4, 5)\) or \((5, 4)\), and we are done.

3. In triangle \(ABC\), the altitude, angle bisector, and median from \(C\) divide \(\angle C\) into four equal angles. Find \(\angle B\).

Solved by Geoffrey A. Kandall, Hamden, CT, USA; D. J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comănești, Romania. We give Smeenk’s solution.

We let \(CD, CE, CF\) be the altitude, angle bisector, and median from \(C\), respectively. It is given that \(\angle ACD = \angle DCE = \angle ECF = \angle FCB = x\).

Then we have \(\alpha = 90^\circ - x, \beta = 90^\circ - 3x\).

It is well known that \(\sin \angle ACF : \sin \angle FCB = \sin \alpha : \sin \beta\), hence

\[
\cos 3x : \sin x = \sin(90^\circ - x) : \sin(90^\circ - 3x),
\]

\[
\cos 3x : \sin x = \cos x : \cos 3x,
\]

\[
\cos 6x = \sin 2x.
\]

This means that \(6x + 2x = 180^\circ\), so \(x = 22.5^\circ\) and \(\angle B = \beta = 22.5^\circ\).
4. Let \( x, y, \) and \( z \) be positive real numbers such that \( x + y + z = 1. \) For a positive integer \( n, \) let \( S_n = x^n + y^n + z^n. \) Also, let \( P = S_2S_{2005} \) and \( Q = S_3S_{2004}. \)

(a) Find the smallest possible value of \( Q. \)

(b) If \( x, y, \) and \( z \) are distinct, determine which of \( P \) or \( Q \) is the larger.

**Solution by Arkady Alt, San Jose, CA, USA.**

(a) By the Power Mean inequality, for any positive integer \( k \) we have

\[
\frac{x^k + y^k + z^k}{3} \geq \left( \frac{x + y + z}{3} \right)^k = \frac{1}{3^k},
\]

\[
S_k \geq x^k + y^k + z^k \geq \frac{1}{3^{k-1}} \geq \frac{1}{3},
\]

where equality occurs if and only if \( x = y = z = \frac{1}{3}. \) Thus the minimum value of \( Q \) is \( \frac{1}{3^2} \cdot \frac{1}{3^{2003}} = \frac{1}{3^{2005}}. \)

(b) We will prove that if \( x, y, z \) are distinct, then

\[
\frac{S_{n+1}}{S_n} > \frac{S_n}{S_{n-1}}
\]

holds for any positive integer \( n. \) Indeed,

\[
S_{n+1}S_{n-1} - S_n^2 = \frac{(x^{n+1} + y^{n+1} + z^{n+1})(x^{n-1} + y^{n-1} + z^{n-1}) - (x^n + y^n + z^n)^2}{3} = \frac{x^{n+1}(y^{n-1} + y^{n-1}) + y^{n+1}(z^{n-1} + x^{n-1}) + z^{n+1}(x^{n-1} + y^{n-1})}{3} -2(x^ny^n + y^nz^n + z^nx^n)
\]

\[
= \sum_{\text{cyclic}} (x^{n+1}y^{n-1} + x^{n-1}y^{n+1} - 2x^ny^n)
\]

\[
= \sum_{\text{cyclic}} x^{n-1}y^{n-1}(x - y)^2 \geq 0.
\]

Therefore, \( \frac{S_{n+1}}{S_n} > \frac{S_m}{S_{m-1}} \) for \( n \geq m \geq 2, \) or \( S_{m-1}S_{n+1} > S_mS_n. \) In particular for \( m = 3, n = 2004 \) we have \( S_2S_{2005} > S_3S_{2004}. \)

Next are solutions to the Hong Kong Team Selection Test 2, given at [2009: 21a–215].

1. Let \( ABCD \) be a cyclic quadrilateral. Show that the orthocentres of \( \triangle ABC, \triangle BCD, \triangle CDA, \) and \( \triangle DAB \) are the vertices of a quadrilateral
congruent to $ABCD$ and show that the centroids of the same triangles are the vertices of a cyclic quadrilateral.

Solved by Michel Bataille, Rouen, France; D.J. Smeenk, Zalkhommel, the Netherlands; and Titu Zvonaru, Comănești, Romania. We give Bataille's solution.

Let $H$, $K$, $L$, and $M$ be the orthocentres of $\triangle ABC$, $\triangle BCD$, $\triangle DCA$, and $\triangle DAB$, respectively and let $O$ be the centre of the circle through $A$, $B$, $C$, $D$. Let $U$ and $V$ be the orthogonal projections of $O$ onto $AC$ and $BD$, respectively and let $W$ be defined by $\overline{UW} = \frac{1}{2} \overline{OM}$ (see the figure). From a well-known property in the triangle, we have $\overline{AM} = 2\overline{OV}$, and similarly $\overline{CK} = 2\overline{OV}$. Hence, $\overline{AM} = \overline{CK}$ and $AMKC$ is a parallelogram.

In addition, since $U$ is the midpoint of $AC$ and $\overline{UW} = \frac{1}{2} \overline{AM}$, $W$ is the midpoint of $CM$ that is, the centre of the parallelogram $AMKC$. In the same way, we obtain that $BHLD$ is a parallelogram with centre $W$ and it follows that the quadrilateral $HKLM$ is the symmetric of $DABC$ about $W$. Thus, $HKLM$ is congruent to $DABC$.

In any triangle with circumcentre $O$, centroid $G$, and orthocentre $H$, we have $\overline{OG} = \frac{1}{3} \overline{OH}$. Here, the triangles $\triangle ABC$, $\triangle BCD$, $\triangle DCA$, and $\triangle DAB$ have the same circumcentre, namely $O$, and therefore the centroids of these triangles are the images of $H$, $K$, $L$, $M$ under the homothety with centre $O$ and factor $\frac{1}{3}$. Since $H$, $K$, $L$, $M$ are concyclic (as $A$, $B$, $C$, $D$ are concyclic), the four centroids are concyclic as well.

2. Let $ABCD$ be a cyclic quadrilateral with $BC = CD$. The diagonals $AC$ and $BD$ intersect at $E$. Let $X$, $Y$, $Z$, and $W$ be the incentres of $\triangle ABE$, $\triangle ADE$, $\triangle ABC$, and $\triangle ADC$, respectively. Show that $X$, $Y$, $Z$, and $W$ are concyclic if and only if $AB = AD$.

Solution by Titu Zvonaru, Comănești, Romania.

Let $AZ$ meet $BC$ at $M$. By the Bisector Theorem, $BM = \frac{BC \cdot AC}{AB + AC}$. 

Since

\[ AM = \frac{2AB \cdot AC \cdot \cos \frac{\angle BAC}{2}}{AB + AC}, \]

applying the Bisector Theorem again in \( \triangle ADB \) with the bisector \( BM \) we obtain \( \frac{AZ}{ZM} = \frac{AB}{BM} \) hence

\[ AZ = \frac{2AB \cdot AC \cdot \cos \frac{\angle BAC}{2}}{AB + BC + AC} \tag{1} \]

We let \( a = BC = CD, x = AB, y = AD, p = BE, q = DE, t = AC, \) and \( m = AE. \)

Since the quadrilateral \( ABCD \) is cyclic we have \( \angle BDC = \angle BAC \) and \( \angle DBC = \angle CAD \), hence \( AE \) is a bisector in \( \triangle ABD \). We have

\[ \frac{BE}{ED} = \frac{AB}{AD} \quad \Rightarrow \quad q = \frac{py}{x}. \tag{2} \]

The quadrilateral \( XZWY \) is cyclic if and only if \( AX \cdot AZ = AY \cdot AW \). Using (1) and (2) we have successively

\[ AX \cdot AZ = AY \cdot AW; \]

\[ \frac{2 \cdot AB \cdot AE \cdot \cos \frac{\angle BAC}{2}}{AB + AE + BE} \cdot \frac{2 \cdot AB \cdot AC \cdot \cos \frac{\angle BAC}{2}}{AB + AC + BC} = \frac{2 \cdot AD \cdot AE \cdot \cos \frac{\angle CAD}{2}}{AD + AE + ED} \cdot \frac{2 \cdot AC \cdot AD \cdot \cos \frac{\angle CAD}{2}}{AD + AC + DC}; \]

\[ \frac{x^2}{(x + m + p)(x + t + a)} = \frac{y^2}{(y + m + q)(y + t + a)}; \]

\[ x^2 [y^2 + y(m + q + t + a) + (m + q)(t + a)] = y^2 [x^2 + x(m + p + t + a) + (m + p)(t + a)]; \]

\[ (x - y)[xy(m + t + a) + m(t + a)(x + y) + py(t + a)] = 0 \]

\[ x - y = 0; \]

hence \( X, Y, Z, \) and \( W \) are concyclic if and only if \( AB = AD \).

3. Points \( A \) and \( B \) lie in a plane and \( \ell \) is a line in that plane passing through \( A \) but not through \( B \). The point \( C \) moves from \( A \) toward infinity along a half-line of \( \ell \). The incircle of \( \triangle ABC \) touches \( BC \) at \( D \) and \( AC \) at \( E \). Show that the line \( DE \) passes though a fixed point.
Solved by Michel Bataille, Rouen, France; Oliver Geipel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geipel's solution.

Let $G$ be the point on the half line from $A$ through $C$ that satisfies $AB = AG$. We prove that the line $DE$ passes through the midpoint of the line segment $BG$.

For the tangential segments $CD$ and $CE$ it holds

$$ CD = CE. \tag{1} $$

Let the incircle of $\triangle ABC$ touch the side $AB$ at point $F$. Then, we have

$$ BD = BF = AB - AF = AG - AE = GE. \tag{2} $$

Let lines $DE$ and $BG$ intersect at the point $S$. By Menelaus' Theorem for $\triangle BCG$ and the line $DE$, and by the equations (1) and (2), we have

$$ 1 = \frac{BD}{CD} \cdot \frac{CE}{GE} \cdot \frac{GS}{BS} = \frac{GS}{BS}. $$

Consequently, $S$ is the midpoint of $BG$, which completes the proof.

Next we look at solutions from our readers to problems of the 20th Nordic Mathematical Olympiad given at [2009 : 215].

1. Let $B$ and $C$ be points on two given rays from the same point $A$, such that $AB + AC$ is constant. Prove that there exists a point $D$ distinct from point $A$ such that the circumcircles of the triangles $ABC$ pass through $D$ for all choices of $B$ and $C$ subject to the given constraint.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's write-up.

Let $m$ be the given constant and let $B_1$, $C_1$ be on the two given rays with $AB_1 = AC_1 = \frac{m}{2}$. Let $D$ be the point of intersection other than $A$ of the circumcircle of $\triangle AB_1C_1$ with the bisector of $\angle B_1AC_1$ (see the figure on the next page).

Now, consider arbitrary points $B$, $C$ satisfying all the constraints. Then $DB_1 = DC_1$, $BB_1 = CC_1$ and in addition, $\angle DB_1B = \angle DC_1C = 90^\circ$ (since $AD$ is a diameter of the circumcircle of $\triangle AB_1C_1$).
It follows that the right-angled triangles \( \triangle DB_1B \) and \( \triangle DC_1C \) are congruent and so \( \angle B_1DB = \angle C_1DC \). As a result,

\[
\angle BDC = \angle B_1DC_1 = 180^\circ - \angle B_1AC_1 = 180^\circ - \angle BAC
\]

and \( A, B, C, D \) are concyclic. In other words, the circumcircle of \( \triangle ABC \) passes through \( D \).

2. The real numbers \( x, y, \) and \( z \) are not all equal and satisfy

\[
x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k.
\]

Determine all possible values of \( k \).

Solved by Arkady Alt, San Jose, CA, USA; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON. We give the solution of the latter.

We prove that \( k = \pm 1 \) and both values are attainable. We have \( z = k - x^{-1} = \frac{kx - 1}{x} \). Since \( z \neq 0 \), substituting for \( z \) in \( y = k - z^{-1} \) yields

\[
y = \frac{k^2x - k - x}{kx - 1}.
\]

Since \( y \neq 0 \), we then have

\[
x = k - y^{-1} = k - \left( \frac{kx - 1}{k^2x - k - x} \right) = \frac{k^3x - k^2 - 2kx + 1}{k^2x - k - x},
\]

or \( k^2x^2 - kx - x^2 = k^3x - k^2 - 2kx + 1 \).

Simplifying, we obtain \( (k^2 - 1)x^2 - k(k^2 - 1)x + k^2 - 1 = 0 \), or \( (k^2 - 1)(x^2 - kx + 1) = 0 \).

If \( x^2 - kx + 1 = 0 \), then \( k = x + x^{-1} \) and \( k = x + y^{-1} \), hence \( x = y \). From \( x + y^{-1} = y + z^{-1} \) we deduce that \( y = z \), so \( x = y = z \), a contradiction.

Thus, \( k^2 = 1 \) since \( x^2 - kx + 1 \neq 0 \).

For each value of \( k \), infinitely many triples \( (x, y, z) \) satisfy the conditions. To see this, let \( x \in \mathbb{R} \setminus \{0, \pm 1\} \) and let \( y = \frac{1}{1 - x} \), \( z = \frac{x - 1}{x} \). Then clearly, \( x + y^{-1} = y + z^{-1} = z + x^{-1} = 1 \). On the other hand, if we let \( y = \frac{-1}{1 + x} \) and \( z = -\frac{1 + x}{x} \), then \( x + y^{-1} = y + z^{-1} = z + x^{-1} = -1 \).
3. The sequence \( \{a_n\} \) of positive integers is defined by \( a_0 = m \) and the recursion \( a_{n+1} = a_n^5 + 487 \) for all \( n \geq 0 \). Determine all values of \( m \) for which the sequence contains as many square numbers as possible.

\[ a_{n+1} = a_n^5 + 487 \]

\( \text{Solution by Oliver Geipel, Brühl, NRW, Germany.} \)

The quadratic residues modulo 4 are 0 and 1. The table below shows the residues of the expression \( x^5 + 487 \) modulo 4, and the above transition diagram demonstrates that only the first and the second term of any \( \{a_n\} \) can be square numbers.

<table>
<thead>
<tr>
<th>( x \pmod{4} )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^5 + 487 \pmod{4} )</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

If \( a_0 = a^2 \) and \( a_1 = b^2 \) for positive integers \( a, b \), then \( b^2 = a^{10} + 487 \); hence \( (b-a^5)(b+a^5) = 487 \). Since 487 is prime, \( b-a^5 = 1 \) and \( b+a^5 = 487 \); thus, \( b = 244, a^5 = 243, A_0 = 9 \), and

\[ a_1 = 9^5 + 487 = 59536 = 244^2. \]

Consequently, \( m = 9 \) is the unique solution.

Next are some solutions to problems of the XXXII Russian Mathematical Olympiad 2005-2006 Final Round 9th Form, given at [2009 : 216-217].

2. Show that there exist four integers \( a, b, c, \) and \( d \) whose absolute values are greater than 1000000 and which satisfy

\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{abcd}. \]

\( \text{Solution by Oliver Geipel, Brühl, NRW, Germany.} \)

We prove that for each positive integer \( N \), there are integers \( a, b, c, d \) with absolute values greater than \( N \) which satisfy the equation. In fact, let \( a = N + 1, b = -N - 2, c = 1 - p, \) and \( d = p(p-1) - 1 \). Then,

\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{p-1} + \frac{1}{p(p-1)-1} \]

\[ = \frac{1}{p - (p-1)} + \frac{1}{p(p-1)-1} \]

\[ = \frac{1}{p(p-1)[p(p-1)-1]} = \frac{1}{(-ab)(-c)d} = \frac{1}{abcd}, \]

which completes the proof.
5. Let $a_1$, $a_2$, $\ldots$, $a_{10}$ be positive integers such that $a_1 < a_2 < \cdots < a_{10}$. Let $b_k$ be the greatest divisor of $a_k$ such that $b_k < a_k$. If $b_1 > b_2 > \cdots > b_{10}$, prove that $a_{10} > 500$.

*Solution by Titu Zvonaru, Comănești, Romania.*

If some $a_i$ with $1 \leq i \leq 9$ is a prime, then $b_i = 1$ and $1 > b_{10}$ a contradiction.

Let $p_i$ be the smallest prime divisor of $a_i$. Then $a_i = p_i b_i$ for each $i$. Since $b_1 > b_2 > \cdots > b_9$ and $a_1 < a_2 < \cdots < a_9$, we deduce that $p_1 < p_2 < \cdots < p_9$.

The first 9 primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, so it follows that $a_9 = p_9 \cdot b_9 \geq 23 \cdot 23 = 529$ (since $p_9 \leq b_9$) and $a_{10} \geq a_9 + 1 \geq 530$.

7. A $100 \times 100$ square board is cut into dominoes (that is, into $2 \times 1$ rectangles). Two players play a game. At each turn, a player may glue together any two adjacent squares if there is a cut between them. A player loses if he or she reconnects the board (thus allowing the board to be lifted by a corner without it falling apart). Who has a winning strategy, the first player or the second player?

*Solution by Oliver Geipel, Brühl, NRW, Germany.*

We generalize to an $m \times n$ rectangular board such that $\frac{mn}{2} - m - n$ is an even integer and $mn + 12 \geq 8(m + n)$. We will prove that the second of two players, $A$ and $B$, has a winning strategy.

Two squares are adjacent if they share an edge. A cut is a common line segment of length 1 between two adjacent unconnected squares. Glueing two squares together is equivalent to removing the cut between them.

A set $C$ of squares is connected if, for any two squares $s$, $s' \in C$, there are squares $s = s_0$, $s_1$, $\ldots$, $s_n = s' \in C$ such that $s_k = s_{k-1}$ is adjacent to $s_k$ and for each $k$ there is no cut between them. A component is a maximal connected set. At any moment the board is partitioned into components, in particular there are $\frac{mn}{2}$ components at the beginning. With each move (turn) the number of components decreases by at most one, and a player loses if he connects the last two components.

$B$'s strategy is as follows.

(1) Remove all cuts having an endpoint on the border of the board. There are at most $2(m + n) - 4$ such cuts, requiring at most $4(m + n) - 8$ moves to be removed. Once this is achieved, then there are no less than

$$\frac{1}{2} mn - 4(m + n) + 8 = \frac{1}{2} (mn + 12 - 8(m + n)) + 2 \geq 2$$

components, one being an external component containing the border of the board, and at least one internal component not touching the border.

Call a set $C$ of squares contiguous if, for any two $s$, $s' \in C$, there are squares $s = s_0$, $s_1$, $\ldots$, $s_n = s' \in C$ such that $s_k = s_{k-1}$ is adjacent to $s_k$ for
$k = 1, 2, \ldots, n$. A cluster is a contiguous union of internal components. It can easily be shown by Mathematical Induction on the number of squares of a cluster, that a cluster has an even number of cuts on its external border.

(2) In each move, remove a cut of a cluster which is not an external border edge of this cluster. This is possible because, before each move of $B$, the number of remaining cuts is odd. (Observe that the initial number of cuts is the even number $\frac{3mn}{2} - m - n$.) The number of clusters is not decreased in this way. Therefore, the number of clusters can only decrease when $A$ moves. Consequently, player $A$ will eventually break up the last cluster.

8. A quadratic polynomial $f(x) = x^2 + ax + b$ is given. Suppose that the equation $f(f(x)) = 0$ has four distinct real roots and that the sum of two of them is equal to $-1$. Prove that $b \leq -\frac{1}{4}$.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille’s solution.

We assume that $a, b \in \mathbb{R}$. Let $x_1, x_2, x_3, x_4$ be the distinct real roots of $f(f(x)) = 0$, so that the numbers $f(x_i)$ are real roots of $f(x) = 0$. Clearly $a^2 \geq 4b$, but if $a^2 = 4b$, then $f(x_i) = -\frac{a}{2}$ for each $i$ and the $x_i$ would be four distinct roots of $x^2 + ax + b + \frac{a}{2} = 0$, a contradiction. Thus, $a^2 > 4b$ and $f(x) = 0$ has two distinct real roots $u_1, u_2$. Note that $\{f(x_1), f(x_2), f(x_3), f(x_4)\} \subset \{u_1, u_2\}$. If three or more of $f(x_1), f(x_2), f(x_3), f(x_4)$ were equal, say to $u_1$, then $x^2 + ax + b - u_1 = 0$ for at least three distinct real values, a contradiction. It follows that we have (say)

$$f(x_1) = f(x_2) = u_1, \quad f(x_3) = f(x_4) = u_2.$$  

Now,

$$f(f(x)) = (x^2 + ax + b)^2 + a(x^2 + ax + b) + b = x^4 + 2ax^3 + \cdots,$$

so that $x_1 + x_2 + x_3 + x_4 = -2a$ with two of $x_1, x_2, x_3, x_4$ adding to $-1$ (by hypothesis). There are just two essential cases: $x_1 + x_2 = -1$ or $x_1 + x_3 = -1$. In the former case, we have $a = -1$ (since $x_1, x_2$ are the roots of $x^2 + ax + b - u_1 = 0$) and $x_3 + x_4 = 1 - 2a = 3$, contradicting the fact that $x_3, x_4$ are the roots of $x^2 + ax + b - u_2 = 0$. Thus, we must be in the latter case $x_1 + x_3 = -1$ with

$$x_1^2 + ax_1 + b - u_1 = 0, \quad x_3^2 + ax_3 + b - u_2 = 0.$$  

Recalling that $u_1 + u_2 = -a$, we deduce $x_1^2 + x_3^2 = -2b$. Since $2(x_1^2 + x_3^2) \geq (x_1 + x_3)^2 = 1$, we obtain $-4b \geq 1$ and the result follows.

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2. Assume that the sum of the cubes of three consecutive positive integers is a cube of some positive integer. Prove that the middle number of these three numbers is divisible by 4.

Solved by Michel Bataille, Rouen, France, and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille’s solution.

Let \( n \) be an integer such that \( n \geq 2 \) and

\[
(n - 1)^3 + n^3 + (n + 1)^3 = m^3
\]

for some positive integer \( m \). We show that \( n \) is divisible by 4.

Since (1) is the same as \( 3n(n^2 + 2) = m^3 \), we see that 3 divides \( m^3 \), hence 3 divides \( m \). With \( m = 3k \), equation (1) becomes

\[
n(n^2 + 2) = 9k^3.
\]

First, we prove that \( n \) is even.

Assume on the contrary that \( n \) is odd. Then, \( n \) and \( n^2 + 2 \) are coprime (any common divisor must be odd and divide \( 2 = (n^2 + 2) - n(n) \)). From (2), it follows that 9 divides \( n \) or \( n^2 + 2 \). In the former case, by comparing the standard prime factorization of each side of (2), we see that \( n^2 + 2 \) must be a cube. However, any cube is congruent to 0, 1, or 8 modulo 9. But \( n^2 + 2 \equiv 2 \pmod{9} \), since 9 divides \( n \), a contradiction.

Similarly, if 9 divides \( n^2 + 2 \), then we see that \( n \equiv 4 \) or \( 5 \pmod{9} \). Since \( n \) must be a cube, we again have a contradiction.

Thus, \( n = 2q \) for a positive integer \( q \) and (2) yields \( 4q(2q^2 + 1) = 9k^3 \). Then 2 divides \( k^3 \) and hence \( k \), so \( k = 2\ell \) and we obtain \( q(2q^2 + 1) = 2 \cdot (9\ell^3) \). Since \( 2q^2 + 1 \) is odd, 2 must divide \( q \), and finally \( n \) is divisible by 4.

8. A 3000 \( \times \) 3000 square is divided into dominoes (that is, into 2 \( \times \) 1 rectangles). Prove that one can paint the dominoes with three colours such that each colour is used equally often and each piece shares a side with no more than two pieces of the same colour.

Solution by Oliver Geupel, Brihl, NRW, Germany.

The statement holds for each \( N \times N \) board where \( 3 \mid N \). We use the Russian national colors white (W), blue (B), and red (R). First, we paint the 1 \( \times \) 1 squares of the board as shown in Figure 1. Now, each domino covers two squares with distinct colors. Second, we paint each domino with the color that does not occur on the the two squares it covers. We will show that this coloring has the desired properties.

Let \( w, b, \) and \( r \) denote the number of white, blue, and red dominoes. A domino is either white or blue, if and only if one of its parts covers one of the \( \frac{2}{3}N^2 \) red cells. Hence, \( w + b = \frac{1}{3}N^2 \). Similar, \( w + r = \frac{1}{3}N^2 \). Thus, \( b = r \). By symmetry, we obtain \( w = b = r \), hence the each color is used equally often.
It remains to show that no domino shares a side with more than two other pieces of its color. By symmetry, it suffices to consider the case in Figure 2. There the central red domino has at most two red neighbors—the two dominos that cover a white and a blue square each.
This proof is complete.


1. Prove that \( \sin \sqrt{x} < \sqrt{\sin x} \) whenever \( 0 < x < \frac{\pi}{2} \).

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give Bataille's version.

We first prove the inequality for \( 0 < x \leq 1 \). Let \( f(t) = \frac{\sin t}{t} \). Then \( f'(t) = \frac{\cos t (t - \tan t)}{t^2} \) is negative on \( (0, 1) \), so \( f \) decreases on this interval.
Also \( 0 < f(x) < 1 \) for \( x \) in this range, so \( \sqrt{f(x)} > f(x) \geq f(\sqrt{x}) \), or
\[
\sqrt{\frac{\sin x}{x}} > \frac{\sin x}{x} \geq \frac{\sin \sqrt{x}}{\sqrt{x}},
\]
and we conclude that \( \sqrt{\sin x} > \sin \sqrt{x} \).

Now, we suppose \( 1 < x < \frac{\pi}{2} \). From \( 0 < \sqrt{x} < \frac{x+1}{2} < \frac{\pi}{2} \), we obtain \( \sin \sqrt{x} < \sin \left( \frac{x+1}{2} \right) \). Hence, it suffices to prove \( \sin \left( \frac{x+1}{2} \right) \leq \sqrt{\sin x} \).

Since \( \cos(x+1) = 1 - 2 \sin^2 \left( \frac{x+1}{2} \right) \), we will equivalently show that
\[
g(x) = 2 \sin x + \cos(x+1) \geq 1 \quad \text{for} \quad 1 \leq x \leq \frac{\pi}{2}. \tag{1}
\]
We have

\[ g'(x) = 2 \cos x - \sin(x + 1), \]
\[ g''(x) = -2 \sin x - \cos(x + 1). \]

Since \( 2 \sin x \geq 2 \sin 1 \) and \( \cos(x + 1) \geq \cos \left(1 + \frac{\pi}{2}\right) = -\sin 1 \), we have \( g''(x) \leq -\sin 1 < 0 \). Thus, \( g' \) decreases from \( g'(1) = 2 \cos 1(1 - \sin 1) > 0 \) to \( g' \left( \frac{\pi}{2} \right) = -\sin \left(1 + \frac{\pi}{2}\right) = -\cos 1 < 0 \). Thus, there exists \( \alpha \in \left(1, \frac{\pi}{2}\right) \) such that \( g'(x) > 0 \) for \( x \in (1, \alpha) \) and \( g'(x) < 0 \) for \( x \in \left(\alpha, \frac{\pi}{2}\right) \). Therefore,

\[ g(x) \geq \min \left\{ g(1), g \left( \frac{\pi}{2} \right) \right\}. \tag{2} \]

Now, \( g \left( \frac{\pi}{2} \right) = 2 - \sin 1 > 1 \) and \( g(1) = 1 + 2 \sin 1(1 - \sin 1) > 1 \), hence the minimum in (2) is greater than 1 and the inequality (1) follows immediately.

4. The angle bisectors \( BB_1 \) and \( CC_1 \) of \( \triangle ABC \) (with \( B_1 \) on \( AC \) and \( C_1 \) on \( AB \)) meet at \( I \). The line \( B_1 C_1 \) meets the circumcircle of \( \triangle ABC \) at \( M \) and \( N \). Prove that the circumradius of \( \triangle MIN \) is twice the circumradius of \( \triangle ABC \).

Partial solution by Geoffrey A. Kandall, Hamden, CT, USA.

Here is a partial solution: the case where \( \triangle ABC \) is isosceles with apex \( A \).

Let \( R \) be the circumradius of \( \triangle ABC \) and \( \triangle AMN \), \( R^* \) be the circumradius of \( \triangle MIN \), \( p = AB = AC \), \( 2q = BC \), \( e = AN \), \( f = IN \), \( h \) be the altitude from \( A \) to \( BC \), and \( x, y, z, w \) be as in the diagram.

Since the product of two sides of a triangle is equal to the product of its circumdiameter and the altitude to the third side, we have

\[ R = \frac{e^2}{2x} = \frac{p^2}{2h} \quad \text{and} \quad R^* = \frac{f^2}{2y}. \]

In view of \( \frac{AC_1}{C_1B} = \frac{AC}{CB} = \frac{p}{2q} \), we have \( \frac{x}{h} = \frac{p}{p + 2q} \) hence

\[ x = \frac{ph}{p + 2q} \quad \text{and} \quad e^2 = \frac{p^3}{p + 2q}; \tag{1} \]

\[ \frac{x}{y} = \frac{x}{y + z}, \quad \frac{y + z}{y} = \frac{p}{2q} \left(1 + \frac{q}{w}\right) = \frac{p + q}{q}. \tag{2} \]
Thus,
\[ y = \frac{q}{p + q}, \quad x = \frac{pqh}{(p + q)(p + 2q)}. \]

Since \( f^2 = e^2 - x^2 + y^2 \) and \( h^2 = p^2 - q^2 \), we have
\[ f^2 = \frac{p^3}{p + 2q} - \frac{p^2(p^2 - q^2)}{(p + 2q)^2} + \frac{p^2q^2(p^2 - q^2)}{(p + q)^2(p + 2q)^2} = \frac{2p^3q}{(p + q)(p + 2q)}. \]

Consequently,
\[ R^* = \frac{1}{2} \cdot \frac{2p^3q}{(p + q)(p + 2q)} \cdot \frac{(p + q)(p + 2q)}{pqh} = \frac{p^2}{h} = 2R. \]

7. The polynomial \((x + 1)^n - 1\) is divisible by a polynomial
\[ P(x) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_1x + c_0 \]
of even degree \( k \) such that \( c_0, c_1, \ldots, c_{k-1} \) are odd integers. Prove that \( n \) is divisible by \( k + 1 \).

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We drop the hypothesis that \( k \) is even.

The result is obvious if \( k = 0 \). If \( k = 1 \), then \((x + 1)^n - 1\) is divisible by \( x + c_0 \); that is \((-c_0 + 1)^n - 1 = 0\). Hence, \( c_0 \in \{0, 2\} \), contradicting the fact that \( c_0 \) is odd. It remains to consider \( k \geq 2 \).

Write \( n = (k + 1)q + r \), where \( q, r \) are integers with \( 0 \leq r \leq k \). Considering the polynomials in the ring \( \mathbb{Z}_2[x] \), we have
\[ (x + 1)^n - 1 = \left(x^k + x^{k-1} + \cdots + x + 1\right)Q(x), \]
\[ (x + 1)^n - x^n = x^{n-k}\left(x^k + x^{k-1} + \cdots + x + 1\right)Q\left(\frac{1}{x}\right), \]
where \( Q(x) \) is a polynomial of degree \( n - k \) and (2) follows from (1) upon replacing \( x \) by \( 1/x \) and multiplying by \( x^n \). Writing \( p_k(x) = x^k + \cdots + x + 1 \), we deduce from (1) and (2) that
\[ x^n - 1 = p_k(x)\left[Q(x) - x^{n-k}Q\left(\frac{1}{x}\right)\right] = p_k(x)S(x), \]
where \( S(x) \in \mathbb{Z}_2[x] \).

Thus, \( p_k(x) \) divides \( x^n - 1 \) in \( \mathbb{Z}_2[x] \). Moreover, \( p_k(x) \) also divides \( x^n - x^r \) in \( \mathbb{Z}_2[x] \), since it divides \( x^{k+1} - 1 \), which divides \( x^{(k+1)q} - 1 \), which in turn divides \( x^{(k+1)q+r} - x^r = x^n - x^r \).

Now, \( p_k(x) \) is of degree \( k \) and it divides \( x^r - 1 = (x^n - 1) - (x^n - x^r) \), which is of degree \( r \). But \( 0 \leq r \leq k \), hence \( r = 0 \), as desired.

That completes the material for this number of the Corner. Send me your nice solutions!