Mayhem Solutions

M394. Proposed by the Mayhem Staff.

The numbers \( a, b, c, d, \) and \( e \) are five consecutive integers, in that order. Prove that the difference between the average of the squares of \( e \) and \( e \) and the average of the squares of \( a \) and \( c \) is equal to four times \( e \).

Solution by all the solvers below indicated by a star.

We write the numbers \( a, b, c, d, \) and \( e \) as \( n - 2, n - 1, n, n + 1, \) and \( n + 2 \), respectively. Then

\[
\frac{1}{2} (e^2 + e^2) - \frac{1}{2} (a^2 + c^2) = \frac{1}{2} (e^2 - a^2) = \frac{1}{2} ((n + 2)^2 - (n - 2)^2) = \frac{1}{2} (2n^2) = 4n = 4e,
\]

as required.

Solved by *EDIN AJANOVIĆ, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; *JACLYN CHANG, student, Western Canada High School, Calgary, AB; *RICHARD I. HESS, Rancho Palos Verdes, CA, USA; *HUGO LUYO SANCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; *RICARD PEIRO, IES "Abastos", Valencia, Spain; *BRUNO SALGUEIRO FANEGO, Viveiro, Spain; *JOSE JAIME SAN JUAN CASTELLANOS, student, Universidad tecnológica de la Mixteca, Oaxaca, Mexico; *JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; *GUSNADI WIYOGA, student, SMPN 5, Yogyakarta, Indonesia; *OSCAR XIA, student, St. George's School, Vancouver, BC; and *KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M395. Proposed by the Mayhem Staff.

The quadrilateral \( ABCD \) is such that each of its sides is tangent to a given circle, as shown. If \( AB = AD \), prove that \( BC = CD \).

Solution by Jadyn Chang, student, Western Canada High School, Calgary, AB.

In the diagram, \( AB = AD \), and \( AB, BC, CD, DA \) are tangent to the circle at \( E, F, G, H \), respectively.

Because of the theorem that says that the two tangents to a circle from a given exterior point have the same length, then \( AE = AH, BE = BF, CF = CG, \) and \( DG = DH \).
Also, since \( AB = AD \), then \( AH + DH = AE + BE \). But \( AH = AE \), so \( DH = BE \). But \( DG = DH \) and \( BE = BF \), so \( DG = BF \). Since \( CF = CG \), then \( BC = BF + CF = DG + CG = CD \), as required.

Also solved by EDIN AJANOVIĆ, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOPOULOS, Messolonghi, Greece; HUGO LUYO SANCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; RICARD PEIRO, IES “Abastos”, Valencia, Spain; BRUNO SALGUEIRO PANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There was one incomplete solution submitted.

**M396. Proposed by the Mayhem Staff.**

The rectangle \( ABCD \) has side lengths \( AB = 8 \) and \( BC = 6 \). Circles with centres \( O_1 \) and \( O_2 \) are inscribed in triangles \( ABD \) and \( BCD \). Determine the distance between \( O_1 \) and \( O_2 \).

**Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.**

We know that \( AD = CB \), \( AB = CD \), and \( BD = DB \). Hence \( \triangle ABD \) is congruent to \( \triangle BCD \). This means that the two incircles have equal radii.

Next, we find the radius, \( r \), of these circles by finding the radius of the incircle of \( \triangle ABD \) (see the figure below). Connect \( O_1 \) to each of \( A \), \( B \), and \( D \). Also, let points \( P \), \( Q \), and \( R \) on \( AB \), \( BD \), and \( DA \), respectively, be the points of tangency of the circle to the sides of the triangle; connect \( O_1 \) to \( P \), \( Q \), and \( R \).

Since \( AD = 6 \) and \( AB = 8 \), then by the Pythagorean Theorem, we have

\[
BD = \sqrt{AD^2 + AB^2} = \sqrt{6^2 + 8^2} = 10.
\]

Also, the area of \( \triangle ABD \) is \( \frac{1}{2}(AD)(AB) = \frac{1}{2}(8)(6) = 24 \). But this area also equals the sum of the areas of \( \triangle AOB \), \( \triangle BOA \), and \( \triangle DOA \). Since \( O_1P \), \( O_1Q \), and \( O_1R \) are perpendicular to \( AB \), \( BD \), and \( DA \), respectively, then these areas equal \( \frac{1}{2}(AB)(O_1P) \), \( \frac{1}{2}(BD)(O_1Q) \), and \( \frac{1}{2}(DA)(O_1R) \), respectively. Therefore, \( \frac{1}{2}(8r) + \frac{1}{2}(10r) + \frac{1}{2}(6r) = 24 \), or \( 12r = 24 \), and so \( r = 2 \).
Lastly, construct rectangle $O_1XO_2Y$ with sides parallel to the sides of the original rectangle (see the second figure on the preceding page). Note that $O_1X = 8 - 2r$, since $O_1$ is $r$ units from $AD$ and $O_2$ is $r$ units from $BC$. Thus, $O_1X = 4$. Similarly, $XO_2 = 6 - 2r = 2$. Therefore,

$$O_1O_2 = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}.$$ 

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JACLYN CHANG, student, Western Canada High School, Calgary, AB; HUGO LUYO SANCHEZ, Pontificia Universidad Catolica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were two incomplete solutions submitted.

**M397. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.**

Determine all pairs $(x, y)$ of integers such that

$$x^4 - x + 1 = y^2.$$ 

**Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.**

We consider four cases: $x \leq -1$, $x = 0$, $x = 1$, and $x \geq 2$.

If $x \leq -1$, then $x < 1$, so $(x^2)^2 = x^4 < x^4 - x + 1$. Also, $x < 0$ and $2x + 1 \leq -1 < 0$, so $x(2x + 1) > 0$, which yields $2x^2 > -x$ and

$$x^4 - x + 1 < x^4 + 2x^2 + 1 = (x^2 + 1)^2.$$ 

Therefore, $(x^2)^2 < x^4 - x + 1 < (x^2 + 1)^2$. Since $x^4 - x + 1$ is strictly between two consecutive perfect squares, then it cannot be a perfect square itself, so it cannot equal $y^2$ in this case.

If $x = 0$, then the equation becomes $y^2 = 1$, so $y = \pm 1$. This yields the solutions $(x, y) = (0, 1)$ and $(0, -1)$.

If $x = 1$, then the equation becomes $y^2 = 1$, so $y = \pm 1$. This yields the solutions $(x, y) = (1, 1)$ and $(1, -1)$.

If $x \geq 2$, then $x > 1$ so $x^4 - x + 1 < x^4 = (x^2)^2$. Also, $x(2x - 1) > 0$ so $-x > -2x^2$, which yields

$$x^4 - x + 1 > x^4 - 2x^2 + 1 = (x^2 - 1)^2.$$ 

Therefore, $(x^2 - 1)^2 < x^4 - x + 1 < (x^2)^2$. Since $x^4 - x + 1$ is again strictly between two consecutive perfect squares, then it cannot be a perfect square itself, so it cannot equal $y^2$ in this case.

This covers all possible cases. Therefore, the solutions are $(0, 1)$, $(0, -1)$, $(1, 1)$, and $(1, -1)$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were three incomplete solutions submitted.
M398. Proposed by the Mayhem Staff.

(a) The cubic equation \( w^3 - bw^2 + cw - d = 0 \) has roots \( r, s, \) and \( t \). Determine \( b, c, \) and \( d \) in terms of \( r, s, \) and \( t \).

(b) Suppose that \( a \) is a real number. Determine all solutions to the system of equations

\[
\begin{align*}
x + y + z &= a, \\
xy + yz + zx &= -1, \\
xyz &= -a.
\end{align*}
\]

Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.

(a) Since \( r, s, \) and \( t \) are the roots of \( w^3 - bw^2 + cw - d = 0 \), then we have

\[
\begin{align*}
w^3 - bw^2 + cw - d &= (w - r)(w - s)(w - t), \\
w^3 - bw^2 + cw - d &= w^3 - (r + s + t)w^2 + (rs + st + rt)w - rst.
\end{align*}
\]

Since these cubics are equal for any value of \( w \), the corresponding coefficients are equal, so we have \( b = r + s + t, \ c = rs + st + rt, \) and \( d = rst \).

(b) Suppose that \( x, y, \) and \( z \) are the roots of the equation

\[
m^3 - (x + y + z)m^2 + (xy + yz + zx)m - xyz = 0.
\]

From the given information, this means that \( x, y, \) and \( z \) are the roots of \( m^3 - am^2 - m + a = 0 \), which can be rewritten as

\[
\begin{align*}
m^2(m - a) - (m - a) &= 0; \\
(m^2 - 1)(m - a) &= 0; \\
(m - 1)(m + 1)(m - a) &= 0.
\end{align*}
\]

Therefore, the roots are \( m = 1, \ m = -1, \) and \( m = a \).

Therefore, the possible values of \( x, y, \) and \( z \) are \( 1, -1, \) and \( a \). In order to satisfy the given equations, \( x, y, \) and \( z \) need to take all three of these values, in some order. Therefore, the possible triples \( (x, y, z) \) are \( (1, -1, a), \) \( (-1, 1, a), \) \( (a, 1, -1), \) \( (a, -1, 1), \) \( (1, a, -1), \) \( (-1, a, 1), \) and \( (-1, -1, 1) \).

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messakonghi, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SANCHEZ, Pontificia Universidad Catolica del Peru, Lima, Peru; RICARD PEIRO, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incomplete solutions submitted.

Zelator noted that some of these solutions are redundant when \( a = 1 \) or \( a = -1 \).

M399. Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Determine all triples \( (a, b, c) \) of positive integers for which \( \frac{3ab - 1}{abc + 1} \) is a positive integer.
Solution by Oscar Xia, student, St. George’s School, Vancouver, BC.

Suppose that $\frac{3ab - 1}{abc + 1} = n$, where $n$ is a positive integer. Then we have

$3ab - 1 = nabc + n$, or $3ab - nabc = n + 1$, or $ab = \frac{n + 1}{3 - nc}$.

Since $a$, $b$, $c$, and $n$ are positive integers, then $3 - nc$ is a positive integer
(since $ab$ is positive) so $(n, c) = (1, 1), (2, 1), (1, 2)$.

If $(n, c) = (1, 1)$, then $ab = 1$ so $(a, b) = (1, 1)$.

If $(n, c) = (2, 1)$, then $ab = 3$ so $(a, b) = (3, 1)$ or $(1, 3)$.

If $(n, c) = (1, 2)$, then $ab = 2$ so $(a, b) = (2, 1)$ or $(1, 2)$.

Therefore, the five triples are $(a, b, c) = (1, 1, 1), (3, 1, 1), (1, 3, 1), (2, 1, 2), (1, 2, 2)$. (We can check that each triple satisfies the requirements.)

Also solved by EDIN AJOVIC, student, First BosnÄk High School, Sarajevo, Bosnia and Herzegovina; GEOFFREY A. RANDALL, Hamden, CT, USA; RICARD PEIRO, IES ‘Abastos’, Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incorrect solutions and one incomplete solution submitted.

M400. Proposed by Mihály Benze, Brasov, Romania.

Suppose that $a$, $b$, and $c$ are positive real numbers. In addition, suppose that $a^n + b^n = c^n$ for some positive integer $n$ with $n \geq 2$. Prove that if $k$ is a positive integer with $1 \leq k < n$, then $a^k$, $b^k$, and $c^k$ are the side lengths of a triangle.

Solution by Bruno Salgueiro Fanegó, Viveiro, Spain, modified by the editor.

Suppose that $a^n + b^n = c^n$ for some positive integer $n \geq 2$. Since the numbers $a$, $b$, and $c$ are positive, then $a < c$ and $b < c$.

Suppose that $k$ is a positive integer with $1 \leq k < n$. To show that $a^k$, $b^k$, and $c^k$ are the side lengths of a triangle, we need to prove three inequalities, namely we need to prove that $a^k + b^k > c^k$, that $a^k + c^k > b^k$, and that $b^k + c^k > a^k$.

Since $a < c$, then $b^k + c^k > a^k$. Since $b < c$, then $a^k + c^k > b^k$. It remains to prove that $a^k + b^k > c^k$.

Since $0 < a < c$ and $0 < b < c$ and $k - n < 0$, then $a^{k-n} > c^{k-n} > 0$ and $b^{k-n} > c^{k-n} > 0$. Therefore,

\[
\begin{align*}
  c^k &= c^{k-n}c^n \\
       &= c^{k-n}(a^n + b^n) \\
       &= c^{k-n}a^n + c^{k-n}b^n \\
       &< a^{k-n}a^n + b^{k-n}b^n \\
       &= a^k + b^k,
\end{align*}
\]

as required.

Therefore, $a^k$, $b^k$, and $c^k$ are the side lengths of a triangle.

There were two incorrect solutions submitted.