THE OLYMPIAD CORNER

No. 286

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We start this number with translations of a number of Olympiads from South America. My thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use and to Leda Sanchez, Executive Assistant to the Vice Provost (International), for helping with the translation. The first set are the problems of the XV Olimpiada Matemática Rioplatense 2006, Nivel 2.

XV OLIMPIADA MATEMÁTICA RIOPLATENSE
San Isidra, 9–10 December 2006
Nivel 2

1. Let $ABC$ be a right triangle with right angle at $A$. Consider all the isosceles triangles $XYZ$ with right angle at $X$, where $X$ lies on the segment $BC$, $Y$ lies on $AB$, and $Z$ is on the segment $AC$. Determine the locus of the medians of the hypotenuses $YZ$ of such triangles $XYZ$.

2. Carlitos listed all the subsets of $\{1, 2, \ldots, 2006\}$ in which the difference between the number of even numbers and the number of odd numbers is a multiple of 3. How many subsets did Carlitos list?

3. A finite number of (possibly overlapping) intervals on a line are given. If the rightmost $1/3$ of each interval is deleted, an interval of length 31 remains. If the leftmost $1/3$ of each interval is deleted, an interval of length 23 remains. Let $M$ and $m$ be the maximum and minimum of the lengths of an interval in the collection, respectively. How small can $M - m$ be?

4. Let $a_1, a_2, \ldots, a_n$ be positive numbers. The sum of all the products $a_i a_j$ with $i < j$ is equal to 1. Show that there is a number among them such that the sum of the remaining numbers is less than $\sqrt{2}$.

5. A circle $\Gamma$ is tangent to the sides $AB$ and $AC$ of triangle $ABC$ at $E$ and $F$, respectively. Let $BF$ and $EC$ intersect at $X$, let $\Gamma$ intersect $AX$ at $H$, and let $EH$ and $FH$ intersect $BC$ at $Z$ and $T$, respectively. The lines $ET$ and $FZ$ intersect at $Q$. Show that $Q$ lies on the line $AX$.

6. For each permutation $(x_1, x_2, \ldots, x_{99})$ of $\{1, 2, \ldots, 99\}$, let

$$L = |x_1 - x_2 \sqrt{3}| + |x_2 - x_3 \sqrt{3}| + \cdots + |x_{98} - x_{99} \sqrt{3}| + |x_{99} - x_1 \sqrt{3}|.$$ 

Determine the maximum value of $L$. How many permutations give rise to this value of $L$?
Next from the same package are the problems of the XV Olimpíada Matemática Rioplatense 2006, Nivel 3. Again, thanks go to Bill Sands and to Leda Sanchez.

XV OLYMPIADA MATEMÁTICA RIOPLATENSE 2006
San Isidra, 9–10 December 2006
Nivel 3

1. (a) For each \( k \geq 3 \), find a positive integer \( n \) that can be represented as the sum of exactly \( k \) mutually distinct positive divisors of \( n \).

   (b) Suppose that \( n \) can be expressed as the sum of exactly \( k \) mutually distinct positive divisors of \( n \) for some \( k \geq 3 \). Let \( p \) be the smallest prime divisor of \( n \).

   Show that
   \[
   \frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{p+k-1} \geq 1.
   \]

2. Let \( ABCD \) be a convex quadrilateral with \( AB = AD \) and \( CB = CD \). The bisector of \( \angle BDC \) intersects \( BC \) at \( L \), and \( AL \) intersects \( BD \) at \( M \), and it is known that \( BL = BM \). Determine the value of \( 2\angle BAD + 3\angle BCD \).

3. The numbers 1, 2, \ldots, 2006 are written around the circumference of a circle. One allowed operation is to exchange two adjacent numbers. After a sequence of such exchanges each number ends up 13 positions to the right of its initial position.

   If the 2006 numbers 1, 2, \ldots, 2006 are partitioned into 1003 distinct pairs, then show that in at least one of the operations two numbers of one of the pairs are exchanged.

4. The acute triangle \( ABC \) with \( AB \neq AC \) has circumcircle \( \Gamma \), circumcentre \( O \) and orthocentre \( H \). The midpoint of \( BC \) is \( M \) and the extension of the median \( AM \) intersects \( \Gamma \) at \( N \). The circle of diameter \( AM \) intersects \( \Gamma \) again at \( A \) and \( P \).

   Show that the lines \( AP \), \( BC \), and \( OH \) are concurrent if and only if \( AH = HN \).

5. Consider a finite number of lines in the plane no two of which are parallel and no three of which are concurrent. These lines divide the plane into finite and infinite regions. In each finite region we write 1 or \(-1\). In one operation, we can choose any triangle made of three of the lines (which may be cut by other lines in the collection) and multiply by \(-1\) each of the numbers in the triangle. Determine if it is always possible to obtain 1 in all the regions by successively applying this operation, regardless of the initial distribution of the numbers 1 and \(-1\).
6. Consider an infinite sequences \( \{x_n\}_{n=1}^{\infty} \) of positive integers that satisfies the recurrence
\[
x_{n+2} = \gcd(x_{n+1}, x_n) + 2006
\]
for each positive integer \( n \), where \( \gcd(u, v) \) is the greatest common divisor of the integers \( u \) and \( v \).

Does there exist a sequence of this type which contains exactly \( 10^{2006} \) distinct numbers?

Continuing with this theme we have the problems of the 21st Olimpiada Iberoamericana de Matemática. Premer Dia, 2006. Thanks again go to Bill Sands and Leda Sanchez for making them available to the Corner.

21 OLIMPIADA IBEROAMERICANA DE MATEMÁTICA
Guayaquil, 26-27 September 2006

1. In the scalene triangle \( ABC \) with \( \angle BAC = 90^\circ \), the tangent line to the circumcircle at at \( A \) intersects the line \( BC \) at \( M \). Let \( S \) and \( R \) be the points where the incircle of \( ABC \) touches \( AC \) and \( AB \), respectively. The line \( RS \) intersects the line \( BC \) at \( N \). The lines \( AM \) and \( SR \) meet at \( U \). Show that triangle \( UMN \) is isosceles.

2. Let \( a_1, a_2, \ldots, a_n \) be real numbers. Let \( d \) be the difference between the smallest and the largest of them, and let \( s = \sum_{i<j} |a_i - a_j| \). Show that
\[
(n - 1)d \leq s \leq \frac{n^2d}{4}
\]
and determine the conditions under which equality holds in each inequality.

3. The numbers \( 1, 2, \ldots, n^2 \) are placed in the cells of an \( n \times n \) board, one number per cell. A coin is initially placed in the cell containing the number \( n^2 \). The coin can move to any of the cells which share a side with the cell it currently occupies.

First, the coin travels from the cell containing the number 1 to the cell containing the number \( n^2 \), using the smallest possible number of moves. Then the coin travels from the cell containing the number 1 to the cell containing the number 2 using the smallest possible number of moves, and then from the cell containing the number 3, and continuing until the coin returns to the initial cell, taking a shortest route each time it travels. The complete trip takes \( N \) steps. Determine the smallest and largest possible values of \( N \).

4. Determine all pairs \((a, b)\) of positive integers such that \( 2a + 1 \) and \( 2b - 1 \) are relatively prime and \( a + b \) divides \( 4ab + 1 \).
5. The circle \(\Gamma\) is inscribed in quadrilateral \(ABCD\) with \(AD\) and \(CD\) tangent to \(\Gamma\) at \(P\) and \(Q\), respectively. If \(BD\) intersects \(\Gamma\) at \(X\) and \(Y\) and \(M\) is the midpoint \(XY\), prove that \(\angle AMP = \angle CMQ\).

6. Let \(n\) be an odd positive integer, and let \(P_0\) and \(P_1\) be two consecutive vertices of a regular \(n\)-gon. For each \(k \geq 2\) define \(P_k\) to be the vertex of the \(n\)-gon that lies on the perpendicular bisector of \(P_{k-1}P_{k-2}\). Determine all \(n\) for which the sequence \(P_0, P_1, P_2, \ldots\) covers all the vertices of the \(n\)-gon.

As the last problem set for this Corner we give the XVIII Olimpiada de Matematica de Paises del Cono Sur. Again, many thanks to Bill Sands and Leda Sanchez.

**XVIII OLIMPIADA DE MATEMÁTICA DE PAISES DEL CONO SUR**

**Atlántida, June 14–15, 2007**

1. Find all pairs \((x, y)\) of nonnegative integers that satisfy

\[x^3y + x + y = xy + 2xy^2.\]

2. Given are 100 positive integers whose sum equals their product. Determine the minimum number of 1's that may occur among the 100 numbers.

3. Let \(ABC\) be an acute triangle with altitudes \(AD, BE, CF\) where \(D, E, F\) lie on \(BC, AC, AB\), respectively. Let \(M\) be the midpoint of \(BC\). The circumcircle of triangle \(AEF\) cuts the line \(AM\) at \(A\) and \(X\). The line \(AM\) cuts the line \(CF\) at \(Y\). Let \(Z\) be the point of intersection of \(AD\) and \(BX\). Show that the lines \(YZ\) and \(BC\) are parallel.

4. Some cells of a 2007 \(\times\) 2007 table are coloured. The table is "charrua" if none of the rows and none of the columns are completely coloured.

(a) What is the maximum number \(k\) of coloured cells that a charrua table can have?

(b) For such \(k\), calculate the number of distinct charrua tables that exist.

5. Let \(ABCDE\) be a convex pentagon that satisfies the following:

(i) There is a circle \(\Gamma\) tangent to each of the sides.

(ii) The lengths of the sides are all positive integers.

(iii) At least one of the sides of the pentagon has length 1.

(iv) The side \(AB\) has length 2.
Let \( P \) be the point of tangency of \( \Gamma \) with \( AB \).

(a) Determine the lengths of the segments \( AP \) and \( BP \).

(b) Give an example of a pentagon satisfying the given conditions.

6. Show that for each positive integer \( n \), there is a positive integer \( k \) such that the decimal representation of each of the numbers \( k, 2k, \ldots, nk \) contains all of the digits 0, 1, 2, \ldots, 9.

Next we look at the solutions to problems of the 55th Czech and Slovak Mathematical Olympiad 2006 given at [2009 : 81–82].

1. (P. Novotný) A sequence \( \{a_n\}_{n=1}^\infty \) of positive integers is defined for \( n \geq 1 \) by \( a_{n+1} = a_n + b_n \), where \( b_n \) is obtained from \( a_n \) by reversing its digits (the number \( b_n \) may start with zeroes). For instance if \( a_1 = 170 \), then \( a_2 = 241 \), \( a_3 = 383 \), \( a_4 = 766 \), \ldots. Decide whether \( a_7 \) can be a prime number.

Solution by Titu Zvonaru, Comănești, Romania, modified by the editor.

The answer is that \( a_7 \) cannot be a prime number.

We use the following lemmas:

Lemma 1. If \( a_n \) has an even number of digits, then 11 divides \( a_n + b_n \).

Proof: Let \( a_n = d_1d_2 \ldots d_{2k} \), \( b_n = d_2 \ldots d_2d_1 \) be the decimal representations of \( a_n \) and \( b_n \). Modulo 11 we have

\[
a_n + b_n = (d_110^{2k-1} + \cdots + d_{2k-1}10 + d_{2k}) + (d_{2k}10^{2k-1} + \cdots + d_210 + d_1)
\]

\[
= d_1[(11 - 1)^{2k-1} + 1] + d_2[(11 - 1)^{2k-2} + (11 - 1)] + \cdots + d_{2k-1}[(11 - 1) + (11 - 1)^{2k-2}] + d_1(1 + (11 - 1)^{2k-1})
\]

\[
\equiv d_1(-1 + 1) + d_2(1 - 1) + \cdots + d_{2k}(1 - 1)
\]

\[
\equiv 0 \pmod{11}.
\]

Lemma 2. If \( a_n \) is divisible by 11, then \( b_n \) is divisible by 11.

Proof: If \( a_n \) has an even number of digits this follows from Lemma 1. If \( a_n \) has an odd number of digits, then as in the proof of Lemma 1 we deduce that \( a_n - b_n \equiv 0 \pmod{11} \), and the result follows.

Clearly, if \( a_n \) has \( k \) digits, then \( a_{n+1} \) has at most \( k + 1 \) digits. Suppose for the sake of contradiction that 11 does not divide \( a_7 \). Then it follows from Lemma 1 and Lemma 2 that \( a_1 \) has an odd number of digits and that \( a_2, a_3, \ldots, a_6 \) each have the same number of digits as \( a_1 \) (otherwise
the first \( a_i \) after \( a_1 \) with more digits than \( a_1 \) has an even number of digits, implying that 11 divides \( a_7 \).

Let \( f \) and \( \ell \) be the first and last digits of \( a_1 \). Then, in order not to have an increase in the number of digits, the first digits of \( a_1, a_2, a_3, a_4, a_5 \) are \( f, f + \ell, 2(f + \ell), 4(f + \ell), 8(f + \ell); \) and then \( a_6 \) has one more digit than \( a_1 \) (since \( f + \ell \geq 1 \)), a contradiction.

Therefore, \( a_7 \) is divisible by 11, and since it is easy to see that \( a_7 > 11 \), this means that \( a_7 \) is not prime.

2. (J. Šimša) Let \( m \) and \( n \) be positive integers such that

\[
(x + m)(x + n) = x + m + n
\]

has at least one integer solution. Prove that \( \frac{1}{2} < \frac{m}{n} < 2 \).

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give Bataille’s version.

Let \( p(x) = x^2 + (m + n - 1)x + (mn - m - n) \). Since

\[
(m + n - 1)^2 - 4(mn - m - n) \\
= (m + n + 1)^2 - 4mn \\
\geq (2\sqrt{mn} + 1)^2 - 4mn > 0,
\]

the equation \( p(x) = 0 \) has two distinct solutions, one of which is an integer (from the hypothesis). As the sum of the solutions is the integer \( 1 - m - n \), the other solution is an integer as well. We denote these solutions by \( a, a' \) with \( a' < a \). Also, we note that \( p(-m) = -n < 0 \), \( p(-n) = -m < 0 \) so that \(-m \) and \(-n \) are between \( a \) and \( a' \) and in particular,

\[
m, n \geq 1 - a. \tag{1}
\]

Lastly, we observe that \( a \leq 0 \), since \( p(x) \neq 0 \) for \( x \geq 1 \) (if \( x \geq 1 \), then \( x^2 \geq x, x(m + n) \geq m + n \) and so \( p(x) \geq mn > 0 \)). Now, we rewrite \( p(a) = 0 \) as

\[
(m - (1 - a))(n - (1 - a)) = 1 - a.
\]

From \( 1 - a \geq 1 \) and the inequalities (1), we see that \( m - (1 - a) \) and \( n - (1 - a) \) are divisors \( d \), \( d' \) of \( 1 - a \) with \( d, d' \geq 1 \) and \( dd' = 1 - a \). Thus, \( m = dd' + d, n = dd' + d' \) and so

\[
2m - n = dd' + 2d - d' = d'(d - 1) + 2d \geq 2d > 0
\]

with \( 2m - n > 0 \) deduced similarly. The desired inequalities follow.

3. (T. Jurik) Triangle \( ABC \) is not equilateral, and the angle bisectors at \( A \) and \( B \) intersect the sides \( BC \) and \( AC \) at \( K \) and \( L \), respectively. Let \( S \) be the incentre, \( O \) be the circumcentre, and \( V \) be the orthocentre of triangle \( ABC \). Prove that the following statements are equivalent:
(a) The line $KL$ is tangent to the circumcircles of triangles $ALS$, $BVS$, and $BKS$.

(b) The points $A$, $B$, $K$, $L$, and $O$ are concyclic.

**Solution by Titu Zvonaru, Comănești, Romania.**

We denote by $\Gamma(XYZ)$ the circumcircle of $\triangle XYZ$, and let $\alpha = \angle BAC$, $\beta = \angle CBA$, and $\gamma = \angle ACB$.

Suppose first that (a) is true. Since $KL$ is tangent to $\Gamma(ALS)$ and $AK$ is the bisector of $\angle BAC$, we have

$$\angle KLS = \angle LAS = \angle SAB \iff \angle KLB = \angle KAB,$$

hence, points $A, B, K, L$ are concyclic \quad (1)

We also deduce that

$$\angle LBK = \angle KAL \iff \alpha = \beta. \quad (2)$$

Suppose that $KL$ is tangent to $\Gamma(BVS)$ at $T$. Taking the power of point $L$ with respect to $\Gamma(BVS)$ and $\Gamma(BKS)$ we obtain $LT^2 = LS \cdot LB = LK^2$, hence $KL$ is tangent to $\Gamma(BVS)$ at $K$, that is, the quadrilateral $VBKS$ is cyclic and

$$\angle VBK + \angle VSK = 180^\circ.$$

By (2) we know that the points $V$, $S$, and $C$ are collinear, so that

$$\angle VBK = \angle KSC \iff 90^\circ - \gamma = \frac{\alpha}{2} + \frac{\gamma}{2} \iff \alpha + 3\gamma = 180^\circ.$$

Since $\alpha = \beta$, $\alpha + \beta + \gamma = 180^\circ$ and $\alpha + 3\gamma = 180^\circ$, hence

$$\alpha = \beta = 72^\circ, \quad \gamma = 36^\circ. \quad (3)$$

Using (3), we deduce that

$$\angle OBD = \frac{180^\circ - \angle BOC}{2} = \frac{180^\circ - 2\alpha}{2} = 18^\circ,$$

$$\angle KAO = \angle KAC - \angle OAC = \frac{\alpha}{2} - \frac{180^\circ - \angle AOC}{2} = 36^\circ - 18^\circ = 18^\circ,$$

hence, quadrilateral $AOKB$ is cyclic. \quad (4)
By (1) and (4) it follows that the statement (b) is true. Conversely, suppose that the statement (b) is true, so that the points \( A, B, K, L \) and \( O \) are concyclic.

Since \( ABKL \) is cyclic, \( \angle LAK = \angle LBK \) is equivalent to \( \alpha = \beta \), which is equivalent to \( LK \parallel AB \); it follows that \( \angle SLK = \angle SBA = \angle SAL \) and \( \angle SKL = \angle SAB = \angle SBK \), and hence

\[
KL \text{ is tangent to } \Gamma(ALS) \text{ and to } \Gamma(BKS). \tag{5}
\]

Since \( ABKO \) is cyclic, we have that \( \angle OBK = \angle KAO \) is equivalent to \( 90^\circ - \alpha = \frac{\alpha}{2} - (90^\circ - \beta) \); but \( \alpha = \beta \), hence \( 90^\circ - \alpha = \frac{\alpha}{2} - 90^\circ + \alpha \) is equivalent to \( \alpha = \beta = 72^\circ \) and \( \gamma = 36^\circ \).

Since \( \alpha = \beta \), we deduce that

\[
\angle SKL = \angle SAB = \frac{\alpha}{2} = 36^\circ = \angle SBK \tag{6}
\]

and \( \angle VBK = 90^\circ - \gamma = 54^\circ \); \( \angle KSC = \frac{\alpha}{2} + \frac{\gamma}{2} = 54^\circ \), hence

\[
\text{the quadrilateral } SVBK \text{ is cyclic.} \tag{7}
\]

By (6) and (7)

\[
LK \text{ is tangent to } \Gamma(BVS) \text{ at point } K. \tag{8}
\]

By (5) and (8) it follows that the statement (a) is true.

4. (J. Švrček) A segment \( AB \) is given in the plane. Find the locus of the centroids of all acute triangles \( ABC \) for which the following holds: the vertices \( A \) and \( B \), the orthocentre \( V \), and the centre \( S \) of the incircle of the triangle \( ABC \) are concyclic.

Solved by Oliver Geulp, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Geulp's solution.

Let \( A' \) and \( B' \) be points on \( AB \) such that \( 3AA' = 3BB' = AB \). Let \( \sigma \) denote the region which is the open strip between the two perpendiculars to \( AB \) through \( A' \) and \( B' \). Let \( \Gamma_1 \) and \( \Gamma_2 \) denote the two circular arcs joining \( A' \) and \( B' \) with peripheral angles of 60°. We will prove that the locus of the centroids \( G \) are the two sub-arcs of \( \Gamma_1 \) and \( \Gamma_2 \) which lie inside \( \sigma \) (see the figure on the next page).

Let \( C \) be any point such that \( \triangle ABC \) is acute. Let \( AA^* \) and \( BB^* \) be the altitudes of \( \triangle ABC \) passing through \( A \) and \( B \). Since the points \( C, B^*, V, \) and \( A^* \) are concyclic, we have

\[
\angle AVB = \angle A^*VB^* = 180^\circ - \angle ACB.
\]
On the other hand,
\[ \angle ASB = 180^\circ - \frac{1}{2}(\angle BAC + \angle ABC) \]
\[ = 90^\circ + \frac{1}{2}\angle ACB. \]

The points \( A, B, V, \) and \( S \) are concyclic if and only if \( \angle AVB = \angle ASB, \) equivalently \( 180^\circ - \angle ACB = 90^\circ + \frac{1}{2}\angle ACB, \) that is, \( \angle ACB = 60^\circ. \)

Therefore, the locus of \( C \) is the union of the two circular arcs joining \( A \) and \( B \) that have peripheral angles of 60\(^\circ\), restricted to the region which is the open strip between the perpendiculars to \( AB \) through \( A \) and \( B \). Finally, if \( M \) is the midpoint of \( AB \), then \( MC = 3MG \), that is, the locus of \( G \) is homothetic to the locus of \( C \) with \( M \) as the centre of the homothety and ratio \( \frac{1}{3}. \)

5. (M. Panák) Find all triples \((p, q, r)\) of distinct prime numbers such that
\[ p|(q + r), \quad q|(r + 2p), \quad r|(p + 3q). \]

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Titu Zvonaru, Comănești, Romania. We give Manes’ solution.

The triples \((p, q, r)\) of distinct primes satisfying the above divisibility conditions are \((5, 3, 2), (2, 11, 7), \) and \((2, 3, 11). \)

Note that \( q \) is an odd prime since \( q = 2 \) and \( q | (r + 2p) \) implies \( r + 2p \) is even, and so \( r = 2, \) a contradiction since \( p, q, \) and \( r \) are distinct. Assume that \( p \) and \( r \) are also odd. Then \( q + r = pa, \) \( r + 2p = qb, \) and \( p + 3q = rc \) for some integers \( a, b, c \) where \( b \) is odd. Therefore, \( b = 2d + 1 \) for some integer \( d. \) Then \( r = qa - q \) and \( r + 2p = qb \) implies \( p(a + 2) = q(b + 1). \) Therefore, \( p | (b + 1) = 2(d + 1), \) so that \( p | (d + 1). \) Multiplying the equation \( r + 2p = q(2d + 1) \) by \( c \) and substituting \( rc = p + 3q \) yields \( p(1 + 2c) = 2q(d - 1), \) whence \( p | (d - 1). \) Thus, \( p | (d + 1) \) and \( p | (d - 1) \) implies \( p = 2, \) a contradiction. Therefore, either \( p \) or \( r \) must equal 2.

Assume \( r = 2 \) with \( p \) and \( q \) odd primes. Then \( p | (q + 2) \) implies either \( p = q + 2 \) or \( p < q + 2. \) If \( p < q + 2, \) then \( q | (r + 2p) = 2(p + 1), \) so that \( q | (p + 1). \) Since \( p \) and \( q \) are both odd, it follows that \( q < p + 1 < q + 3. \) Therefore, either \( p + 1 = q + 1 \) or \( p + 1 = q + 2, \) both of which are contradictions since \( p \) and \( q \) are distinct odd primes. Hence, \( p = q + 2. \) Then \( q | (r + 2p) = 2(p + 1), \) and so \( q | (p + 1) = q + 3, \) whence \( q = 3 \) and \( p = 5. \) This yields the first triple \((5, 3, 2). \)
Finally, assume $p = 2$ with $q$ and $r$ odd primes. The divisibility conditions for this case are

\begin{align*}
q + r & = 2a, \\
r + 4 & = qb, \\
3q + 2 & = rc,
\end{align*}

for some positive integers $a$, $b$, $c$ with $b$ and $c$ odd. Assume $r < q$. Then $q \mid (r + 4)$ implies $q \leq r + 4$. Therefore, $r < q \leq r + 4$. Since $q$, $r$ are odd primes, it follows that the only possible values for $q$ are $q = r + 2$ and $q = r + 4$. If $q = r + 2$, then $q \mid (r + 2p) = r + 4$ implies $(r + 2) \mid (r + 4)$, a contradiction since $r > 0$. Therefore, $q = r + 4$ so that in (3), $3(r + 4) + 2 = rc$ implies $r(c - 3) = 14$. Hence, $r \mid 14$ so that $r = 7$ and $q = r + 4 = 11$. Thus, the second triple is $(2, 11, 7)$.

On the other hand if $r > q$, let $r = q + 2k$ for some integer $k$. Note that $k > 1$ since $r = q + 2$ and $q \mid (r + 4) = 1 + 6$ imply $q = 3$ and $r = 5$. However, these values do not satisfy $r \mid (3q + 2)$. In (3), $3q + 2 = (q + 2k)c$ implies $q(3 - c) = 2(ck - 1) > 0$. Therefore, $2 \mid (3 - c)$ $> 0$ and $c$ is odd yield $c = 1$. Hence, $q \mid 6$, so that $q = 3$ and $r = 3q + 2 = 11$. This yields the last triple $(2, 3, 11)$.

Now we turn to the files for the April 2009 number of the Corner and solutions from our readers to problems of the Scientific and Technical Research Institute of Turkey, Team Selection Examination for the International Mathematical Olympiad given at [2009 : 144].

2. Let $n$ be a positive integer. In how many different ways can a $2 \times n$ rectangle be partitioned into rectangles with sides of integer length?

**Solution by Oliver Geipel, Brühl, NRW, Germany.**

Consider the rectangle with vertices $(0, 0)$, $(0, 2)$, $(n, 0)$, and $(n, 2)$ in the Cartesian plane. A partition can be characterized by the set $E$ of line segments $\langle (j, k), (j + 1, k) \rangle$ and $\langle (j, k), (j, k + 1) \rangle$ which constitute the borders of the small rectangles. We call a partition type $A$ if $\langle (n - 1, 1), (n, 1) \rangle \in E$; we call it type $B$ if $\langle (n - 1, 1), (n, 1) \rangle \not\in E$. For each partition $E$, the set $E' = E - \{(k, n - 1), (k, n)\}, \langle (n, j), (n, j + 1)\rangle | 0 \leq k \leq 2, 0 \leq j \leq 1\}$

$\cup \{(n - 1, 0), (n - 1, 1), (n - 1, 1), (n - 1, 2)\}$

constitutes a partition of the $2 \times (n - 1)$ rectangle with vertices $(0, 0)$, $(0, 2)$, $(n - 1, 0)$, and $(n - 1, 2)$.

If $E'$ is of type $A$, that means $\langle (n - 2, 1), (n - 1, 1) \rangle \in E'$, then there are five corresponding sets $E$ possible, four of type $A$ and one of type $B$; see Figure 1. Otherwise, if $E'$ is of type $B$, then there are three corresponding sets $E$ possible, one of type $A$ and two of type $B$; see Figure 2.
Let $A_n$ and $B_n$ denote the number of type A and type B partitions, respectively, and let $C_n = A_n + B_n$. We obtain $A_n = 4A_{n-1} + B_{n-1}$ and $B_n = A_{n-1} + 2B_{n-1}$ for $n \geq 2$. For $n \geq 3$ we derive

$$C_n = A_n + B_n = 5A_{n-1} + 3B_{n-1}$$

$$= 23A_{n-2} + 11B_{n-2} = 6C_{n-1} - 7C_{n-2}.$$ 

The initial values $C_1 = 2$ and $C_2 = 8$ are easy to check. We have obtained a linear recursion for $C_{n-1}$ which can be solved with repertoire methods, thus yielding the desired number of partitions

$$C_n = \frac{2 + \sqrt{2}}{2} \left(3 + \sqrt{2}\right)^{n-1} + \frac{2 - \sqrt{2}}{2} \left(3 - \sqrt{2}\right)^{n-1}.$$ 

3. Let $x, y, z$ be positive real numbers with $xy + yz + zx = 1$. Prove that

$$\frac{27}{4}(x + y)(y + z)(z + x) \geq (\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x})^2 \geq 6\sqrt{3}.$$ 

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille’s write-up.

From the constraint, we have

$$(x + y)(y + z) = y^2 + 1,$$

$$(y + z)(z + x) = z^2 + 1,$$

$$(z + x)(x + y) = x^2 + 1,$$

so that the right inequality can be rewritten as

$$x + y + z + \sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq 3\sqrt{3}. \quad (1)$$

Now, $(x + y + z)^2 = x^2 + y^2 + z^2 + 2 \geq xy + yz + zx + 2 = 3$, hence

$$x + y + z \geq \sqrt{3}. \quad (2)$$

Also, the function $f(t) = \sqrt{t^2 + 1}$ is a convex function (its second derivative satisfies $f''(t) = (t^2 + 1)^{-3/2} > 0$). Thus,

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq 3\sqrt{\left(\frac{x + y + z}{3}\right)^2 + 1}.$$
and using (2) we obtain
\[
\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq 2\sqrt{3}.
\] (3)

Adding (2) and (3) yields (1). As for the left inequality, it is equivalent to
\[
\frac{1}{\sqrt{x^2 + 1}} + \frac{1}{\sqrt{y^2 + 1}} + \frac{1}{\sqrt{z^2 + 1}} \leq \frac{3\sqrt{3}}{2}.
\] (4)

The constraint allows us to write \(x = \tan \frac{\alpha}{2}, y = \tan \frac{\beta}{2}, z = \tan \frac{\gamma}{2}\) where \(\alpha, \beta, \gamma\) are the angles of a triangle. Then, (4) can be rewritten as
\[
\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{2},
\]
which holds because from the concavity of \(\cos\) on \((0, \frac{\pi}{2})\) we have
\[
\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \leq 3 \cos \left( \frac{\alpha + \beta + \gamma}{6} \right) = \frac{3\sqrt{3}}{2}.
\]

5. Given a circle with diameter \(AB\) and a point \(Q\) on the circle different from \(A\) and \(B\), let \(H\) be the foot of the perpendicular dropped from \(Q\) to \(AB\). Prove that if the circle with centre \(Q\) and radius \(QH\) intersects the circle with diameter \(AB\) at \(C\) and \(D\), then \(CD\) bisects \(QH\).

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We give the version of Amengual Covas.

Let \(O\) be the centre of the circle on \(AB\) as diameter, and let \(Q'\) be the point on this circle diametrically opposite to \(Q\).

Let the common chord \(CD\) of the two given circles intersects \(QH\) and \(QO\) at points \(M\) and \(N\), respectively.

Since this common chord is perpendicular to the line of centres \(QO\), we see that, in right triangle \(DQQ'\), \(DN\) is the altitude to the hypotenuse.

By a standard mean proportion we then have
\[
QD^2 = QQ' \cdot QN,
\]
that is,
\[
QH^2 = 2QO \cdot QN.
\]
Since \( \triangle QNM \) is similar to \( \triangle QHO \), we also have \( \frac{QM}{QN} = \frac{QO}{QH} \), and hence \( QM \cdot QH = QO \cdot QN \).

Therefore, \( QM \cdot QH = \frac{1}{2} QH^2 \); whence \( QM = \frac{1}{2} QH \), as required.


1. Let \( a, b, c, \) and \( d \) be real numbers. Prove that

\[
\sqrt{a^4 + c^4} + \sqrt{a^4 + d^4} + \sqrt{b^4 + c^4} + \sqrt{b^4 + d^4} \geq 2\sqrt{2}(ab + bc).
\]

Solved by Arkady Alt, San Jose, CA, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's contribution.

The stated inequality is incorrect. A simple counterexample is given by \( a = b = c = 1 \) and \( d = 0 \). We prove the following correct version:

\[
\sqrt{a^4 + c^4} + \sqrt{a^4 + d^4} + \sqrt{b^4 + c^4} + \sqrt{b^4 + d^4} \geq 2\sqrt{2}(ab + cd). \tag{1}
\]

By the AM–GM Inequality and the Cauchy–Schwarz Inequality, we have

\[
\sqrt{a^4 + c^4} + \sqrt{b^4 + d^4} \geq 2\sqrt[4]{(a^4 + c^4)(b^4 + d^4)} \geq 2\sqrt{a^2b^2 + c^2d^2}. \tag{2}
\]

Since \( 2(a^2b^2 + c^2d^2) - (ab + cd)^2 = (ab - cd)^2 \geq 0 \) we have

\[
\frac{\sqrt{2(a^2b^2 + c^2d^2)}}{2\sqrt{a^2b^2 + c^2d^2}} \geq \frac{ab + cd}{\sqrt{2(ab + cd)}}. \tag{3}
\]

From (2) and (3) we obtain

\[
\sqrt{a^4 + c^4} + \sqrt{b^4 + d^4} \geq \sqrt{2}(ab + cd). \tag{4}
\]

Similarly, we have

\[
\sqrt{a^4 + d^4} + \sqrt{b^4 + c^4} \geq \sqrt{2}(ab + dc). \tag{5}
\]

Adding (4) and (5), inequality (1) follows.

2. In a triangle \( ABC \) with \( |AB| < |AC| < |BC| \), the perpendicular bisector of \( AC \) intersects \( BC \) at \( K \) and the perpendicular bisector of \( BC \) intersects \( AC \) at \( L \). Let \( O, O_1 \), and \( O_2 \) be the circumcentres of the triangles \( ABC, CKL \), and \( OAB \), respectively. Prove that \( OCO_1O_2 \) is a parallelogram.
Solution by Titu Zvonaru, Comănești, Romania.

As usual write $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$ and $a = BC$, $b = CA$, $c = AB$.

Since $c < b < a$, it follows that $\gamma < \beta < \alpha$, and it is easy to see that $\beta < 90^\circ$ and $\alpha + \gamma > 90^\circ$. Let $M$ and $N$ be the midpoints of the sides $BC$ and $AC$, respectively.

In $\triangle CML$ and $\triangle CNK$ we have

$$CL = \frac{a}{2 \cos \gamma}; \quad CK = \frac{b}{2 \cos \gamma}.$$

Since $\frac{a}{CL} = \frac{b}{CK}$ and $\angle BCA = \angle LCK$, it follows that $\triangle CLK$ and $\triangle ABC$ are similar, hence $LK = \frac{c}{2 \cos \gamma}$.

By the Law of Sines in $\triangle CLK$, we obtain

$$\frac{CO_1}{LK} = \frac{c}{2 \sin \gamma \cos \gamma} = \frac{c}{2 \sin 2\gamma} \quad \text{(1)}.$$

By the Law of Sines in $\triangle OAB$, we have

$$\frac{OO_2}{AB} = \frac{c}{2 \sin \angle AOB} = \frac{c}{2 \sin 2\gamma} \quad \text{(2)}.$$

By (1) and (2) we have that $CO_1 = OO_2$.

If $\alpha \geq 90^\circ$, then $\angle CLK > 90^\circ$ and

$$\angle O_1 CL = \frac{180^\circ - \angle LO_1 C}{2} = 90^\circ - \left(\frac{360^\circ - 2 \angle LKC}{2}\right) = \alpha - 90^\circ.$$

If $\alpha < 90^\circ$, then we obtain

$$\angle O_1 CL = \frac{180^\circ - \angle LO_1 C}{2} = 0^\circ - \frac{2 \angle LKC}{2} = 90^\circ - \alpha.$$
In any case, it is easy to see that \( O_1CB = \alpha + \gamma - 90^\circ = 90^\circ - \beta \), hence \( CO_1 \perp AB \). This implies that \( CO_1 || OO_2 \), because \( O_2 \) belongs to the perpendicular bisector of \( AB \). It follows that \( OCO_1O_2 \) is a parallelogram.

4. Find all triples \((m, n, k)\) of nonnegative integers such that \( 5^m + 7^n = k^3 \).

Solved by Oliver Geupel, BniHL, NRW, Germany; and by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Geupel’s solution.

The unique solution is \((m, n, k) = (0, 1, 2)\).

For nonnegative integers \(i\), we have

\[
5^{2i} \equiv 1 \pmod{8}, \quad 5^{2i+1} \equiv -3 \pmod{8}, \quad 7^{2i} \equiv 1 \pmod{8}, \quad 7^{2i+1} \equiv -1 \pmod{8}, \quad i^3 \equiv \pm2, 4 \pmod{8}.
\]

Therefore, if \(m, n, \) and \(k\) are nonnegative integers with \( 5^m + 7^n = k^3 \), then there are nonnegative integers \(s, t, \) and \(u\) such that \(m = 2s, n = 2t + 1\) and \(k = 2u\); hence

\[
25^s + 7 \cdot 49^t = 8u^3.
\] (1)

We claim that \(3 \mid s\).

If \(t = 0\), then \(25^s \equiv 2 - u^3 \pmod{9}\). For nonnegative integers \(i\), it holds that \(25^{3i+1} \equiv 7 \pmod{9}\) and \(25^{3i+2} \equiv 4 \pmod{9}\). On the other hand, however, \(2 - u^3 \equiv 1, 2, 3 \pmod{9}\), hence \(3 \mid s\).

Otherwise, if \(t > 0\), then \(25^s \equiv 8u^3 \pmod{49}\); hence \(\gcd(u, 7) = 1\). By Euler’s Totient Theorem, \(25^{14s} \equiv (2u)^{42} \equiv (2u)^{\varphi(49)} \equiv 1 \pmod{49}\). It is tedious but straightforward to check that \(5^t \equiv 1 \pmod{49}\) if and only if \(42 \mid i\). Thus, \(3 \mid s\), which completes the proof of our claim.

Substituting \(s = 3v\), we obtain from (1) that

\[
7 \cdot 49^t = (2u)^3 - 25^{3v} = (2u - 25^v) \left((2u)^2 + 2u \cdot 25^v + 25^{2v}\right). \] (2)

Therefore, there exists a nonnegative integer \(w\) such that

\[
(2u)^2 + 2u \cdot 25^v + 25^{2v} = 7^{2t+1-w}
\] (3)

and \(2u - 25^v = 7^w\); thus

\[
(2u)^2 - 4u \cdot 25^v + 25^{2v} = 7^w.
\] (4)

From (3) and (4) it follows that \(6u \cdot 25^v = 7^{2t+1-w} - 7^w\). If \(w \geq 1\) then \(7 \mid u\), and \(7\) would be a divisor of \(2u - 7^w = 25^v\), which is impossible. Consequently, \(w = 0\). It follows that \(2u = 25^v + 1\); hence by (2):

\[
25^{3v} + 7 \cdot 49^t = (25^v + 1)^3 = 25^{3v} + 3 \cdot 25^{2v} + 3 \cdot 25^v + 1,
\]

\[
7 \cdot 49^t = 3 \cdot 25^{2v} + 3 \cdot 25^v + 1,
\]

and thus \(25^v \mid (7 \cdot 49^t - 1)\).
Now, if \( v \geq 1 \), then \( 5 \mid (7 \cdot 49^v - 1) \). However, the residues of \( 7 \cdot 49^v \) modulo 5 are \( \pm 2 \), which is a contradiction. We conclude that \( v = 0 \) and therefore \( u = 1 \) and \( (m, n, k) = (0, 1, 2) \).

5. Let \( a, b, \) and \( c \) be the side lengths of a triangle whose incircle has radius \( r \). Prove that

\[
\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}.
\]

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give the comment and reference from Bataille.


Next we turn to solutions to problems of the 2005 Australian Mathematical Olympiad given at [2009 : 146-147].

1. Let \( ABC \) be a right-angled triangle with the right angle at \( C \). Let \( BCDE \) and \( ACFG \) be squares external to the triangle. Furthermore, let \( AE \) intersect \( BC \) at \( H \), and let \( BG \) intersect \( AC \) at \( K \). Find the size of \( \angle DKH \).

Solved by Oliver Geupel, Brühl, NRW, Germany; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comănești, Romania. We give Kandall’s solution.

Let \( BC = a \) and \( AC = b \). Triangle \( KCB \) is similar to triangle \( GFB \) and triangle \( HCA \) is similar to triangle \( EDA \). Therefore,

\[
\frac{KC}{b} = \frac{a}{a+b} \quad \text{and} \quad \frac{HC}{a} = \frac{b}{a+b}.
\]

Consequently, \( KC = HC = \frac{ab}{a+b} \), hence \( \angle DKH = 45^\circ \).

3. Let \( n \) be a positive integer, and let \( a_1, a_2, \ldots, a_n \) be positive real numbers such that \( a_1 + a_2 + \cdots + a_n = n \). Prove that

\[
\frac{a_1}{a_1^2 + 1} + \frac{a_2}{a_2^2 + 1} + \cdots + \frac{a_n}{a_n^2 + 1} \leq \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1}.
\]
Solved by George Apostolopoulos, Messolonghi, Greece; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Henry Ricardo, Tappan, NY, USA. We give Ricardo’s write-up.

We need two easily established facts: (a) \( x + \frac{1}{x} \geq 2 \) for positive \( x \), and (b) \( f(t) = \frac{1}{t+1} \) is a convex function for nonnegative \( t \). Then for each \( k \) we have

\[
\frac{a_k}{a_k + 1} = \frac{1}{\frac{a_k^2 + 1}{a_k}} \leq \frac{1}{2},
\]

and so

\[
\sum_{k=1}^{n} \frac{a_k}{a_k^2 + 1} \leq \frac{n}{2} = n f(1) = n f\left( \frac{\sum_{k=1}^{n} \frac{a_k}{n}}{n} \right) \leq \sum_{k=1}^{n} f(a_k) = \sum_{k=1}^{n} \frac{1}{a_k + 1}.
\]

It is easy to see that equality holds if and only if \( a_k = 1 \) for each \( k \).

4. Prove that for each positive integer \( n \) there exists a positive integer \( x \) such that \( \sqrt{x + 2004^n} + \sqrt{x} = (\sqrt{2005} + 1)^n \).

Solved by Arkady Alt, San Jose, CA, USA, and Michel Bataille, Rouen, France. We give Bataille’s version.

First we solve for \( x \) the given equation. Squaring yields

\[
2 \sqrt{x(x + 2004^n)} = \left( \sqrt{2005} + 1 \right)^{2n} - 2004^n - 2x.
\]

and squaring again yields

\[
x = \frac{\left( \left( \sqrt{2005} + 1 \right)^{2n} - 2004^n \right)^2}{4 \left( \sqrt{2005} + 1 \right)^{2n}}.
\]

Observing that \( 2004 = \left( \sqrt{2005} + 1 \right) \left( \sqrt{2005} - 1 \right) \), we finally see that

\[
x = \frac{1}{4} \left( \left( \sqrt{2005} + 1 \right)^n - \left( \sqrt{2005} - 1 \right)^n \right)^2
\]

is the unique real solution to the given equation. To complete the proof, it is sufficient to show that for any positive integers \( n \) and \( a \) the number \( A = \left( (\sqrt{a} + 1)^n - (\sqrt{a} - 1)^n \right)^2 \) is an integer multiple of 4.

From the Binomial Theorem, we have

\[
A = \left( \sum_{k=0}^{n} \binom{n}{k} (\sqrt{a})^{n-k} (1 + (-1)^{k+1}) \right)^2 = \left( 2 \sum_{k=0 \text{ odd}} \binom{n}{k} (\sqrt{a})^{n-k} \right)^2.
\]
Now, if \( n \) is odd, then \( n - k \) is even for each odd \( k \) and \( \sum_{k \text{ odd}} \binom{n}{k} (\sqrt{a})^{n-k} \) is an integer so that \( A \) is an integer multiple of 4.

If \( n \) is even, then \( 2 \sum_{k \text{ odd}} \binom{n}{k} (\sqrt{a})^{n-k} = 2(\sqrt{a}) \cdot B \) for some integer \( B \) and \( A = 4aB^2 \) is an integer multiple of 4 as well.

6. Let \( ABC \) be a triangle. Let \( D, E, \) and \( F \) be points on the line segments \( BC, CA, \) and \( AB \), respectively, such that line segments \( AD, BE, \) and \( CF \) meet in a single point. Suppose that \( ACDF \) and \( BCEF \) are cyclic quadrilaterals. Prove that \( AD \) is perpendicular to \( BC, BE \) is perpendicular to \( AC \), and \( CF \) is perpendicular to \( AB \).

**Solution by Geoffrey A. Kandall, Hamden, CT, USA.**

Let \( P \) be the point at which \( AD, BE, \) and \( CF \) meet.

Since \( ACDF \) is cyclic, \( \angle ACF = \angle ADF \); since \( BCEF \) is cyclic, \( \angle ECF = \angle EBF \). Therefore, \( PDBF \) is cyclic. Analogously, \( PEAF \) is cyclic.

Now, \( \angle EFA = \angle EPA = \angle DFB \). Also, \( \angle PFE = \angle PAE = \angle PFD \) (the latter equality holds since \( ACDF \) is cyclic). Thus, \( \angle CFA = \angle CFB = 90^\circ \).

It follows that \( \angle BEA \) and \( \angle ADB \) are each \( 90^\circ \).

7. Let \( a_0, a_1, a_2, \ldots \) and \( b_0, b_1, b_2, \ldots \) be two sequences of integers such that \( a_0 = b_0 = 1 \) and for each nonnegative integer \( k \)

(a) \( a_{k+1} = b_0 + b_1 + b_2 + \cdots + b_k \), and

(b) \( b_{k+1} = (0^2 + 0 + 1)a_0 + (1^2 + 1 + 1)a_1 + \cdots + (k^2 + k + 1)a_k \).

For each positive integer \( n \) show that

\[
a_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n}.
\]

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We use Alt's solution.

The recursions (a) and (b) can be rewritten as follows:

\[
a_{n+1} = a_n + b_n, \]
\[
b_{n+1} = (n^2 + n + 1)a_n + b_n; \quad n \geq 1.
\]
By making the substitutions \( b_n = a_{n+1} - a_n \) and \( b_{n+1} = a_{n+2} - a_{n+1} \)
in \( b_{n+1} = (n^2 + n + 1) a_n + b_n \) we obtain successively
\[
\begin{align*}
a_{n+2} - a_{n+1} &= (n^2 + n + 1) a_n + a_{n+1} - a_n , \\
a_{n+2} &= 2a_{n+1} + n (n + 1) a_n , \\
a_{n+2} &= 2a_{n+1} + n (n + 1) a_n ; \quad n \geq 1.
\end{align*}
\]
where \( a_0 = 1 \) and \( a_1 = b_0 = 1 \).

Using (2) we get \( a_2 = 2, a_3 = 6, a_4 = 24, \) and \( a_5 = 120, \) suggesting that \( a_n = n! \), and we confirm this by using Mathematical Induction.

Indeed, supposing that \( a_n = n! \) and \( a_{n-1} = (n-1)! \) and using (2) we obtain, for any \( n \geq 1 \),
\[
a_{n+1} = 2a_{n} + (n - 1)na_{n-1} = 2n! + (n - 1)n(n - 1)!
= 2n! + (n - 1)n! = (n - 1 + 2)n! = (n + 1)!
\]

Since \( a_n = n! \), then \( b_n = a_{n+1} - a_n = (n + 1)! - n! = n \cdot n! = na_n, \) and therefore
\[
\frac{b_1b_2\cdots b_n}{a_1a_2\cdots a_n} = \frac{n!a_1a_2\cdots a_n}{a_1a_2\cdots a_n} = n! = a_n.
\]


1. (E. Barabanov) Is it possible to partition the set of all integers into three nonempty pairwise disjoint subsets so that for any two numbers \( a \) and \( b \) from different subsets
   
   (a) there is a number \( c \) in the third subset such that \( a + b = 2c \)?
   
   (b) there are numbers \( c_1 \) and \( c_2 \) in the third subset such that \( a + b = c_1 + c_2 \)?

**Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.**

(a) This is impossible. Suppose \( Z = A \cup B \cup C \) is a partition of \( Z \) satisfying the given condition. Without loss of generality, assume \( 1 \in A \). If \( B \) contains any even integer \( b \), then \( 1 + b \) is odd. Since \( 2c \) is even for all \( c \in C \), we have a contradiction. Hence, \( B \) contains no even integers. Then \( 2 \in A \) or \( 2 \in C \). In either case, \( 2 + b \) is odd for any \( b \in B \), again a contradiction.

(b) This is possible. Let \( Z \) be partitioned as \( Z = U \cup V \cup W \) where \( U = \{3k \mid k \in \mathbb{Z}\}, V = \{3k + 1 \mid k \in \mathbb{Z}\}, \) and \( W = \{3k + 2 \mid k \in \mathbb{Z}\} \). Let \( a \) and \( b \) be two numbers from different subsets in the partition. There are three cases to consider:

If \( a \in U, b \in V \), then write \( a = 3k_1 \) and \( b = 3k_2 + 1 \), and take \( c_1 = 3k_1 + 2 \) and \( c_2 = 3(k_2 - 1) + 2 \) as the required elements in \( W \).
If \( a \in U, b \in W \), then write \( a = 3k_1 \), and \( b = 3k_2 + 2 \), and take \( c_1 = 3k_1 + 1 \) and \( c_2 = 3k_2 + 1 \) as the required elements in \( V \).

If \( a \in V, b \in W \), then write \( a = 3k_1 + 1 \) and \( b = 3k_2 + 2 \), and take \( c_1 = 3k_1 \) and \( c_2 = 3(k_2 + 1) \) as the required elements in \( U \).

Therefore, \( U, V, \) and \( W \) satisfy the prescribed condition.

3. (V. Karamzin) Let \( a, b, \) and \( c \) be positive real numbers such that \( abc = 1 \). Prove that \( 2(a^2 + b^2 + c^2) + a + b + c \geq ab + bc + ca + 6 \).

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Alt’s version.

Since \( a + b + c \geq 3\sqrt[3]{abc} = 3 \) and \( ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3 \) by the AM–GM Inequality, then we have

\[
2(a^2 + b^2 + c^2) + a + b + c - (ab + bc + ca) - 6
= 2(a^2 + b^2 + c^2 - ab - bc - ca) + a + b + c + ab + bc + ca - 6
= (a - b)^2 + (b - c)^2 + (c - a)^2
\quad + (a + b + c - 3) + (ab + bc + ca - 3) \geq 0.
\]

5. (I. Voronovich) Let \( AA_1, BB_1, \) and \( CC_1 \) be the altitudes of an acute triangle \( ABC \). Prove that the feet of the perpendiculars from \( C_1 \) to the segments \( AC, BC, BB_1, \) and \( AA_1 \) are collinear.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonari, Comănești, Romania. We give Kandall’s version.

Let \( P, Q, R, S \) be the feet of the perpendiculars from \( C_1 \) to \( AC, BC, BB_1, AA_1 \), respectively, and let the orthocentre of \( ABC \) be \( H \). Draw \( PS \) and \( SR \).

The quadrilaterals \( APSC_1 \) and \( SHRC_1 \) are cyclic, and so \( \angle PSA = \angle PC_1A = 90^\circ - \angle CAB \) and \( \angle HSR = \angle HC_1R = 90^\circ - \angle RC_1B = \angle RBA = 90^\circ - \angle CAB \). Thus, \( \angle PSA = \angle HSR \), that is, the points \( P, S, \) and \( R \) are collinear. The proof that \( S, R, \) and \( Q \) are collinear is analogous. Therefore, \( P, S, R, \) and \( Q \) are collinear.
7. (I. Zhuk) Let $x$, $y$, and $z$ be real numbers greater than 1 such that

$$xy^2 - y^2 + 4xy + 4x - 4y = 4004,$$
$$xz^2 - z^2 + 6xz + 9x - 6z = 1009.$$ 

Determine all possible values of $xyz + 3xy + 2xz - yz + 6x - 3y - 2z$.

Solved by Arkady Alt, San Jose, CA, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Zelator’s solution.

The first equation is equivalent to $x(y^2 + 4y + 4) = 4004 + y^2 + 4y$, or $x(y + 2)^2 = 4000 + (y + 2)^2$, and we obtain

$$x = \frac{4000}{(y + 2)^2} + 1. \tag{3}$$

By similar manipulations of the second equation we obtain

$$x = \frac{1000}{(z + 3)^2} + 1. \tag{4}$$

Note that both (3) and (4) are consistent with the hypothesis that $x > 1$, $y > 1$, and $z > 1$.

By (3) and (4) we have

$$\frac{4000}{(y + 2)^2} = \frac{1000}{(z + 3)^2} \iff \left(\frac{y + 2}{z + 3}\right)^2 = 4,$$

and since $y + 2 > 0$ we have $\frac{y + 2}{z + 3} = 2$ and $y = 2z + 4$.

Next, we write

$$Q(x, y, z) = xyz + 3xy + 2xz - yz + 6x - 3y - 2z$$
$$= (xyz + 3xy + 2xz + 6x) + (-yz - 3y - 2z)$$
$$= Q_1(x, y, z) + Q_2(x, y, z). \tag{5}$$

We have $Q_1(x, y, z) = x(yz + 3y + 2z + 6)$. Substituting $y = 2z + 4$ yields $Q_1(x, y, z) = 2x(z + 3)^2$, and then by (4) we obtain

$$Q_1(x, y, z) = 2000 + 2(z + 3)^2. \tag{6}$$

Next we substitute $y = 2z + 4$ into $Q_2(x, y, z) = -yz - 3y - 2z$ to obtain

$$Q_2(x, y, z) = 6 - 2(z + 3)^2. \tag{7}$$

By virtue of (5), (6), and (7) we have $Q(x, y, z) = 2006$.

Thus, the expression $Q(x, y, z)$ has a fixed value, namely 2006, so the set of all possible values of $Q(x, y, z)$ is the singleton set \{2006\}.
To finish the file of readers' solutions for the April 2009 number of the Corner we look at solutions to problems of the 56th Belarusian Mathematical Olympiad 2006, Category B, Final Round, given at [2009 : 148–149].

1. (I. Voronovich) Given a convex quadrilateral $ABCD$ with $DC = a$, $BC = b$, $\angle DAB = 90^\circ$, $\angle DCB = \varphi$, and $AB = AD$, find the length of the diagonal $AC$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Kandall's solution.

Let $AB = AD = t$ and $\angle BDC = \theta$.
Then $DB = t\sqrt{2}$ and $\angle ADB = 45^\circ$.

By the Law of Cosines,

$$AC^2 = a^2 + t^2 - 2at \cos(\theta + 45^\circ) = a^2 + t^2 - \sqrt{2}at(\cos \theta - \sin \theta).$$

In $\triangle BCD$ we have the relations

$$\cos \theta = \frac{a^2 + 2t^2 - b^2}{2at\sqrt{2}}$$
$$\sin \theta = \frac{b \sin \varphi}{t\sqrt{2}}.$$

Now we substitute these and simplify:

$$AC = \left(\frac{a^2 + b^2 + 2ab \sin \varphi}{2}\right)^{1/2}.$$

3. (I. Biznets) Let $a$, $b$, and $c$ be positive real numbers. Prove that

$$\frac{a^3 - 2a + 2}{b + c} + \frac{b^3 - 2b + 2}{c + a} + \frac{c^3 - 2c + 2}{a + b} \geq \frac{3}{2}.$$

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and by Titu Zvonaru, Comănești, Romania. We use Zvonaru's presentation.

Since $a^3 - 2a + 2 = a^3 - 3a + 2 + a = (a - 1)^2(a + 2) + a$, the given inequality is the same as

$$\left(\frac{(a - 1)^2(a + 2)}{b + c}\right) + \left(\frac{(b - 1)^2(b + 2)}{c + a}\right) + \left(\frac{(c - 1)^2(c + 2)}{a + b}\right) + \left(\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}\right) \geq \frac{3}{2}.$$

But this inequality is true, as the first sum is obviously nonnegative and the second sum is greater than $\frac{3}{2}$ by Nesbitt's inequality.
6. (I. Voronovich) A sequence \( \{ (a_n, b_n) \}_{n=1}^{\infty} \) of pairs of real numbers is such that

\[
(a_{n+1}, b_{n+1}) = (a_n^2 - 2b_n, b_n^2 - 2a_n)
\]

for all \( n \geq 1 \). Find \( 2^{512}a_{10} - b_{10} \) if \( 4a_1 - 2b_1 = 7 \).

**Solution by Michel Bataille, Rouen, France.**

Let \( p(x) = x^3 - a_1x^2 + b_1x - 1 \) and let \( \alpha, \beta, \gamma \) be the complex roots of this polynomial. Then, \( p(x) = (x - \alpha)(x - \beta)(x - \gamma) \) and

\[
a_1 = \alpha + \beta + \gamma,
\]

\[
b_1 = \alpha\beta + \beta\gamma + \gamma\alpha,
\]

\[
1 = \alpha\beta\gamma.
\]

Now, easy calculations yield

\[
-p(x)p(-x) = (x^2 - \alpha^2)(x^2 - \beta^2)(x^2 - \gamma^2)
\]

as well as \(-p(x)p(-x) = q(x^2)\), where

\[
q(x) = x^3 - (a_1^2 - 2b_1)x^2 + (b_1^2 - 2a_1)x - 1
\]

\[
= x^3 - a_2x^2 + b_2x - 1.
\]

Thus, the roots of \( x^3 - a_2x^2 + b_2x - 1 \) are \( \alpha^2, \beta^2, \gamma^2 \) and so

\[
a_2 = \alpha^2 + \beta^2 + \gamma^2,
\]

\[
b_2 = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2.
\]

Continuing this way, an easy induction argument yields

\[
a_n = \alpha^{2^{n-1}} + \beta^{2^{n-1}} + \gamma^{2^{n-1}},
\]

\[
b_n = (\alpha\beta)^{2^{n-1}} + (\beta\gamma)^{2^{n-1}} + (\gamma\alpha)^{2^{n-1}}.
\]

for all positive integers \( n \).

Since \( p(2) = 7 - (4a_1 - 2b_1) = 0 \), we have that 2 is a root of \( p(x) \).

Taking \( \alpha = 2 \), then \( \beta\gamma = \frac{1}{2} \) and the above formulas give

\[
a_{10} = 2^{2^9} + \beta^{2^9} + \gamma^{2^9},
\]

\[
b_{10} = \frac{1}{2^{2^9}} + 2^{2^9}(\beta^{2^9} + \gamma^{2^9})
\]

It follows that

\[
2^{512}a_{10} - b_{10} = 2^{2^9}a_{10} - b_{10} = 2^{2^9} - \frac{1}{2^{2^9}} = 2^{1024} - \frac{1}{2^{512}}.
\]

That completes this number of the Corner. Send me your nice solutions, generalizations, and comments.