SKOLIAD No. 125

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Please send your solutions to problems in this Skoliad by 1 Oct, 2010. A copy of *CRUX with Mayhem* will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

The deadline for Skoliad 124 solutions in the previous issue (*CRUX with MAYHEM* Vol. 36, No. 3) is 1 Sept, 2010 NOT 1 July, 2010; our apologies.

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Our contest for this month is the Baden-Württemberg Mathematics Contest, 2009. Our thanks go to the Landeswettbewerb Mathematik Baden Württemberg for providing this contest and for permission to publish it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d’avoir traduit ce concours.

**Concours mathématique**
**Baden-Württemberg 2009**

1. Déterminer tous les entiers naturels $n$ tels que la somme de $n$ et de ses chiffres décimaux est 2010.

2. Un polygone régulier à 18 côtés est découpé en pentagones congrus, tel qu’illustré. Déterminer les angles internes d’un tel pentagone.

3. Dans la figure à droite, $\triangle ABE$ est isocèle avec base $AB$, $\angle BAC = 30^\circ$, et $\angle ACB = \angle AFC = 90^\circ$. Déterminer le ratio entre la surface du $\triangle ESC$ et la surface du $\triangle ABC$. 
4. À partir de deux nombres non nuls $z_1$ et $z_2$, soit $z_n$ égal à $\frac{z_{n-1}}{z_{n-2}}$ pour $n > 2$. Alors $z_1$, $z_2$, $z_3$, ... forment une suite. Démontrer que si on multiplie n'importe quels 2009 termes consécutifs de cette suite, le produit fait lui-même partie de la suite.

5. Soit $\triangle ABC$ un triangle isocèle tel que $\angle ACB = 90^\circ$. Un cerce avec centre $C$ coupe $AC$ en $D$ et $BC$ en $E$. Tracer la ligne $AE$. La perpendiculaire à $AE$ passant par $C$ coupe la ligne $AB$ en $F$, tandis que la perpendiculaire à $AE$ passant par $D$ coupe la ligne $AB$ en $G$. Démontrer que la longueur de $BF$ égale la longueur de $GF$.

6. Une machine choisit un des diviseurs de 2009 et vous mizes sur le chiffre en position unitaire de ce diviseur. Sur quel chiffre mizez-vous?

**Baden-Württemberg Mathematics Contest 2009**

1. Find all natural numbers $n$ such that the sum of $n$ and the digit sum of $n$ is 2010.

2. A regular 18-gon can be cut into congruent pentagons as in the figure below. Determine the interior angles of such a pentagon.

3. In the figure on the right, $\triangle ABE$ is isosceles with base $AB$, $\angle BAC = 30^\circ$, and $\angle ACB = \angle BAC = 90^\circ$. Find the ratio of the area of $\triangle ESC$ to the area of $\triangle ABC$. 

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**Diagram:**

- Diagram of a triangle $ABC$ with a circle around it. The circle is divided into 18 congruent pentagons. One pentagon is highlighted.

- Diagram of a triangle $ABE$ with the angles $\angle BAC = 30^\circ$, $\angle ACB = \angle BAC = 90^\circ$. Points $E$, $F$, $S$, and $C$ are labeled. The triangle $ESC$ is highlighted. 

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4. Given two nonzero numbers $z_1$ and $z_2$, let $z_n$ be $\frac{z_{n-1}}{z_{n-2}}$ for $n > 2$. Then $z_1$, $z_2$, $z_3$, . . . form a sequence. Prove that if you multiply any 2009 consecutive terms of the sequence, then the product is itself a member of the sequence.

5. Let $\triangle ABC$ be an isosceles triangle such that $\angle ACB = 90^\circ$. A circle with centre $C$ cuts $AC$ at $D$ and $BC$ at $E$. Draw the line $AE$. The perpendicular to $AE$ through $C$ cuts the line $AB$ at $F$, and the perpendicular to $AE$ through $D$ cuts the line $AB$ at $G$. Show that the length of $BF$ equals the length of $GF$.

6. A gaming machine randomly selects a divisor of $2^{2010}$ and displays its ones digit. Which digit should you gamble on?


1. The sum of a four-digit number and its four digits is 2005. What is this four-digit number?

Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

Let $n$ denote the desired number. Surely $n \leq 2005$. Since the sum of three digits is at most 27, the digit sum of $n$ is at most 29. Therefore $n \geq 1976$.

Let $d$ represent a digit, and let $S$ be the sum of $n$ and its digits.

If $n = 2000 + d$, then $S = 2000 + 2 + 2d$ which is even and thus cannot equal 2005.

If $n = 1990 + d$, then $S = 1990 + 2d$ which is too large.

If $n = 1980 + d$, then $S = 1998 + 2d$ which is even and thus cannot equal 2005.


Hence, $n = 1979$.

Also solved by Michael Cheung, student, Port Moody Secondary School, Port Moody, BC; LENA CHOL, student, Ecole Baning Middle School, Coquitlam, BC; Timothy Chu, student, R.C. Palmer Secondary School, Richmond, BC; Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC; WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC; and Lisa Wang, student, Port Moody Secondary School, Port Moody, BC.

2. In triangle $ABC$, $AB = 10$ and $AC = 18$. $M$ is the midpoint of $BC$, and the line through $M$ parallel to the bisector of $\angle CAB$ cuts $AC$ at $D$. Find the length of $AD$. 

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

Let L denote the point on BC such that AL is the bisector of ∠CAB.
The Sine Law in △ABL yields that \[ \frac{BL}{\sin \angle BAL} = \frac{10}{\sin \angle ALB}, \]
and therefore \[ BL = 10 \left( \frac{\sin \angle BAL}{\sin \angle ALB} \right). \]
Likewise, using the Sine Law in △ALC yields \[ \frac{CL}{\sin \angle CAL} = \frac{18}{\sin \angle ALC}. \]
Thus, \[ CL = 18 \left( \frac{\sin \angle CAL}{\sin \angle ALC} \right). \]
But it is also true that ∠CAL = ∠BAL and ∠ALC = 180° − ∠ALB, so \[ CL = 18 \left( \frac{\sin \angle BAL}{\sin \angle ALB} \right). \]
Let z denote the fraction \[ \frac{\sin \angle BAL}{\sin \angle ALB}. \]
Then \[ BL = 10z \] and \[ CL = 18z. \]
Therefore, \[ BC = 28z \] and \[ CM = 14z. \]
As △ACL is similar to △DCM, it follows that \[ \frac{DC}{AC} = \frac{CM}{CL}, \]
so \[ \frac{DC}{18} = \frac{14z}{18z}, \] so \[ DC = 14. \]
Hence, \[ AD = 4. \]

3. Let \( x, y, z \) be positive numbers such that \( x + y + xy = 8, y + z + yz = 15, \) and \( z + x + zx = 35. \) Find the value of \( x + y + z + xy. \)

Solution by Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC.

Since \( x + y + xy = 8, \) it follows that \( x(1 + y) = 8 - y, \) so \( x = \frac{8 - y}{y + 1}. \)
Likewise, since \( y + z + yz = 15, \) it follows that \( z(1 + y) = 15 - y, \) so \( z = \frac{15 - y}{y + 1}. \)
Substituting these into the third given equation yields that
\[
\frac{15 - y}{y + 1} + \frac{8 - y}{y + 1} + \left( \frac{15 - y}{y + 1} \right) \left( \frac{8 - y}{y + 1} \right) = 35,
\]
so
\[
\frac{23 - 2y}{y + 1} + \frac{120 - 23y + y^2}{(y + 1)^2} = 35
\]
and \((23 - 2y)(y + 1) + 120 - 23y + y^2 = 35(y + 1)^2.\) Therefore,
\[
23y + 23 - 2y^2 - 2y + 120 - 23y + y^2 = 35y^2 + 70y + 35,
\]
so \(0 = 36y^2 + 72y - 108 = 36(y^2 + 2y - 3) = 36(y - 1)(y + 3).\) Thus, \( y = 1 \) or \( y = -3.\) Since \( y \) is given to be positive, \( y = 1, \) and, thus, \( x = \frac{8 - y}{y + 1} = \frac{7}{2}.\)
and \( z = \frac{15 - y}{y + 1} = 7. \) Hence \( x + y + z + xy = \frac{7}{2} + 1 + 7 + \frac{7}{2} \cdot 1 = 15. \)
Also solved by MICHAEL CHEUNG, student, Port Moody Secondary School, Port Moody, BC.

While our solver’s brute force solution shows admirable stamina, a more elegant solution is also possible: If \( x + y + xy = 8 \), then \( x + y + xy + 1 = 9 \). and now the left-hand side can be factored: \( (x + 1)(y + 1) = 9 \). Similarly the other two given equations yield that \( (y + 1)(z + 1) = 16 \) and that \( (z + 1)(x + 1) = 36 \). Multiplying the last two of these equations and dividing by the first yields that

\[
\frac{(y + 1)(z + 1)^2(x + 1)}{(x + 1)(y + 1)} = \frac{16 \cdot 36}{9},
\]

so \( (z + 1)^2 = 64 \). so \( z + 1 = \pm 8 \), so \( z = 7 \) or \( z = -9 \). Again, \( z \) is positive, so \( z = 7 \). It now follows from the first of the given equations that \( x + y + z + xy = 8 + 7 = 15 \).

4. The number of mushrooms gathered by 11 boys and \( n \) girls is \( n^2 + 9n - 2 \), with each person gathering exactly the same number. Determine the positive integer \( n \).

Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

Each of the \( n + 11 \) children must gather \( \frac{n^2 + 9n - 2}{n + 11} \) mushrooms. Now

\[
n^2 + 9n - 2 = (n + 11)(n - 2) + 20,
\]

so the number of mushrooms is

\[
n - 2 + \frac{20}{n + 11}.
\]

This must be an integer, so \( n + 11 \) must divide 20. Since \( n \) is nonnegative, \( n = 9 \).

Also solved by MICHAEL CHEUNG, student, Port Moody Secondary School, Port Moody, BC; TIMOTHY CHU, student, R.C. Palmer Secondary School, Richmond, BC; VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

One can use polynomial division to find that \( n^2 + 9n - 2 = (n + 11)(n - 2) + 20 \), or you can use guess and check: If \( n^2 + 9n - 2 = (n + 11)P + R \), then \( P \) must contain an \( n \) to get \( n^2 \) on the other side. Thus \( n^2 + 9n - 2 = (n + 11)(n + ?) + R \). The question mark must be \(-2\) to get \( 9n \) on the other side, so \( R = 20 \) follows.

5. The positive integer \( x \) is such that both \( x \) and \( x + 99 \) are squares of integers. Find the sum of all such integers \( x \).

Solution by Ellen Chen, student, Burnaby North Secondary School, Burnaby, BC.

Say \( x = n^2 \) and \( x + 99 = m^2 \). Then \( 99 = m^2 - n^2 = (m + n)(m - n) \), so 99 is written as the product of two integers. This is only possible in three ways:

<table>
<thead>
<tr>
<th>( m + n )</th>
<th>( m - n )</th>
<th>( m )</th>
<th>( n )</th>
<th>( x = n^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>99</td>
<td>1</td>
<td>50</td>
<td>49</td>
<td>2401</td>
</tr>
<tr>
<td>33</td>
<td>3</td>
<td>18</td>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>11</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Sum: 2627

Also solved by TIMOTHY CHU, student, R.C. Palmer Secondary School, Richmond, BC; WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.
6. The side lengths of a right triangle are all positive integers, and the length of one of the legs is at most 20. The ratio of the circumradius to the inradius of this triangle is 5 : 2. Determine the maximum value of the perimeter of this triangle.

Solution by the editors.

First let us review a few facts from geometry. The angle between a tangent to a circle and the radius to the point of tangency is 90°. Therefore you can use the Pythagorean Theorem in each of the two triangles in the figure: The square of the length of the dotted line equals both $x^2 + r^2$ and $y^2 + r^2$. Therefore $x = y$, that is, intersecting tangents are equal.

Consider the right-angled triangle $\triangle ABC$. Let $M$ be the midpoint of $AC$, and let $N$ be the midpoint of $AB$. Then $MN$ is parallel to $BC$, so $\triangle ANM$ is also right-angled. Using the Pythagorean Theorem in $\triangle ANM$ and in $\triangle BNM$ it follows that $AM = BM$. Thus $M$ is the centre of the circle through $A$, $B$, and $C$.

Now we can attack the problem. You have just seen that since the triangle is right-angled, its hypotenuse is a diameter for the circumscribed circle, whose radius is therefore $c/2$. Let $r$ be the radius of the inscribed circle. Note that two of the radii in the figure together with parts of the left and bottom sides of the triangle form a square. Therefore, the length of the remaining part of the left side is $a - r$ and the length of the remaining part of the bottom side is $b - r$. Since intersecting tangents are equal, this means that $c = a - r + b - r$. Thus $r = (a + b - c)/2$.

Since the ratio of the circumradius to the inradius is 5 : 2,

$$\frac{c/2}{(a + b - c)/2} = \frac{5}{2}.$$ 

Therefore, \( \frac{c}{a + b - c} = \frac{5}{2} \) so \( 2c = 5a + 5b - 5c \). Since \( c = \frac{5}{7}(a + b) \). By the Pythagorean Theorem, \( a^2 + b^2 = c^2 = \frac{25}{49}(a + b)^2 = \frac{25}{49}(a^2 + 2ab + b^2) \). Hence, \( 49a^2 + 49b^2 = 25a^2 + 50ab + 25b^2 \), so \( 24a^2 - 50ab + 24b^2 = 0 \), so \( 2(4a - 3b)(3a - 4b) = 0 \). Thus \( a : b = 3 : 4 \) or \( a : b = 4 : 3 \). Either way the given triangle is a 3-4-5 triangle.
The shortest side is given to be at most 20. The largest multiple of 3 less than or equal to 20 is 18. Thus, the sides are 18, 24, and 30, and the maximum value of the perimeter is 72.

7. Let $\alpha$ be the larger root of $(2004a)^2 - 2003 \cdot 2005a - 1 = 0$ and $\beta$ be the smaller root of $x^2 + 2003x - 2004 = 0$. Determine the value of $\alpha - \beta$.

**Solution by Timothy Chu, student, R.C. Palmer Secondary School, Richmond, BC.**

The constant term of a quadratic polynomial is the product of its roots. Both polynomials have negative constant terms, so both must have one positive and one negative root. Since $2003 \cdot 2005 = (2004 - 1)(2004 + 1) = 2004^2 - 1$ and $2004^2 - (2004^2 - 1) - 1 = 0$, one of the roots of the first polynomial is 1. Since the other root is negative, $\alpha = 1$. The second polynomial is easily factored as $(x - 1)(x + 2004)$, whence $\beta = -2004$. Therefore $\alpha - \beta = 2005$.

Also solved by WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC.

To see that the constant term of a quadratic polynomial is indeed the product of its roots, consider that $(x - a)(x - b) = x^2 - (a + b)x + ab$. A similar property holds for higher degree polynomials.

Once you realise that $2003 \cdot 2005 = 2004^2 - 1$, the first polynomial is also easy to factor as $(2004^2a + 1)(x - 1)$.

8. Let $a$ be a positive number such that $a^2 + \frac{1}{a^2} = 5$. Determine the value of $a^3 + \frac{1}{a^3}$.

**Solution by the editors.**

Since $(a + \frac{1}{a})^2 = a^2 + 2 + \frac{1}{a^2}$, it follows from the given equation that $(a + \frac{1}{a})^2 = 7$, and so $a + \frac{1}{a} = \sqrt{7}$ since $a$ is positive. Similarly,

\[
(a + \frac{1}{a})^3 = (a + \frac{1}{a})^2 (a + \frac{1}{a}) = (a^2 + 2 + \frac{1}{a^2}) (a + \frac{1}{a})
\]

\[
= a^3 + 2a + \frac{1}{a} + a + \frac{2}{a} + \frac{1}{a^2} = a^3 + 3 (a + \frac{1}{a}) + \frac{1}{a^2}.
\]

Therefore, $a^3 + \frac{1}{a^3} = (a + \frac{1}{a})^3 - 3(a + \frac{1}{a}) = (\sqrt{7})^3 - 3\sqrt{7} = 4\sqrt{7}$.

9. In the figure, $ABCD$ is a rectangle with $AB = 5$ such that the semicircle with diameter $AB$ cuts $CD$ at two points. If the distance from one of them to $A$ is 4, find the area of $ABCD$.
Solution by Lena Choi, student, École Banting Middle School, Coquitlam, BC.

Since $AB$ is a diameter and $P$ is on the circle, $\angle APB = 90^\circ$. Since $AP = 4$ and $AB = 5$, it follows that $BP = 3$. Hence the area of $\triangle ABP$ is $\frac{3 \cdot 4}{2} = 6$. If you instead use $AB$ as the base of the triangle, then the height equals the length of $BC$. Therefore, the area of the rectangle is twice the area of the triangle, so the area of the rectangle is 12.

Also solved by KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC.

Our solver used the fact that if $P$ is on the circle with diameter $AB$, then $\angle APB = 90^\circ$. To prove this fact, rotate the triangle around the centre of the circle to obtain the dotted part in the figure on the right. By construction, the four sided polygon is a parallelogram. Since both diagonals are diameters and therefore equal, the parallelogram must be a rectangle, whence $\angle APB = 90^\circ$.

10. Let $a$ be $9 \left( n \left( \frac{10}{9} \right)^n - 1 - \frac{10}{9} - \left( \frac{10}{9} \right)^2 - \cdots - \left( \frac{10}{9} \right)^{n-1} \right)$ where $n$ is a positive integer. If $a$ is an integer, determine the maximum value of $a$.

Solution by Krisitan Hansen, student, Burnaby North Secondary School, Burnaby, BC.

The sum of the geometric series is

$$1 + \frac{10}{9} + \left( \frac{10}{9} \right)^2 + \cdots + \left( \frac{10}{9} \right)^{n-1} = \frac{1 - \left( \frac{10}{9} \right)^n}{1 - \frac{10}{9}} = -9 \left( 1 - \left( \frac{10}{9} \right)^n \right).$$

Therefore,

$$a = 9 \left( n \left( \frac{10}{9} \right)^n + 9 \left( 1 - \left( \frac{10}{9} \right)^n \right) \right) = 9 \left( n - 9 \left( \frac{10}{9} \right)^n + 9 \right) = 9(n - 9) \left( \frac{10}{9} \right)^n + 81.$$

For this to be an integer, either $n = 1$ or $n = 9$. (If $n > 1$, then the denominator contains too many copies of 9 except when $n = 9$ and the numerator is zero by a lucky miracle.) If $n = 1$, then $a = 1$; if $n = 9$, then $a = 81$. The larger of these is 81, which is the maximum value of $a$.

11. In a two-digit number, the tens digit is greater than the ones digit. The product of these two digits is divisible by their sum. What is this two-digit number?
Solution by Michael Cheung, student, Port Moody Secondary School, Port Moody, BC.

Any (two-digit) multiple of ten satisfies the condition. Otherwise, if the number contains the digit 1 and the digit d, the condition is that d is divisible by d + 1 which is impossible. This leaves just 28 numbers to consider: 32, 42, 43, 52, 53, 54, 62, 63, 64, 65, 72, 73, 74, 75, 76, 82, 83, 84, 85, 86, 87, 92, 93, 94, 95, 96, 97, and 98. These are easily checked one by one; only 63 works out. Thus the solutions are 10, 20, 30, 40, 50, 60, 63, 70, 80, and 90.

Also solved by TIMOTHY CHU, student, R.C. Palmer Secondary School, Richmond, BC.

12. In the figure, PQRS is a rectangle of area 10. A is a point on RS and B is a point on PS such that the area of triangle QAB is 4. Determine the smallest possible value of PB + AR.

Solution by Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC.

Label the lengths as in the figure. Since the area of \( \triangle QAB \) is 4, the areas of the remaining three triangles must add up to 6. That is,

\[
\frac{(10 - x) (x - y)}{2} + \frac{10y}{2x} + \frac{xz}{2} = 6.
\]

Multiplying by 2 and expanding yields

\[
10 - \frac{10y}{x} - xz + yz + \frac{10y}{x} + xz = 12,
\]

so \( yz = 2 \).

The smallest possible value of \( PB + AR = y + z \) subject to the constraint that \( yz = 2 \) is obtained when \( y = z \). Then \( y = z = \sqrt{2} \) and \( PB + AR = 2\sqrt{2} \).

Also solved by KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC.

This issue’s prize of one copy of CRUX with MAYHEM for the best solutions goes to Timothy Chu, student, R.C. Palmer Secondary School, Richmond, BC.

We congratulate our solvers on their success with a rather difficult contest and hope that they and other readers will continue to submit solutions to our problems.