MATHMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Eric Robert (Leo Hayes High School, Fredericton, NB).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le 15 septembre 2010. Les solutions recues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précedera le français, et dans les numéros 2, 4, 6 et 8, le français précedera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M438. Proposé par l'Équipe de Mayhem.

Trouver toutes les paires de nombres réels \((x, y)\) telles que
\[
x^2 + (y^2 - y - 2)^2 = 0.
\]

M439. Proposé par Eric Schmutz, Université Drexel, Philadelphia, PA, É-U.

Trouver l'entier positif \(x\) pour lequel on a
\[
\frac{1}{\log_2 x} + \frac{1}{\log_5 x} = \frac{1}{100}.
\]

M440. Proposé par l'Équipe de Mayhem.

On donne un trapèze \(ABCD\) avec \(AB\) parallèle à \(DC\) et \(AD\) perpendiculaire à \(AB\). Si \(AB = 20, BC = 5x, CD = x^2 + 3x\) et \(DA = 3x\), trouver la valeur de \(x\).

M441. Proposé par Katherine Tsuji et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.

Quel est le nombre maximal de rois non menaçants qu'on peut placer sur un échiquier \(n \times n\)? (Un «roi» est une pièce d'échecs qu'on peut déplacer d'une seule case horizontalement, verticalement ou diagonalement.)
Dans le tableau carré suivant

\[
\begin{pmatrix}
1 & 2 & \cdots & n - 1 & n \\
n + 1 & n + 2 & \cdots & 2n - 1 & 2n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(n - 1)n + 1 & (n - 1)n + 2 & \cdots & n^2 - 1 & n^2 \\
\end{pmatrix}
\]

construit en écrivant sur \( n \) lignes consécutives la liste des nombres de 1 à \( n^2 \), déterminer la somme des nombres sur chaque diagonale. Comparer cette somme à la «constante magique» obtenue en réarrangeant les \( n^2 \) éléments pour former un carré magique.

Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.

On note \([x]\) le plus grand entier n'excédant pas \( x \). Ainsi, \([3.1] = 3\) et \([-1.4] = -2\). On désigne par \( \{x\} \) la partie fractionnaire du nombre réel \( x \) (c'est-à-dire \( \{x\} = x - [x] \)). Par exemple, \( \{3.1\} = 0.1 \) et \( \{-1.4\} = 0.6 \). Trouver tous les nombres réels positifs \( x \) tels que

\[
\left\{ \frac{2x + 3}{x + 2} \right\} + \left\lfloor \frac{2x + 1}{x + 1} \right\rfloor = \frac{14}{9}.
\]

Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.

Soit \( a \) et \( b \) deux nombres réels. Montrer que

\[
\sqrt{a^2 + b^2 + 6a - 2b + 10} + \sqrt{a^2 + b^2 - 6a + 2b + 10} \geq 2\sqrt{10}.
\]

Proposé par the Mayhem Staff.

Trouvez toutes les paires de nombres réels \((x, y)\) telles que

\[
x^2 + (y^2 - y - 2)^2 = 0.
\]

Proposé par Eric Schmutz, Drexel University, Philadelphia, PA, USA.

Déterminez le plus grand entier \( x \) pour lequel

\[
\frac{1}{\log_2 x} + \frac{1}{\log_5 x} = \frac{1}{100}.
\]

Proposé par the Mayhem Staff.

Dans le trapezium \( ABCD \), \( AB \) est parallèle à \( DC \) et \( AD \) est perpendiculaire à \( AB \). Si \( AB = 20 \), \( BC = 5x \), \( CD = x^2 + 3x \), et \( DA = 3x \), déterminez la valeur de \( x \).
**M441.** Proposed by Katherine Tsuji and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

What is the maximum number of non-attacking kings that can be placed on an \( n \times n \) chessboard? (A “king” is a chess piece that can move horizontally, vertically, or diagonally from one square to an adjacent square.)

**M442.** Proposed by Carl Libis, Cumberland University, Lebanon, TN, USA.

Consider the square array

\[
\begin{pmatrix}
1 & 2 & \cdots & n-1 & n \\
n+1 & n+2 & \cdots & 2n-1 & 2n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(n-1)n+1 & (n-1)n+2 & \cdots & n^2-1 & n^2 \\
\end{pmatrix}
\]

formed by listing the numbers 1 to \( n^2 \) in order in consecutive rows. Determine the sum of the numbers on each diagonal. How does this sum compare to the “magic constant” that would be obtained if the \( n^2 \) entries were rearranged to form a magic square?

**M443.** Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Let \( \lfloor x \rfloor \) denote the greatest integer not exceeding \( x \). For example, \( \lfloor 3.1 \rfloor = 3 \) and \( \lfloor -1.4 \rfloor = -2 \). Let \( \{ x \} \) denote the fractional part of the real number \( x \) (that is, \( \{ x \} = x - \lfloor x \rfloor \)). For example, \( \{ 3.1 \} = 0.1 \) and \( \{ -1.4 \} = 0.6 \). Find all positive real numbers \( x \) such that

\[
\left\lfloor \frac{2x+3}{x+2} \right\rfloor + \left\lfloor \frac{2x+1}{x+1} \right\rfloor = \frac{14}{9}.
\]

**M444.** Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let \( a \) and \( b \) be real numbers. Prove that

\[
\sqrt{a^2 + b^2 + 6a - 2b + 10} + \sqrt{a^2 + b^2 - 6a + 2b + 10} \geq 2\sqrt{10}.
\]

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**Mayhem Solutions**

**M381.** Correction. Proposed by Mihály Bencze, Brasov, Romania.

Determine all of the solutions to the equation

\[
\frac{1}{x-1} + \frac{2}{x-2} + \frac{6}{x-6} + \frac{7}{x-7} = x^2 - 4x - 4.
\]
Solution by Sonthaya Senamontree, Thesaban 2 Mukhamontree School,
Udonthani, Thailand.

From the given equation

\[
\frac{1}{x - 1} + \frac{2}{x - 2} + \frac{6}{x - 6} + \frac{7}{x - 7} = x^2 - 4x - 4; \\
\left(\frac{1}{x - 1} + 1\right) + \left(\frac{2}{x - 2} + 1\right) + \left(\frac{6}{x - 6} + 1\right) + \left(\frac{7}{x - 7} + 1\right) = x^2 - 4x; \\
\frac{x}{x - 1} + \frac{x}{x - 2} + \frac{x}{x - 6} + \frac{x}{x - 7} = x^2 - 4x.
\]

Since \(x\) is a common factor of both sides, then \(x = 0\) is a solution. We can continue by assuming that \(x \neq 0\) and dividing by \(x\) to obtain

\[
\frac{1}{x - 1} + \frac{1}{x - 2} + \frac{1}{x - 6} + \frac{1}{x - 7} = x - 4; \\
\left(\frac{1}{x - 1} + \frac{1}{x - 7}\right) + \left(\frac{1}{x - 2} + \frac{1}{x - 6}\right) = x - 4; \\
\frac{2x - 8}{(x - 1)(x - 7)} + \frac{2x - 8}{(x - 2)(x - 6)} = x - 4; \\
\frac{2x - 8}{x^2 - 8x + 7} + \frac{2x - 8}{x^2 - 8x + 12} = x - 4.
\]

Since \(x = 4\) makes both sides 0, then \(x = 4\) is a solution. We can continue by assuming that \(x \neq 4\) and dividing by \(x - 4\) to obtain:

\[
\frac{2}{x^2 - 8x + 7} + \frac{2}{x^2 - 8x + 12} = 1,
\]

and then make the substitution \(a = x^2 - 8x\) to obtain

\[
\frac{2}{a + 7} + \frac{2}{a + 12} = 1; \\
2(a + 12) + 2(a + 7) = (a + 7)(a + 12); \\
2a + 24 + 2a + 14 = a^2 + 19a + 84; \\
0 = a^2 + 15a + 46.
\]

The quadratic formula yields \(a = \frac{-15 \pm \sqrt{15^2 - 4(1)(46)}}{2} = \frac{-15 \pm \sqrt{41}}{2}\).
Since \( a = x^2 - 8x \), then
\[
x^2 - 8x = \frac{-15 \pm \sqrt{41}}{2};
\]
\[
x^2 - 8x + 16 = \frac{17 \pm \sqrt{41}}{2};
\]
\[
(x - 4)^2 = \frac{17 \pm \sqrt{41}}{2};
\]
\[
x = 4 \pm \sqrt{\frac{17 \pm \sqrt{41}}{2}}.
\]

Therefore, \( x = 0 \) or \( x = 4 \) or \( x = 4 \pm \sqrt{\frac{17 \pm \sqrt{41}}{2}} \), with all four combinations of signs being possible.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; G.C. GREUBEL, Newport News, VA, USA; Konstantinos Al. Nakos, Agrinio, Greece; Ricard Peiro, IES "Abastos", Valencia, Spain; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

M401. Proposed by the Mayhem Staff.

Graham and Vazz were marking out a new lawn at CRUX Headquarters. Graham said: "If you make the lawn 9 metres longer and 8 metres narrower, the area will be the same". Vazz said: "If you make it 12 metres shorter and 16 metres wider, the area will still be the same". What are the dimensions of the lawn?

Solution by Jadyn Chang, student, Western Canada High School, Calgary, AB.

Let \( x \) be the length of the lawn and \( y \) be the width of the lawn. Thus, the area of the lawn is \( xy \). We can translate Graham's and Vazz's statements into equations.

According to Graham, \( xy = (x + 9)(y - 8) = xy - 8x + 9y - 72 \), and so \( 8x - 9y = -72 \).

According to Vazz, \( xy = (x - 12)(y + 16) = xy + 16x - 12y - 192 \), and so \( 16x - 12y = 192 \) or \( 8x - 6y = 96 \).

Subtracting the first linear equation from the second one, we obtain \( 3y = 168 \), or \( y = 56 \). We can substitute \( y = 56 \) into either equation to obtain \( x = 54 \).

Therefore, the lawn is 54 m long and 56 m wide.

Also solved by George Apostolopoulos, Messolonghi, Greece; Winda Kirana, student, SMPN 8, Yogyakarta, Indonesia; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Mridul Singh, student, Kendriya Vidyalaya School, Shillong, India; Mrinal Singh, student, Kendriya Vidyalaya School, Shillong, India; Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON; and Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia. There were two incorrect solutions submitted.
M402. Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Determine all ordered pairs \((a, b)\) of positive integers such that
\[a^b b^a + a^b + b^a = 89.\]

Solution by Winda Kirana, student, SMPN 8, Yogyakarta, Indonesia and Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia, independently.

Since \(a^b b^a + a^b + b^a = 89\), then we have that \(a^b b^a + a^b + b^a + 1 = 90\), or \((a^b + 1)(b^a + 1) = 90\).

Since \(a\) and \(b\) are positive integers, then \(a^b + 1\) and \(b^a + 1\) are both positive integer divisors of 90 and each of these divisors is larger than 1.

We make a table of the possible values of \(a^b\) and \(b^a\):

<table>
<thead>
<tr>
<th>(a^b + 1)</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>9</th>
<th>10</th>
<th>15</th>
<th>18</th>
<th>30</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b^a + 1)</td>
<td>45</td>
<td>30</td>
<td>18</td>
<td>15</td>
<td>10</td>
<td>9</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

If \(a^b = 2\), then \(a = 2\) and \(b = 1\), which does not give \(b^a = 29\).

If \(a^b = 4\), then \((a, b) = (4, 1)\) or \((a, b) = (2, 2)\), neither of which gives \(b^a = 17\). If \(a^b = 5\), then \(a = 5\) and \(b = 1\), which does not give \(b^a = 14\).

Similar reasoning shows that \(a^b\) cannot be 14, 17, or 29.

If \(b^a = 44\), then \(b = 44\) and \(a = 1\), which does give \(a^b = 1\). Thus, \((a, b) = (1, 44)\) is a solution. Similarly, \((a, b) = (44, 1)\) is a solution from the last row.

If \(a^b = 8\), then \((a, b) = (8, 1)\) or \((a, b) = (2, 3)\). The second of these gives \(b^a = 9\), so \((a, b) = (2, 3)\) is a solution, as is \((a, b) = (3, 2)\) from the following row.

Therefore, the solutions are \((a, b) = (1, 44), (44, 1), (2, 3), (3, 2)\).

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; RICARDO PEIRO, IES "Abastos", Valencia, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were six incorrect solutions submitted.

All of the incorrect solutions missed the cases \((a, b) = (44, 1)\) and \((a, b) = (1, 44)\).

M403. Proposed by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA.

Jason wrote a computer program that tests if an integer greater than 1 is prime. His devious sister Alice has edited the code so that if the input is odd, the probability that the program gives the correct output is 52% and if the input is even, the probability that the program gives the correct output is 98%. Jason tests the program by inputting two random integers each greater than 1. What is the probability that both outputs are correct?
Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The probability that the first random input is even is 0.5, in which case there is a 98% chance that the output is correct. The probability that the first random input is odd is 0.5, in which case there is a 52% chance that the output is correct. Thus, the probability that the first output is correct is 

\[(0.5)(0.98) + (0.5)(0.52) = 0.75.\]

The probability that the second output is correct is also 0.75. Therefore, the probability that both outputs are correct is 

\[(0.75)^2 = 0.5625 = 9/16.\]

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; CARL LIBIS, Cumberland University, Lebanon, TN, USA; and RICARD PEIRO, IES "Abastos", Valenc'a, Spain.

M404. Proposed by Bill Sands, University of Calgary, Calgary, AB.

A store sells copies of a certain item at $x$ each, or at $a$ items for $y$, or at $b$ items for $z$, where $a$ and $b$ are positive integers satisfying $1 < a < b$ and $x$, $y$, and $z$ are positive real numbers. To make "a items for $y$" a sensible bargain, $y$ should be less than buying $a$ separate items; in other words we should have $y < ax$. To make "b items for $z$" also a sensible bargain, we could insist on one of two conditions:

(a) \( \frac{z}{b} < \frac{y}{a} \); that is, the average price of an item under the "b items for $z$" deal is less than under the "a items for $y" deal.

(b) Whenever we can write $b = qa + r$ for nonnegative integers $q$ and $r$, then $z < qa + rx$ holds; that is, it should always cost more to buy $b$ items by buying a combination of $a$ items plus individual items, than by choosing the "b items for $z" deal.

Show that if condition (a) is true, then condition (b) is also true. Give an example to show that condition (b) could be true while condition (a) is false.

Solution by the proposer.

First, we prove by contradiction that if condition (a) is true, then condition (b) is true.

Suppose that \( \frac{z}{b} < \frac{y}{a} \); that is, assume that $az < by$. Assume that (b) is not true; that is, that there exist nonnegative integers $q$ and $r$ with $b = qa + r$ but with $z \geq qa + rx$.

Then $az \geq aqy + arx$, so $aqy + arx \leq az < by = y(qa + r) = aqy + ry$. Therefore, $arx < ry$. Since $r \geq 0$ and the inequality is not true if $r = 0$, then $r > 0$, so $ax < y$, which contradicts the given information.

Therefore, if condition (a) is true, then condition (b) is true.

If $a = 3$, $b = 5$, $x = 2$, $y = 3$, and $z = 6$, then $1 < a < b$ and $y < ax$, but \( \frac{z}{b} > \frac{y}{a} \), so (a) is not true. But condition (b) is true, since the only ways to write $b = 5$ in the form $b = qa + r$ are $5 = 0(3) + 5$ and $5 = 1(3) + 2$, which gives $qy + rx = 0(3) + 5(2) = 10 > 6 = z$ and $qy + rx = 1(3) + 2(2) = 7 > 6 = z$, so condition (b) is true.
**M405. Proposed by George Apostolopoulos, Messolonghi, Greece.**

Determine a closed form expression for the sum

\[
17 + 187 + 1887 + 18887 + \cdots + 188\ldots87,
\]

where the last term contains exactly \( n \) 8's.

**Solution by Geoffrey A. Kandall, Hamden, CT, USA.**

We note first that \( 17(1) = 17 \) and \( 17(11) = 187 \) and \( 17(111) = 1887 \). Then \( 18887 = 17000 + 1887 = 17(1000 + 111) = 17(1111) \). We can continue this argument inductively to show that the integer \( 188\ldots87 \) (containing \( n \) copies of 8) is equal to \( 17(11\ldots1) \) (containing \( n \) copies of 1 inside the parentheses).

Therefore,

\[
\begin{align*}
(17 + 187 + 1887 + 18887 + \cdots + (188\ldots87)) &= 17(1 + 11 + 111 + 1111 + \cdots + (11\ldots1)) \\
&= \frac{17}{9}(9 + 99 + 999 + \cdots + (99\ldots9)) \\
&= \frac{17}{9}(10(1 + 10 + 10^2 + \cdots + 10^n) - (n + 1)) \\
&= \frac{17}{9}
\left(10 \cdot \frac{10^n - 1}{9} - (n + 1)\right) \\
&= \frac{17}{81}(10^{n+2} - 10 - 9n - 9) \\
&= \frac{17}{81}(10^{n+2} - 9n - 19).
\end{align*}
\]

*Also solved by LUIS J. BLANCO (student) and ANGEL PLAZA, University of Las Palmas de Gran Canaria, Spain; JOAQUÍN GÓMEZ REY, IES Luis Buñuel, Alcorcón, Madrid, Spain; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PEDRO HENRIQUE O. PANTOJA, UFRN, Brazil; RICARD PEIXO, IES "Abastos", Valencia, Spain; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were three incorrect solutions submitted.*

**M406. Proposed by Constantino Ligouras, student, E. Majorana Scientific High School, Putignano, Italy.**

Square \( ABCD \) is inscribed in one-eighth of a circle of radius 1 and centre \( O \) so that there is one vertex on each radius and two vertices \( B \) and \( C \) on the arc. Square \( EFGH \) is inscribed in \( \triangle DOA \) so that \( E \) and \( H \) lie on the radii, and \( F \) and \( G \) lie on \( AD \). In problem M295 [2007 : 200, 202; solution 2008 : 203-204], we saw that the area of square \( ABCD \) is \( \frac{2 - \sqrt{2}}{3} \). Determine the area of square \( EFGH \).
Solution by Ricard Peirò, IES “Abastos”, Valencia, Spain, modified by the editor.

In problem M295, we saw that \( AD^2 = \frac{2 - \sqrt{2}}{3} \).
Since \( \tan 45^\circ = 1 \), then
\[
1 = \tan 45^\circ = \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ}.
\]
Setting \( u = \tan 22.5^\circ \), we have that \( 1 - u^2 = 2u \), or \( u^2 + 2u - 1 = 0 \). Using the quadratic formula, we obtain
\[
u = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}.
\]
Since \( u = \tan 22.5^\circ > 0 \), then \( \tan 22.5^\circ = \sqrt{2} - 1 \).

Let \( x \) be the side length of square \( EFGH \). Then \( EF = FG = x \).

By symmetry, \( AF = DG \), so \( AF = \frac{AD - FG}{2} = \frac{AD - x}{2} \). Since \( \triangle DOA \) is isosceles, then \( \angle DAO = \frac{1}{2}(180^\circ - 45^\circ) = 67.5^\circ \). Since \( \triangle EFA \) is right-angled, then \( \angle FEA = 90^\circ - 67.5^\circ = 22.5^\circ \). Therefore,
\[
\tan 22.5^\circ = \frac{AF}{EF};
\]
\[
\sqrt{2} - 1 = \frac{AD - x}{2x};
\]
\[
(2\sqrt{2} - 2)x = AD - x;
\]
\[
(2\sqrt{2} - 1)x = AD;
\]
\[
x = \frac{AD}{2\sqrt{2} - 1}.
\]

Therefore \( x^2 \), the area of square \( EFGH \), is equal to
\[
\frac{AD^2}{(2\sqrt{2} - 1)^2} = \frac{2 - \sqrt{2}}{3} \cdot \frac{1}{9 - 4\sqrt{2}} = \frac{(2 - \sqrt{2})(9 + 4\sqrt{2})}{3[9^2 - 4^2(2)]} = \frac{10 - \sqrt{2}}{147}.
\]

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and GEOFFREY A. RANDALL, Hamden, CT, USA.
Problem of the Month

Ian VanderBurgh

A popular type of geometry problem involves folding paper. A folding problem usually involves a sheet of paper of specific dimensions and the method of folding. We are then asked to determine one or more lengths in the resulting configuration.

Problem (UK Intermediate Challenge 1999) A rectangular sheet of paper with sides 1 and $\sqrt{2}$ has been folded once as shown, so that one corner just meets the opposite long edge. What is the value of the length $d$?

Feel free to actually try this out! If you’re in the UK, you’ll have a much easier time finding a sheet of paper with dimensions in the ratio $\sqrt{2} : 1$.

How should we start? One of the very first problem solving strategies that we learn is “draw a diagram”. This strategy should almost always be extended very slightly by adding the clause “...and label it carefully”. As it turns out, this is the key to solving this problem.

Solution We redraw the given diagram by adding the “phantom” edges of the paper (the dotted lines) and labelling the relevant points on the diagram.

We then label as many lengths as we possibly can. I suggest that you follow along by labelling each new length that we determine. Make sure that you understand why each length is what it is before moving on to the next step. Since the paper has length $\sqrt{2}$, then $AB = DC = \sqrt{2}$.

Can you see another length that equals $\sqrt{2}$? In fact, $A'B = \sqrt{2}$ since this is the folded image of $AB$.

Can you determine the length of $AE$ in terms of $d$? Since $AD = 1$ and $ED = d$, then $AE = 1 - d$.

Can you find another line segment of length $1 - d$? Since $AE$ becomes $A'E$ after folding, then $A'E = 1 - d$.

Can you see any triangles where we know two of the three side lengths? In $\triangle A'CB$, we have $A'B = \sqrt{2}$ and $BC = 1$.

How can we determine the third side length of $\triangle A'CB$? This triangle is right-angled at $C$, so we can use the Pythagorean Theorem to conclude
that \( A'C^2 = A'B^2 - BC^2 = (\sqrt{2})^2 - 1^2 = 1; \) since \( A'C > 0, \) then \( A'C = \sqrt{1} = 1. \)

Can we use this to determine another length? Yes! Since \( DC = \sqrt{2} \) and \( A'C = 1, \) then \( A'D = \sqrt{2} - 1. \) Now \( \triangle EDA' \) is right-angled at \( D. \) We know one of the three side lengths, namely, \( A'D = \sqrt{2} - 1, \) and we know the other two side lengths in terms of \( d, \) namely, \( ED = d \) and \( EA' = 1 - d. \)

What should we do to try to solve for \( d? \) Let’s apply the Pythagorean Theorem again. (Spoiler alert: There is a better way! If you are uncomfortable squaring expressions like \( 1 - d \) or have never even done this before, skip down to just after the end of the solution for a simpler approach.) We obtain

\[
\begin{align*}
A'E^2 &= ED^2 + A'D^2; \\
(1 - d)^2 &= d^2 + (\sqrt{2} - 1)^2; \\
1 - 2d + d^2 &= d^2 + 2 - 2\sqrt{2} + 1; \\
-2d &= 2 - 2\sqrt{2}; \\
d &= \sqrt{2} - 1.
\end{align*}
\]

Therefore, \( d = \sqrt{2} - 1. \)

My apologies for the spoiler alert above. We were on such a roll that I didn’t want to interrupt our Pythagorean flow.

Do you see a different approach that we could have taken? You may note that \( A'D = ED = \sqrt{2} - 1. \) Can you see a reason why this should be the case? Let’s go back and do some angle-chasing.

Triangle \( \triangle ABC \) has sides of lengths 1, 1, and \( \sqrt{2}. \) What are its angles? Since it is isosceles and right-angled, then \( \angle BA'C = \angle A'BC = 45^\circ. \) Thus,

\[ \angle DA'E = 180^\circ - \angle EA'B - \angle BA'C = 180^\circ - 90^\circ - 45^\circ = 45^\circ. \]

What does that say about \( \triangle A'DE? \) This tells us that this is also isosceles and right-angled! (If you’re not convinced, calculate \( \angle DEA' \).) Therefore, \( ED = A'D \) and we know that \( A'D = \sqrt{2} - 1. \) This allows us to conclude that \( d = ED = \sqrt{2} - 1, \) as required.

This gives us two different ways of handling this problem. Knowing two different approaches is really useful, because it means that if we don’t see one of the approaches in a problem that we’re working on, we might just see the other.

For those of you wanting more of a challenge, here’s a follow-up problem to work on:

A rectangular sheet of paper \( ABCD \) has \( AB = 8 \) and \( BC = 6. \)

The paper is folded so that corner \( A \) coincides with the midpoint, \( M, \) of \( DC. \) What is the length of the fold?