SOLUTIONS

Aucun problème n’est immuable. L’éditeur est toujours heureux d’envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

We acknowledge a correct solution to problem 3354 by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Due to switched \( \text{\LaTeX} \) commands, we instead incorrectly credited Gottfried Perz, Pestalozzigymnasium, Graz, Austria. We further acknowledge a correct solution to problem 3412 by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. Our apologies for these errors.

\[ \text{3415. [2009 : 108, 111]} \quad \text{Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.} \]

Let \( a, b, \) and \( c \) be positive real numbers such that \( abc = 1. \) Prove that

\[ \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \leq \sqrt[3]{3(a + b + c + ab + bc + ca)}. \]

**Comment**: Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina informs us that this problem has appeared as problem 098 in the journal *Mathematical Reflections*, Vol. 5 (2008), pp. 38-39, by the same proposer. Three solutions are given there using a variety of techniques.

\[ \text{Solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Kiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneci, Romania; and the proposer.} \]

\[ \text{3416. [2009 : 108, 111]} \quad \text{Proposed by Michel Bataille, Rouen, France.} \]

Let the sequence \( (a_n) \) be defined by \( a_0 = 6 \) and the recursion

\[ a_{n+1} = \frac{1}{13} \left( 8a_n \sqrt[3]{3a_n^2 + 13} - 6a_n^2 - 13 \right) \]

for \( n \geq 0. \) Prove that each \( a_n \) is a positive integer, and that \( a_n^2 - a_{n+1} \) is divisible by 13 for each \( n \geq 0. \)

**Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.**

Set

\[ f(x) = \frac{2}{13} x \left( 4 \sqrt{3x^2 + 13} - 3x \right) - 1; \]
by hypothesis, $a_{n+1} = f(a_n)$. Let

$$\omega = 2 + \sqrt{3}, \quad \overline{\omega} = 2 - \sqrt{3};$$
$$\alpha = \frac{1}{2}(4 + \sqrt{3}), \quad \overline{\alpha} = \frac{1}{2}(4 - \sqrt{3}).$$

As an aid to calculating, we observe that $\omega \overline{\omega} = 1$, $\alpha \overline{\alpha} = \frac{13}{4}$, and $\alpha - \overline{\alpha} = \sqrt{3}$.

**Part 1.** Define increasing sequences of positive integers $(u_n)$ and $(v_n)$ by

$$u_0 = 4, \quad u_1 = 11, \quad u_{n+2} = 4u_{n+1} - u_n;$$
$$v_0 = 1, \quad v_1 = 6, \quad v_{n+2} = 4v_{n+1} - v_n.$$

Solving for $u_n$ and $v_n$, we obtain

$$u_n = \alpha \omega^n + \overline{\alpha} \overline{\omega}^n,$$
$$v_n = \frac{1}{\sqrt{3}}(\alpha \omega^n - \overline{\alpha} \overline{\omega}^n). \tag{1}$$

From (1) we easily deduce that $3v_n^2 + 13 = u_n^2$, so that $u_n = \sqrt{3v_n^2 + 13}$; hence, $f(v_n) = \frac{2}{13}v_n(4u_n - 3v_n) - 1$. Using (1) again we have

$$4u_n - 3v_n = \alpha \left(4 - \sqrt{3}\right)\omega^n + \overline{\alpha} \left(4 + \sqrt{3}\right)\overline{\omega}^n$$
$$= 2\alpha \overline{\alpha}(\omega^n + \overline{\omega}^n) = \frac{13}{2}(\omega^n + \overline{\omega}^n).$$

Hence,

$$f(v_n) = \frac{2}{13} \cdot \frac{1}{\sqrt{3}}(\alpha \omega^n - \overline{\alpha} \overline{\omega}^n) \cdot \frac{13}{2}(\omega^n + \overline{\omega}^n) - 1$$
$$= \frac{1}{\sqrt{3}}(\omega^n + \overline{\omega}^n)(\alpha \omega^n - \overline{\alpha} \overline{\omega}^n) - 1$$
$$= \frac{1}{\sqrt{3}}(\omega^{2n} - \overline{\alpha} \overline{\omega}^{2n}). \tag{2}$$

For $n \geq 0$ define the positive integer $b_n = v_{2n}$. Using (2) and (1) we have $f(b_n) = f(v_{2n}) = \frac{1}{\sqrt{3}}(\alpha \omega^{2n+1} - \overline{\alpha} \overline{\omega}^{2n+1}) = v_{2n+1} = b_{n+1}$. Finally, from

$$a_0 = b_0 = 6, \quad a_{n+1} = f(a_n), \quad b_{n+1} = f(b_n),$$
induction yields that $a_n = b_n$ for $n \geq 0$, and the result follows.

**Part 2.** For $n \geq 0$ set $\theta_n = \frac{1}{12}(\omega^{2n+1} + \overline{\omega}^{2n+1} - 2)$. Since $\theta_n$ is the solution of the recursion $\theta_{n+1} = 12\theta_n^2 + 4\theta_n$ with initial condition $\theta_0 = 1$, it is a positive integer. As $a_n = v_{2n}$, then from (1) we have

$$a_n = \frac{1}{\sqrt{3}}(\alpha \omega^{2n} - \overline{\alpha} \overline{\omega}^{2n}).$$
It follows that \( a_n^2 - a_{n+1} = \frac{13}{12}(\omega^{2n+1} + \overline{\omega}^{2n+1} - 2) = 13\theta_n \), whence 13 divides \( a_n^2 - a_{n+1} \), as desired.

Also solved by ARKADY ALT, San Jose, CA, USA; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MADHAV R. MODAK, formerly of Sir Parasrambhu College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Bernlingen, Switzerland; and the proposer. There was one incomplete solution submitted.

Many solvers began with the general Pell equation \( B^2 - 3A^2 = 13 \). With \( u_n \) and \( v_n \) as defined in the featured solution above, we see that the pair \( B = u_n \), \( A = v_n \), provides an infinite family of solutions to this Pell equation. Deiermann began instead with the equation \( B^2 - dA^2 = N \), where \( d > 0 \) is a nonsquare integer and \( N > 0 \) is an integer for which the pair \( B = B_0 \), \( A = A_0 \) is the minimal positive solution for an equivalence class of solutions, while the pair \( B_1 \) and \( A_1 \) is the next largest solution. He proved that if \( (a_n) \) is defined by \( a_0 = A_1 \) and

\[
a_{n+1} = \frac{1}{N} \left( 2\sqrt{dA_0^2 + N} \cdot a_n \sqrt{dA_0^2 + N} - 2A_0a_n^2 + A_0N \right)
\]

for \( n \geq 0 \), then each \( a_n \) is a positive integer and \( a_n^2 - A_0a_{n+1}^2 \) is divisible by \( N \) for each \( n \geq 0 \). (In our problem \( N = 13 \), \( d = 3 \), \( A_0 = 1 \), \( B_0 = 4 \), \( A_1 = 6 \), and \( B_1 = 11 \).) Hess proved a similar generalization.


Let \( S_p(n) = 1^p + 2^p + \cdots + n^p \). Prove that

\[
\sum_{n=1}^{\infty} \frac{(S_{-1}(n))^2}{S_1(n)} = \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_3(n)} \right) + 2 \left( \sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_1(n)} \right).
\]

Solution by Joel Schlosberg, Bayside, NY, USA.

Let \( H_n = 1^{-1} + 2^{-1} + \cdots + n^{-1} \) be the \( n \)th harmonic number. Then \( H_n = H_{n-1} + \frac{1}{n} \). As usual let \( \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \) be the Riemann zeta function.

A known identity due to Euler (see Jonathan Sondow and Eric W. Weisstein "Harmonic Number," http://mathworld.wolfram.com/HarmonicNumber.html) states that

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).
\]

It is well known that \( H_n \) is asymptotic to \( \ln n \) and that \( \ln n = o(n^k) \) for any positive \( k \). Therefore,

\[
\lim_{n \to \infty} \frac{H_n}{n+1} = 0
\]

and

\[
\lim_{n \to \infty} \frac{H_n^2}{n+1} = 0.
\]
By (2),
\[
\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{H_n}{n(n+1)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]
\[
= \lim_{N \to \infty} \left( \frac{H_1 \cdot 1}{1} + \sum_{n=2}^{N} \frac{1}{n} (H_n - H_{n-1}) - \frac{H_N}{N+1} \right)
\]
\[
= \lim_{N \to \infty} \left( \frac{1}{1^2} + \sum_{n=2}^{N} \frac{1}{n^2} \right) - \lim_{N \to \infty} \frac{H_N}{N+1}
\]
\[
= \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2). \tag{4}
\]

By (1),
\[
\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^2} = \sum_{n=2}^{\infty} \frac{H_n - \frac{1}{n}}{n^2} = \sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n^2}
\]
\[
= \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3} = 2\zeta(3) - \zeta(3) = \zeta(3). \tag{5}
\]

By (1), (4), and (5),
\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)^2} = \sum_{n=1}^{\infty} H_n \left( \frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n(n+1)} \right)
\]
\[
= \sum_{n=1}^{\infty} \frac{H_n}{n^2} + \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} - 2 \sum_{n=1}^{\infty} \frac{H_n}{n(n+1)}
\]
\[
= 2\zeta(3) + \zeta(3) - 2\zeta(2) = 3\zeta(3) - 2\zeta(2). \tag{6}
\]

By (1) and (3),
\[
\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)}
\]
\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{H_n^2}{n(n+1)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} H_n^2 \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]
\[
= \lim_{N \to \infty} \left( H_1^2 \cdot \frac{1}{1} + \sum_{n=2}^{N} \frac{1}{n} (H_n^2 - H_{n-1}^2) - \frac{H_N^2}{N+1} \right)
\]
\[
= \lim_{N \to \infty} \left( 1 + \sum_{n=2}^{N} \frac{1}{n} (H_n + H_{n-1})(H_n - H_{n-1}) \right) - \lim_{N \to \infty} \frac{H_N^2}{N+1}
\]
\[
= \lim_{N \to \infty} \left( 1 + \sum_{n=2}^{N} \frac{1}{n} (2H_n - \frac{1}{n}) (\frac{1}{n}) \right)
\]
\[
\lim_{N \to \infty} \left( 2 + 2 \sum_{n=2}^{N} \frac{H_n}{n^2} \right) - \lim_{N \to \infty} \left( 1 + \sum_{n=2}^{N} \frac{1}{n^3} \right) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3} = 2 \cdot 2 \zeta(3) - \zeta(3) = 3 \zeta(3). \tag{7}
\]

Note that \( S_{-1}(n) = H_n \). By the well-known formulas for the sums of the first \( n \) positive integers and the first \( n \) cubes,

\[
S_1(n) = \frac{n(n+1)}{2} \quad \text{and} \quad S_3(n) = \frac{n^2(n+1)^2}{4}. \tag{8}
\]

By (4), (6), and (8),

\[
\frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_3(n)} \right) + 2 \left( \sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_1(n)} \right) = 2 \sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)^2} + 4 \sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} = 2(3 \zeta(3) - 2 \zeta(2)) + 4 \zeta(2) = 6 \zeta(3). \tag{9}
\]

By (7),

\[
\sum_{n=1}^{\infty} \frac{(S_{-1}(n))^2}{S_1(n)} = 2 \sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)} = 6 \zeta(3),
\]

\[
\frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_3(n)} \right) + 2 \left( \sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_1(n)} \right) = 6 \zeta(3),
\]

hence the two sides are equal with a common value of \( 6 \zeta(3) \).

Also solved by ARKADY ALT, San Jose, CA, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinenhochschule, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.


Let \( I(\phi) \) be the set of all antiderivatives of a continuous function \( \phi \).

(a) Determine the continuous function \( f: I_p \subset \mathbb{R} \to \mathbb{R}\setminus\{0\} \) such that \( f(0) = 1 \) and \( f^{-p} \in I(f) \), where \( p \) is an odd natural number and the interval \( I_p \) contains zero and is maximal for the given properties of \( f \).

(b) Prove that \( p = q \) if and only if \( I_p = I_q \).
Solution by Michel Bataille, Rouen, France.

(a) Let $I$ be an interval containing 0 and let $f : I \rightarrow \mathbb{R} \setminus \{0\}$ be such that $f(0) = 1$ and $f^{-p} \in \mathcal{I}(f)$. Then $f^{-p}$ is differentiable on the interval $I$ and $(f^{-p})'(x) = f(x)$ for all $x \in I$.

Since $p$ is an odd natural number, the function $h(x) = x^p$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$. Moreover, its inverse is differentiable except at 0. The composition $h \circ f^{-1}(x) = f^{-p}(x)$ is differentiable, thus the function $f^{-1}(x) = \frac{1}{f(x)}$ is differentiable, and so is the function $f$. Note also that $f(x) > 0$ for all $x \in I$ (since $f$ is continuous, does not vanish, and $f(0) > 0$).

Now let $p = 2m - 1$, where $m$ is a positive integer. Since the function $f$ is differentiable,

$$f(x) = (f^{-p})'(x) = -pf^{-p-1}(x)f'(x) = -(2m-1)\frac{f'(x)}{f(x)^{2m}},$$

so that $f$ satisfies

$$(2m-1)f'(x) + \left(f(x)\right)^{2m+1} = 0 \quad (x \in I).$$

An easy calculation shows that the function $g(x) = f(x)^{-m}$ satisfies

$$g'(x)g(x) = \frac{m}{2m-1}.$$ Integrating and substituting $g(0) = 1$, it follows that $\left(g(x)\right)^2 = \frac{2mx}{2m-1} + 1.$

Thus, $\frac{2mx}{2m-1} + 1 > 0$ and $g(x) = \left(\frac{2mx}{2m-1} + 1\right)^{1/2}$ for $x \in I$. We conclude that

$$I \subset I_p = \left(-\frac{p}{p+1}, \infty\right) \quad \text{and} \quad f(x) = \left(\frac{(p+1)x}{p} + 1\right)^{-\frac{p}{p+1}}. \quad \quad (1)$$

Conversely, it is easy to check that the function $f$ defined by (1) on $I_p$ satisfies

$$(f^{-p})'(x) = \left[\left(\frac{(p+1)x}{p} + 1\right)^{-\frac{p}{p+1}}\right]' = f(x)$$

for all $x \in I_p$.

(b) $p = q$ is equivalent to $\frac{p}{p+1} = \frac{q}{q+1}$, hence it is equivalent to $I_p = I_q$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvaro Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposers.

The differentiability of the function $f$ was not addressed by any of the other solvers.

Let \( a, b, \) and \( c \) be positive real numbers.

(a) Prove that \( \sum_{cyclic} \sqrt{ \frac{a^2 + 4bc}{b^2 + c^2} } \geq 2 + \sqrt{2}. \)

(b) Prove that \( \sum_{cyclic} \sqrt{ \frac{a^2 + bc}{b^2 + c^2} } \geq 2 + \frac{1}{\sqrt{2}}. \)

Solution to (a) by Oliver Geupel, Brühl, NRW, Germany; and the proposer.

We will show that the inequality holds even if one of \( a, b, c \) is zero. Without loss of generality, assume that \( a \geq b \geq c \geq 0. \)

Case 1. If \( 4b^3 \geq a^2c, \) then we have

\[
\frac{a^2 + 4bc}{b^2 + c^2} - \frac{a^2}{b^2} = \frac{c(4b^3 - a^2c)}{b^2(b^2 + c^2)} \geq 0,
\]

\[
\frac{b^2 + 4ca}{c^2 + a^2} - \frac{b^2}{a^2} = \frac{c(4a^3 - b^2c)}{a^2(c^2 + a^2)} \geq 0,
\]

and

\[
\frac{c^2 + 4ab}{a^2 + b^2} \geq \frac{4ab}{a^2 + b^2}:
\]

The result now follows after several applications of the AM–GM Inequality:

\[
\sum_{cyclic} \sqrt{ \frac{a^2 + 4bc}{b^2 + c^2} } \geq \frac{a}{b} + \frac{b}{a} + 2\sqrt{\frac{ab}{a^2 + b^2}}
\]

\[
= \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{a^2 + b^2}{ab} + \left( \frac{a^2 + b^2}{2\sqrt{2}ab} + 2\sqrt{\frac{ab}{a^2 + b^2}} \right)
\]

\[
\geq 2 \left( 1 - \frac{1}{\sqrt{2}} \right) + 2\sqrt{\frac{2(a^2 + b^2)}{ab}}
\]

\[
\geq 2 \left( 1 - \frac{1}{\sqrt{2}} \right) + 2\sqrt{2} = 2 + \sqrt{2}.
\]

Case 2. If \( 4b^3 \leq a^2c, \) then \( a \geq 2b \) and we have

\[
\frac{a^2 + 4bc}{b^2 + c^2} - \frac{a^2 + 4b^2}{2b^2} = \frac{(b - c)[a^2(b + c) - 4b^2(b - c)]}{2b^2(b^2 + c^2)} \geq 0.
\]

Hence,

\[
\sum_{cyclic} \sqrt{ \frac{a^2 + 4bc}{b^2 + c^2} } \geq \sqrt{\frac{a^2 + 4b^2}{2b^2}} + \frac{b}{a} + 2\sqrt{\frac{ab}{a^2 + b^2}}.
\]
Setting $x = \frac{a}{b} \geq 2$, we need to prove that

$$f(x) = \sqrt{\frac{x^2}{2} + 2 + \frac{1}{x}} + 2\sqrt{\frac{x}{x^2 + 1}} \geq 2 + \sqrt{2}.$$ 

Since $x \geq 2$, we have

$$x(x^2 + 1) - (x + 1)^2 = x^3 \left(1 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}\right)$$

$$\geq x^3 \left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8}\right) = \frac{1}{8}x^3 > 0;$$

hence $\sqrt{x(x^2 + 1)} > x + 1$, and then

$$f'(x) = \frac{x}{2\sqrt{\frac{x^2}{2} + 2}} - \frac{1}{x^2} - \frac{x^2 - 1}{(x^2 + 1)\sqrt{x(x^2 + 1)}}$$

$$> \frac{1}{2\sqrt{\frac{1}{2} + \frac{2}{x^2}}} - \frac{1}{x^2} - \frac{x^2 - 1}{(x^2 + 1)(x + 1)}$$

$$\geq \frac{1}{2} - \frac{1}{x^2} - \frac{x - 1}{x^2 + 1} = \frac{x^2 - 4}{4x^2} + \frac{(x - 2)^2 + 1}{4(x^2 + 1)} > 0.$$ 

Thus, $f(x)$ is an increasing function on $[2, \infty)$, and therefore,

$$f(x) \geq f(2) = \frac{5}{2} + 2\sqrt{\frac{2}{5}} > 2 + \sqrt{2}.$$ 

There was one incorrect solution submitted.

Geupel credits the proposer and his publication [1] at the MathLinks website for his solution to (a). The proposer submitted a solution to part (b) as well; however, his argument relies on a graphical computational package result, which has not been verified otherwise. Geupel remarked that the author has published a solution to part (b) in [3]. Geupel also mentioned a similar inequality,

$$\sum_{cyclic} \frac{a^2 + bc}{b^2 + c^2} \geq \sqrt{\frac{2(a^2 + b^2 + c^2)}{ab + bc + ca} + \frac{1}{\sqrt{2}}},$$ 

proved by the proposer in [2].

References


Prove that

\[
\prod_{k=1}^{n} \left( \frac{(k + 1)^2}{k(k + 2)} \right)^{k+1} < n + 1 < \prod_{k=1}^{n} \left( \frac{k^2 + k + 1}{k(k + 1)} \right)^{k+1}.
\]

Similar solutions by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam and Oliver Geipel, Brühl, NRW, Germany.

We recognize that the product on the left is a telescoping product, thus

\[
\prod_{k=1}^{n} \left( \frac{(k + 1)^2}{k(k + 2)} \right)^{k+1} = \frac{2(n + 1)^{n+1}}{(n + 2)^{n+1}} = \frac{2(n + 1)}{(1 + \frac{1}{n+1})^{n+1}}.
\]

By the Bernoulli Inequality, we have

\[
\left( 1 + \frac{1}{n + 1} \right)^{n+1} > 1 + (n + 1) \frac{1}{n + 1} = 2,
\]

which proves the left inequality.

Again by the Bernoulli Inequality, we have

\[
\left( \frac{k^2 + k + 1}{k(k + 1)} \right)^{k+1} = \left( 1 + \frac{1}{k(k + 1)} \right)^{k+1} > \left( 1 + \frac{1}{k} \right) = \frac{k + 1}{k}.
\]

Hence,

\[
\prod_{k=1}^{n} \left( \frac{k^2 + k + 1}{k(k + 1)} \right)^{k+1} > \prod_{k=1}^{n} \frac{k + 1}{k} = n + 1.
\]

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMMINIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursuline Gymnasium, Innsbruck, Austria; SALEM MALIKI, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhai College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the propose.
Solution by Arkady Alt, San Jose, CA, USA.

After the substitution \((a, b, c) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)\), the inequality becomes equivalent to

\[
\frac{y^2}{x(y + 1)} + \frac{z^2}{y(z + 1)} + \frac{x^2}{z(x + 1)} \geq \frac{3}{2},
\]

where \(x, y, \) and \(z\) are positive real numbers with \(xyz \geq 1\).

Since by the Cauchy-Schwartz Inequality we have

\[
\left((xy + x) + (yz + y) + (zx + z)\right)\left(\frac{y^2}{x(y + 1)} + \frac{z^2}{y(z + 1)} + \frac{x^2}{z(x + 1)}\right) \geq (x + y + z)^2,
\]

it suffices to prove that

\[
\frac{(x + y + z)^2}{xy + yz + zx + x + y + z} \geq \frac{3}{2},
\]

or

\[
2(x + y + z)^2 \geq 3(xy + yz + zx) + 3(x + y + z).
\]

However, the last inequality follows immediately from

\[
(x + y + z)^2 \geq 3(xy + yz + zx),
\]

and

\[
(x + y + z)^2 = (x + y + z)(x + y + z) \geq 3 \sqrt[3]{xyz}(x + y + z) \geq 3(x + y + z).
\]

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; SALEEM MALIK, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect solution and two incomplete solutions submitted.


Let \(a, b, \) and \(c\) be positive real numbers such that \(a + b + c \leq 1\). Prove that

\[
\frac{a}{a^3 + a^2 + 1} + \frac{b}{b^3 + b^2 + 1} + \frac{c}{c^3 + c^2 + 1} \leq \frac{27}{31}.
\]
Solution by Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.

Set \( a = \frac{x}{3}, \ b = \frac{y}{3}, \) and \( c = \frac{z}{3}. \) Then \( x, y, \) and \( z \) are positive real numbers satisfying \( x + y + z \leq 3, \) and the inequality is equivalent to

\[
\sum_{\text{cyclic}} \frac{x}{3x^3 + 3x^2 + 27} \leq \frac{3}{31}.
\]

By the AM-GM Inequality, we have

\[
x^3 + 1 + 1 \geq 3x, \quad \text{and} \quad 3(x^2 + 1) \geq 6x.
\]

Therefore, it suffices for us to prove that

\[
\sum_{\text{cyclic}} \frac{x}{9x + 22} \leq \frac{3}{31} \iff 3 - \sum_{\text{cyclic}} \frac{22}{9x + 22} \leq \frac{27}{31}.
\]

By the Cauchy-Schwarz Inequality,

\[
\left( \sum_{\text{cyclic}} \frac{1}{9x + 22} \right) \left( \sum_{\text{cyclic}} 9x + 22 \right) \geq 9,
\]

so it follows that

\[
\sum_{\text{cyclic}} \frac{22}{9x + 22} \geq \frac{198}{9(x + y + z) + 66}
\]

\[
\geq \frac{198}{27 + 66}
\]

\[
= \frac{31}{31}.
\]

which settles the inequality on the right of the double implication above.

Equality holds if and only if \( x = y = z = 1, \) or \( a = b = c = \frac{1}{3}. \)

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEH MALKIK, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Janous defined the numbers \( x_1 = 1.100203739 \ldots \) and \( x_M = 0.657298106 \ldots \) the respective (unique) real roots of \( 3x^4 + 3x^3 + x^2 - 6x - 3 \) and \( 2x^3 + x^2 - 1, \) and he showed that if \( a, b, \) and \( c \) are positive reals with \( a + b + c = K \) and \( 0 < K/3 \leq x_1, \) then

\[
\sum_{\text{cyclic}} \frac{a}{\alpha^3 + \alpha^2 + 1} \leq \begin{cases} 
\frac{27K}{K^3 + 3K^2 + 27}, & \text{if } 0 < K/3 \leq x_M, \\
\frac{3x_M}{x_M^3 + x_M^2 + 1}, & \text{if } x_M < K/3 \leq x_1.
\end{cases}
\]
Let $n \geq 2$ be an integer and $x_1, x_2, \ldots, x_n$ positive real numbers such that $x_1 + x_2 + \cdots + x_n = 2n$. Prove that
\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} \frac{x_j}{\sqrt{x_i^3 + 1}} \right) \geq \frac{2n(n-1)}{3}.
\]

Solution by Arkady Alt, San Jose, CA, USA.

Let $x_i = 2t_i$, $1 \leq i \leq n$. Then the given inequality is equivalent to
\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} \frac{t_j}{\sqrt{1 + 8t_i^3}} \right) \geq \frac{n(n-1)}{3},
\]
where each $t_i$ is positive and $t_1 + t_2 + \cdots + t_n = n$.

Since $\sqrt{1 + 8t_i^3} \leq 1 + 2t_i^2$ is equivalent to $1 + 8t_i^3 \leq 1 + 4t_i^2 + 4t_i^4$, or $2t_i \leq 1 + t_i^2$, which is clearly true, we then have
\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} \frac{t_j}{\sqrt{1 + 8t_i^3}} \right) \geq \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \frac{t_j}{1 + 2t_i^2} \right).
\]

Hence, to establish (1), it suffices to show that
\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} \frac{t_j}{1 + 2t_i^2} \right) \geq \frac{n(n-1)}{3}.
\]

Now,
\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} \frac{t_j}{1 + 2t_i^2} \right) = \sum_{j=1}^{n} \left( - \frac{t_j}{1 + 2t_j^2} + \sum_{i \neq j}^{n} \frac{t_j}{1 + 2t_i^2} \right)
= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{t_j}{1 + 2t_i^2} \right) - \sum_{j=1}^{n} \frac{t_j}{1 + 2t_j^2}
= \sum_{i=1}^{n} \frac{n}{1 + 2t_i^2} - \sum_{j=1}^{n} \frac{t_j}{1 + 2t_j^2}
= \sum_{i=1}^{n} \frac{n - t_i}{1 + 2t_i^2}.
\]
Thus, (2) becomes
\[
\sum_{i=1}^{n} \frac{n-t_i}{1+2t_i^2} \geq \frac{n(n-1)}{3}. \tag{3}
\]

To prove (3) we show that the inequality below holds for all positive real numbers \(x\) and all positive integers \(n\), with equality if and only if \(x = 1\):
\[
\frac{n-x}{1+2x^2} \geq \left(\frac{7n-4}{9}\right) - \left(\frac{4n-1}{9}\right)x. \tag{4}
\]

Note that (4) is equivalent, in succession, to
\[
9(n-x) \geq [(7n-4)-(4n-1)x] (1+2x^2),
0 \leq (8n-2)x^3 - (14n-8)x^2 + (4n-10)x + (2n+4),
0 \leq [(4n-1)x + (n+2)](x-1)^2,
\]
and clearly the last inequality is true.

Using (4) and the fact that \(t_1 + t_2 + \cdots + t_n = n\), we then have
\[
\sum_{i=1}^{n} \frac{n-t_i}{1+2t_i^2} \geq \sum_{i=1}^{n} \left(\frac{7n-4}{9} - \left(\frac{4n-1}{9}\right)t_i\right) = \frac{(7n-4)n}{9} - \frac{(4n-1)n}{9} = \frac{n(3n-3)}{9} = \frac{n(n-1)}{3},
\]
establishing (3) and completing our proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.


For a positive integer \(m\), let \(\sigma\) be the permutation of \(\{0, 1, 2, \ldots, 2m\}\) defined by \(\sigma(2i) = i\) for each \(i = 0, 1, 2, \ldots, m\) and \(\sigma(2i-1) = m+i\) for each \(i = 1, 2, \ldots, m\). Prove that there exists a positive integer \(k\) such that \(\sigma^k = \sigma\) and \(1 \leq k \leq 2m+1\).

Solution by Oliver Geupel, Brühl, NRW, Germany.

If \(0 \leq i \leq m\), then \(\sigma^{-1}(i) = 2i\); and if \(m+1 \leq i \leq 2m\), then \(\sigma^{-1}(i) = 2(i-m) - 1 \equiv 2i \pmod{2m+1}\). Therefore, we have
\[
\sigma^{-1}(i) \equiv 2i \pmod{2m+1} \tag{1}
\]
for each \(i \in \{0, 1, \ldots, 2m\}\.\)
By Euler’s theorem, we have
\[
2^{\phi(2m+1)} \equiv 1 \pmod{2m+1},
\]
where \( \phi \) denotes Euler’s totient function.

Using (1) and (2), we then have
\[
\sigma^{-\phi(2m+1)}(i) \equiv 2^{\phi(2m+1)}i \equiv i \pmod{2m+1},
\]
and it follows that \( \sigma^{\phi(2m+1)}(i) = i \) for each \( i \).

Hence, \( \sigma^{\phi(2m+1)+1} = \sigma \), and since \( 2 \leq \phi(2m+1) + 1 \leq 2m+1 \), an appropriate choice for \( k \) would be \( k = \phi(2m+1) + 1 \).

Also solved by MICHEL BATAILLE, Rouen, France; EDMUND SWYLAN, Riga, Latvia; and the proposer.


For real \( x \neq -1 \), let \( f(x) = \frac{e^x}{x+1} \). Prove that if \( f(x) = f(y) \) for some \( x \neq y \), then
\[
\left( \sqrt{x+1} - \sqrt{y+1} \right)^2 \geq \ln f(y).
\]

Solution by Albert Stadler, Herrliberg, Switzerland.

The function \( f(x) \) is negative and decreasing on \((-\infty, -1)\), positive and decreasing on \((-1, 0]\), and positive and increasing on \([0, \infty)\). So, if \( f(x) = f(y) \) for some \( x \neq y \), then we can assume that \(-1 < x \leq 0 < y\). Upon setting \( u = x + 1 \) and \( v = y + 1 \), the problem statement becomes:

Let \( g(u) = \frac{e^u}{u}, u > 0 \). Prove that if \( g(u) = g(v) \) for some \( u \neq v \), then
\[
\left( \sqrt{u} - \sqrt{v} \right)^2 \geq v - 1 - \ln v.
\]

Thus, \( g(1) = e \) is a local minimum value, and \( 0 < u \leq 1 < v \) with \( u \neq v \).

Case 1. Here we assume that \( v \geq 2 \). Then \( 0 < u < 0.5 \), since \( g(2) > 3.6 \) and \( g(0.5) < 3.3 \). From \( g(u) = g(v) \) we obtain \( \frac{1}{u} \leq \frac{e^u}{u} \leq \frac{e^v}{v} \leq \frac{e^{0.5}}{v} \), and hence \( ve^{-v} \leq u \leq ve^{0.5-v} \). We claim that
\[
ve^{-v} + 1 + \ln v \geq 2ve^{\frac{1}{2}-\frac{v}{2}}.
\]

holds for \( v \geq 2 \). Indeed, the inequality (2) holds for \( v = 2 \) and also
\[
\frac{d}{dv} \left( ve^{-v} + 1 + \ln v - 2ve^{\frac{1}{2}-\frac{v}{2}} \right)
\]
\[
= -(v-1)e^{-v} + \frac{1}{v} - (v-2)e^{\frac{1}{2}-\frac{v}{2}}
\]
\[
= \frac{e^{-v}}{v} \left( v(v-2)e^{\frac{1}{2}+\frac{v}{2}} + e^{v} - v(v-1) \right) \geq 0,
\]
since \( e^v \geq v(v - 1) \) for \( v > 0 \); inequality (2) follows from these two facts.

We conclude that \( u + 1 + \ln v \geq ve^{v} + 1 + \ln v \geq 2ve^{\frac{v}{2} - \frac{1}{2}} \geq 2\sqrt{uv} \), which is equivalent to (1). This concludes Case 1.

**Case 2.** Here we assume that \( 1 \leq v \leq 2 \). We will need two lemmas in order to settle this case.

**Lemma 1** If \( 0 \leq t \leq 0.8 \), then

\[
1 - \sqrt{1 - t} \geq \sqrt{\frac{t^2 + 0.5t - \ln(t^2 + 0.5t + 1)}{t^2 + 0.5t + 1}}. \tag{3}
\]

**Proof:** Both sides of (3) are positive, so we obtain an equivalent inequality by squaring and rearranging terms:

\[
1 - t - 2\sqrt{1 - t} \geq \sqrt{\frac{-1 - \ln(t^2 + 0.5t + 1)}{t^2 + 0.5t + 1}},
\]

so we want to show that

\[
\Psi(t) = 2\ln(t^2 + 0.5t + 1) - 2t^3 + t^2 - t + 4 - (4t^2 + 2t + 4)\sqrt{1 - t} \geq 0.
\]

We have \( \Psi(0) = 0 \) and

\[
\frac{\Psi'(t)}{t} = \frac{(-12t^3 + 2t^2 + 12t - 11)\sqrt{1 - t} + 5(2t - 1)(2t^2 + t + 2)}{(2t^2 + t + 2)\sqrt{1 - t}},
\]

so we are left to prove that the numerator is nonnegative for \( 0 < t \leq 0.8 \). For \( 0 \leq t \leq 0.6 \) we have \( 11 - 12t^3 - 2t^2 - 12t \) and \( \sqrt{1 - t} \geq 1 - \frac{2}{3}t \), so that

\[
(-12t^3 + 2t^2 + 12t - 11)\sqrt{1 - t} + 5(2t - 1)(2t^2 + t + 2) \\
\geq (-12t^3 + 2t^2 + 12t - 11)(1 - \frac{2}{3}t) + 5(2t - 1)(2t^2 + t + 2) \\
= \frac{1}{3}(24t^4 + 28t^3 + 18t^2 - 13t + 3) \geq \frac{1}{3}(28t^2 - 13t + 3) > 0,
\]

and the last inequality holds for \( t > 0 \).

For \( 0.6 \leq t \leq 0.8 \) we have

\[
\left|(-12t^3 + 2t^2 + 12t - 11)\sqrt{1 - t} + 5(2t - 1)(2t^2 + t + 2)\right| \\
\geq 5(2t - 1)(2t^2 + t + 2) - |12t^3 + 2t^2 + 12t - 11| \\
\geq \left\{ \begin{array}{ll}
3.32 - 2.49 > 0, & \text{if } 0.6 \leq t \leq 0.7,
7.63 - 6.1 > 0, & \text{if } 0.7 \leq t \leq 0.8.
\end{array} \right.
\]

This concludes the proof of Lemma 1. \( \blacksquare \)
Let \( t = \frac{v - u}{v} \). Then \( 0 < t < 1 \), and we deduce from \( e^v = \frac{e^u}{v} \) that \( v - u = \ln \left( \frac{v}{u} \right) \), or equivalently

\[
t = -\frac{1}{v} \ln(1 - t) = \frac{1}{v} \sum_{k=1}^{\infty} \frac{t^k}{k}.
\]

(4)

Dividing by \( t \) we obtain

\[
v - 1 = \sum_{k=2}^{\infty} \frac{t^{k-1}}{k}.
\]

(5)

The function \( \varphi(x) = \sum_{k=2}^{\infty} \frac{x^{k-1}}{k} \) satisfies \( \varphi(0) = 1 \), is strictly increasing on \((0, 1)\), and \( \varphi(x) \to \infty \) as \( x \to 1 \). Thus, for each \( v \geq 1 \), there is a unique \( t = t(v) \) that satisfies (4) and \( t(v) \) strictly decreases to 0 as \( v \) tends to 1. We now estimate \( t(v) \) from below.

**Lemma 2** If \( 1 \leq v \leq 2 \), then \( t = \frac{v - u}{v} \) satisfies \( t^2 + \frac{t}{2} + 1 \geq v \), and this latter inequality holds if and only if \( t \geq -\frac{1}{4} + \frac{1}{4} \sqrt{1 + 16(v - 1)} \).

**Proof:** Since \( g(0.4) > g(2) \), we have \( 0 < t(2) < 0.8 \). Then we obtain that \( t = \frac{v - u}{v} < \frac{2 - 0.4}{2} = 0.8 \). We deduce from (4) and (5) that

\[
v - 1 = \sum_{k=2}^{\infty} \frac{t^{k-1}}{k}
\]

\[
= \frac{t}{2} + t^2 \sum_{k=3}^{19} \frac{t^{k-3}}{k} + t^{18} \sum_{k=20}^{\infty} \frac{t^{k-19}}{k}
\]

\[
\leq \frac{t}{2} + t^2 \sum_{k=3}^{19} \frac{t^{k-3}}{k} - t^{18} \ln(1 - t)
\]

\[
= \frac{t}{2} + t^2 \sum_{k=3}^{19} \frac{t^{k-3}}{k} + t^{19} v \leq \frac{t}{2} + t^2,
\]

since \( \sum_{k=3}^{19} \frac{0.8^{k-3}}{k} + 2(0.8)^{17} < 1 \). This yields \( t^2 + \frac{t}{2} + 1 \geq v \). By straightforward calculations using the quadratic formula, we see that \( t^2 + \frac{t}{2} - (v - 1) \geq 0 \) is equivalent to \( t \geq -0.5 + \sqrt{0.25 + 4(v - 1)} \),

\[
\geq -\frac{1}{4} + \frac{1}{4} \sqrt{1 + 16(v - 1)}.
\]

Now \( \phi(x) = \frac{x - 1 - \log x}{x} \) is monotonically increasing for \( x > 1 \). Then

\[
\frac{t^2 + 0.5t - \ln(t^2 + 0.5t + 1)}{t^2 + 0.5t + 1} \geq \frac{v - 1 - \log v}{v},
\]
because \( t^2 + \frac{t}{2} + 1 \geq v \) by Lemma 2. Finally, by using Lemma 1, we obtain

\[
\sqrt{v} - \sqrt{u} = \sqrt{v} \left( 1 - \sqrt{1 - \frac{u}{v}} \right) \\
= \sqrt{v}(1 - \sqrt{1 - t}) \\
\geq \sqrt{v} \sqrt{\frac{t^2 + 0.5t - \ln(t^2 + 0.5t + 1)}{t^2 + 0.5t + 1}} \\
\geq \sqrt{v} \sqrt{\frac{v - 1 - \log v}{v}} \\
= \sqrt{v - 1 - \log v},
\]

and squaring yields inequality (1).

This concludes Case 2, and all is proved.

One incomplete solution was received which used the method of Lagrange multipliers but without beforehand establishing the existence of extrema.

The proposer commented that he discovered the result by “some other considerations” and that he sought an elementary solution to the problem.