THE OLYMPIAD CORNER
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We start this issue with the problems of the four Team Selection Tests for BMO 2007 and IMO 2007 of the Republic of Moldova. My thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

REPUBLIC OF MOLDOVA
Team Selection Tests for BMO 2007 and IMO 2007
First Test — 5 March 2007

1. In triangle $ABC$ the points $M$, $N$, and $P$ are the midpoints of the sides $BC$, $AC$, and $AB$, respectively. The lines $AM$, $BN$, and $CP$ intersect the circumcircle of $ABC$ at $A_1$, $B_1$, and $C_1$, respectively. Prove that the area of the triangle $A_1B_1C_1$ does not exceed the sum of the areas of the triangles $BA_1C$, $AB_1C$, and $AC_1B$.

2. Let $p$ be a prime number, $p \neq 2$, and $m_1, m_2, \ldots, m_p$ consecutive positive integers, and $\sigma$ a permutation of the set $A = \{1, 2, \ldots, p\}$. Prove that $A$ contains two distinct numbers $k, l$ such that $p \mid (m_km_{\sigma(k)} - m_lm_{\sigma(l)})$.

3. Inside the triangle $ABC$ there is a point $T$ such that

$$\angle ATB = \angle BTC = \angle CTA = 120^\circ.$$ 

Prove that the Euler lines of the triangles $ATB$, $BTC$, $ATC$ are concurrent.

4. Let $P = A_1A_2\ldots A_n$ be a convex polygon. For any point $M$ in the interior, let $B_i$ be the point where $A_iM$ intersects the perimeter. We say that $P$ is balanced if for some such $M$ the points $B_1, B_2, \ldots, B_n$ are interior to distinct sides of $P$. Determine all $n$ for which there exists a balanced polygon with $n$ sides.

Second Test — 23 March 2007

5. Determine the smallest positive integers $m$ and $k$ such that

(a) there exist $2m + 1$ consecutive positive integers whose cubes sum to a perfect cube;

(b) there exist $2k + 1$ consecutive positive integers whose squares sum to a perfect square.
6. Let $I$ be the incentre of triangle $ABC$ and let $R$ be its circumradius. Prove that $AI + BI + CI \leq 3R$.

7. Let $U$, $V$ be two points inside the angle $BAC$ such that $\angle BAU = \angle CAV$. Let $X_1$, $X_2$ be the projections of $U$ onto the angle sides $AC$, $AB$; and let $Y_1$, $Y_2$ be the projections of $V$ onto the angle sides $AC$, $AB$. Let the lines $X_2Y_1$ and $X_1Y_2$ intersect at $W$. Prove that $U$, $V$, $W$ are collinear.

8. The convex hull of five points in the plane has area $S$. Prove that three of these points form a triangle of area not greater than $\left(\frac{5 - \sqrt{5}}{10}\right)S$.

Third Test — 24 March 2007

9. Let $a_1$, $a_2$, \ldots, $a_n$ ($n \geq 2$) be real numbers in the interval $[0, 1]$. Let $S = a_1^3 + a_2^3 + \cdots + a_n^3$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{2n + 1 + S - a_i^3} \leq \frac{1}{3}.$$

10. Find all polynomials $f$ with integer coefficients, such that $f(p)$ is a prime for every prime $p$.

11. Let $ABC$ be a triangle with $a = BC$, $b = AC$, $c = AB$, inradius $r$, and circumradius $R$. Let $r_A$, $r_B$, and $r_C$ be the radii of the excircles of the triangle $ABC$. Prove that

$$a^2 \left(\frac{2}{r_A} - \frac{r}{r_B} + \frac{r}{r_C} - \frac{r}{r_A}\right) + b^2 \left(\frac{2}{r_B} - \frac{r}{r_C} - \frac{r}{r_B} + \frac{r}{r_A}\right) + c^2 \left(\frac{2}{r_C} - \frac{r}{r_A} - \frac{r}{r_B} + \frac{r}{r_C}\right) = 4(R + 3r).$$

12. Consider $n$ distinct points in the plane, $n \geq 3$, arranged such that the number $r(n)$ of segments of length $l$ is maximized. Prove that $r(n) \leq \frac{n^2}{3}$.

Fourth Test — 25 March 2007

13. Prove that the plane cannot be covered by the inner regions of finitely many parabolas.

14. Let $b_1$, $b_2$, \ldots, $b_n$ ($n \geq 1$) be nonnegative real numbers at least one of which is positive. Prove that $P(X) = X^n - b_1X^{n-1} - \cdots - b_{n-1}X - b_n$ has a single positive root $p$, which is simple, and that the absolute value of each root of $P(X)$ is not greater than $p$.

15. A circle is tangent to the sides $AB$ and $AC$ of the triangle $ABC$ and to its circumcircle at $P$, $Q$, and $R$ respectively. Prove that if $PQ \cap AR = \{S\}$, then $\angle SBA = \angle SCA$. 
16. Prove that there are infinitely many primes $p$ for which there exists a positive integer $n$ such that $p$ divides $n! + 1$ and $n$ does not divide $p - 1$.

Next we give selected problems of the Thai Mathematical Olympiad Examinations 2006. Thanks again go to Bill Sands, Canadian team Leader to the IMO in Vietnam, for collecting them for us.

**THAI MATHEMATICAL OLYMPIAD EXAMINATIONS 2006**

**Selected Problems**

1. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying
$$f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17$$
for all real $x$. Find $f(85)$.

2. Evaluate
$$\sum_{k=84}^{8000} \binom{k}{84} \binom{8084 - k}{84}. $$

3. Find all integers $n$ such that $n^2 + 59n + 881$ is a perfect square.

4. Find the least positive integer $n$ such that
$$\sqrt{3}z^{n+1} - z^n - 1 = 0$$
has a complex root $z$ with $|z| = 1$.

5. Let $p_k$ denote the $k$th prime number. Find the remainder when
$$\sum_{k=2}^{2550} p_k^{p_k^2 - 1}$$
is divided by 2550.

6. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
$$\sum_{i=1}^{2005} f(x_i + x_{i+1}) + f\left(\sum_{i=1}^{2006} x_i\right) \leq \sum_{i=1}^{2006} f(2x_i)$$
for all real numbers $x_1, x_2, \ldots, x_{2006}$.

7. A triangle has perimeter $2s$, inradius $r$, and the distances from its incentre to the vertices are $s_a$, $s_b$, and $s_c$. Prove that
$$\frac{3}{4} + \frac{r}{s_a} + \frac{r}{s_b} + \frac{r}{s_c} \leq \frac{s^2}{12r^2}. $$
8. Let \( \mathbb{N} \) be the set of positive integers. Is there a bijection \( f : \mathbb{N} \to \mathbb{N} \) with the three properties below?

(a) \( f(n + 2006) = f(n) + 2006 \) for all \( n \in \mathbb{N} \);

(b) \( f(f(n)) = n + 2 \) for \( n = 1, 2, 3, \ldots, 2004 \);

(c) \( f(2549) > 2550 \).

9. Find all prime numbers \( p \) such that \( \frac{2^{p-1} - 1}{p} \) is a perfect square.

10. In a school yard 229 boys and 271 girls are divided into 10 groups of 50 students each, with the students in each group numbered from 1 to 50. Four students are selected from two groups such that two pairs of students have identical numbers and the number of girls is odd. Show that the number of ways to select four students in this manner is odd.

11. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
    f(x + \cos(2007y)) = f(x) + 2007 \cos(f(y))
\]

for all real numbers \( x \) and \( y \).

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Next we give the problems of the Turkish Mathematical Olympiad 2006. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for providing these for our use.

**14th TURKISH MATHEMATICAL OLYMPIAD 2006**  
December 16–17, 2006

1. Let \( E \) and \( F \) be points on the side \( CD \) of a convex quadrilateral \( ABCD \) satisfying \( 0 < DE = FC < CD \). The circumcircles of triangles \( ADE \) and \( ACF \) intersect at \( K \neq A \), and the circumcircles of triangles \( BDE \) and \( BCF \) intersect at \( L \neq B \). Prove that \( A, B, K, \) and \( L \) are concyclic.

2. Find the largest real number \( t \) such that, in any school with 2006 students and 14 teachers where every student is acquainted with at least one teacher, a student and a teacher can be found such that they are acquainted and the ratio of the number of students who are acquainted with the teacher to the number of teachers who are acquainted with the student is at least \( t \).

3. Find all positive integers \( n \) for which every coefficient of the polynomial

\[
    P_n(x) = (x^2 + x + 1)^n - (x^2 + x)^n - (x^2 + 1)^n - (x + 1)^n + x^{2n} + x^n + 1
\]

is divisible by 7.
4. Let $a_1, a_2, \ldots, a_n (n \geq 2)$ be positive real numbers satisfying the relation
$t = a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2$. Prove that
$$\sum_{i \neq j} \frac{a_i}{a_j} \geq \frac{(n - 1)^2 t}{t - 1}.$$  

5. Let $A_1, B_1, C_1$ be the feet of the altitudes from vertices $A, B, C$ in
acute triangle $ABC$, respectively, and let $O_A, O_B, O_C$ be the incentres of
the triangles $AB_1C_1, BC_1A_1, CA_1B_1$, respectively. Let $T_A, T_B, T_C$ be
the points of tangency of the incircle of $ABC$ to the sides $BC, CA, AB$, respectively. Show $T_AO_CT_BO_AT_CO_B$ is a regular hexagon.

6. Prove that there exists no triangle whose side lengths, area, and angles
(measured in degrees) are all rational numbers.

As a final set for this number we give the Turkish Team Selection Test
for the IMO 2007. Thanks again go to Bill Sands, Canadian Team Leader to
the IMO in Vietnam, for collecting them for us.

**TURKISH TEAM SELECTION TEST FOR IMO 2007**

*March 24–25, 2007*

1. An airline company is planning to run two-way flights between some of
the six cities $A, B, C, D, E,$ and $F$. Determine the number of ways these
flights can be arranged so that it is possible to travel between any two of
these six cities using only the flights of this company.

2. Let $A$ and $B$ be distinct points on a circle $\Gamma$. For a variable point $P$ on $\Gamma$
distinct from $A$ and $B$, find the locus of the point $M$ such that $PM$ is
the opposite ray to the angle bisector of $\angle APB$ and $MP = AP + PB$.

3. Let $a, b, c$ be positive real numbers such that $a + b + c = 1$. Prove that
$$\frac{1}{ab + 2c^2 + 2c} + \frac{1}{bc + 2a^2 + 2a} + \frac{1}{ca + 2b^2 + 2b} \geq \frac{1}{ab + bc + ca}.$$  

4. The acute triangle $ABC$ is similar to the triangle $A_1B_1C_1$ whose ver-
tices $B_1, C_1, A_1$ lie on the rays $AC, BA, CB$, respectively. Prove that the
orthocentre of $A_1B_1C_1$ is the circumcentre of $ABC$.

5. Determine all positive odd integers $n$ for which there exist odd integers
$x_1, x_2, \ldots, x_n$ such that
$$x_1^2 + x_2^2 + \cdots + x_n^2 = n^4.$$
6. In how many ways can the numbers 1 and −1 be assigned to the unit squares of a 2007 × 2007 chessboard so that the absolute value of the sum of the numbers in any square made up from the unit squares of the chessboard does not exceed 1?

Next we return to solutions from our readers to problems proposed but not used at the 47th International Mathematical Olympiad 2006 in Slovenia given at [2008: 461–464].

G2. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE; \quad \angle ABC = \angle ACD = \angle ADE.$$ 

The diagonals $BD$ and $CE$ intersect at $P$. Prove that the line $AP$ bisects the side $CD$.

*Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's version.*

Let $S$ denote the direct similarity with centre $A$ transforming $B$ into $C$. From the hypotheses, we have $S(C) = D$ and $S(D) = E$. Let $U$ be the point of intersection of the line segments $AC$ and $BD$. Since $S(AC) = AD$ and $S(BD) = CE$, the image $V$ of $U$ under $S$ is the point of intersection of $AD$ and $CE$. It follows that

$$\frac{UA}{UC} = \frac{VA}{VD}. \quad (1)$$

Now, if $AP$ meets $CD$ at $W$, we have from Ceva's theorem

$$\frac{AU}{UC} \cdot \frac{DV}{VA} \cdot \frac{CW}{WD} = 1$$

and using (1), it follows that $\frac{CW}{WD} = 1$. This means that $W$ is the midpoint of $CD$, so the proof is complete.
G3. A point $D$ is chosen on the side $AC$ of a triangle $ABC$ with

$$\angle ACB < \angle BAC < 90^\circ$$

in such a way that $BD = BA$. The incircle of $ABC$ is tangent to $AB$ and $AC$ at points $K$ and $L$, respectively. Let $J$ be the incentre of triangle $BCD$. Prove that the line $KL$ intersects the line segment $AJ$ at its midpoint.

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geupel’s write-up.

It is a basic fact that, if the incircle of $\triangle PQR$ is tangent to the side $PQ$ at the point $T$, then $2PT = PQ + PR - QR$. (See, for example, H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, The Mathematical Association of America, 1967, Theorem 1.4.1, page 11.)

Let $L'$ be the point on the side $AC$ such that $JL' \parallel KL$. Denote the point of intersection of $AJ$ and $KL$ by $M$. Let the incircle of $\triangle BCD$ meet $CD$ at point $T$. Because $J$ is on the internal bisector of $\angle BDC$, we have that $\angle JDL' = 90^\circ - \frac{1}{2}\angle ADB = \angle DL'J$; hence $DL' = 2DT$. Using the basic fact above, we obtain

$$DL' = 2DT = BD + CD - BC = AB + CD - BC$$

and

$$2AL = AB + AC - BC.$$  \hspace{1cm}

Consequently,

$$AL' = AC - CD + DL'$$

$$= AC - CD + (AB + CD - BC) = 2AL.$$  \hspace{1cm}

We conclude that $\frac{AJ}{AM} = \frac{AL'}{AL} = 2$, which completes the proof.

G4. In triangle $ABC$, let $J$ be the centre of the excircle tangent to side $BC$ at $A_1$ and to the extensions of sides $AC$ and $AB$ at $B_1$ and $C_1$, respectively. Suppose that the lines $A_1B_1$ and $AB$ are perpendicular and intersect at $D$. Let $E$ be the foot of the perpendicular from $C_1$ to line $DJ$. Determine the angles $\angle BEA_1$ and $\angle AEB_1$. 

\[\begin{figure}[h]
\centering
\includegraphics{g4.png}
\end{figure}\]
Solution by Titu Zvonaru, Comĕnești, Romania.

As usual we write \( a = BC \), \( b = CA \), \( c = AB \), and \( s = \frac{1}{2}(a + b + c) \).

It is known that the following hold: \( BA_1 = BC_1 = s - c \), \( AB_1 = AC_1 = s \), and \( CA_1 = CB_1 = s - b \). Since \( B_1D \perp AB \), we have

\[
AD = s \cos A, \quad DB = (s - c) \cos B.
\]

By Menelaus’ theorem we obtain

\[
\frac{B_1C}{B_1A} \cdot \frac{DA}{DB} \cdot \frac{A_1B}{A_1C} = 1 \iff \frac{s - b}{s} \cdot \frac{s \cos A}{(s - c) \cos B} \cdot \frac{s - c}{s - b} = 1 \iff \cos A = \cos B,
\]

hence the given triangle \( ABC \) is isosceles with \( \angle A = \angle B \).

We denote by \( h \) the altitude from \( C \) to the line \( AB \), and by \([ABC]\) the area of \( \triangle ABC \). We now have the following calculations and deductions:

\[
JC_1 = \frac{[ABC]}{s - a} = \frac{ch}{c} = h; \quad h^2 = a^2 - \frac{c^2}{4}.
\]

\[
DC_1 = DB + BC_1 = (s - c) \cos B + (s - c)
\]

\[
= (s - c) \left(1 + \frac{c}{2a}\right) = \left(a - \frac{c}{2}\right) \left(a + \frac{c}{2}\right) \cdot \frac{1}{a} = \frac{h^2}{a}.
\]

\[
DJ^2 = h^2 + \frac{h^4}{a^2} \implies DJ = \frac{h\alpha}{a}, \quad \text{where } \alpha = \sqrt{a^2 + h^2}.
\]

\[
DE = \frac{DC^2}{DJ} = \frac{h^3}{a\alpha}; \quad EJ = \frac{JC^2}{DJ} = \frac{ah}{\alpha};
\]

\[
DE \cdot EJ \cdot DJ = \frac{h^5}{a\alpha} \cdot \frac{ah}{\alpha} \cdot \frac{h\alpha}{a} = \frac{h^5}{a\alpha}.
\]

By Stewart’s theorem we obtain the following four deductions:

\[
A_1D^2 \cdot EJ - A_1E^2 \cdot DJ + A_1J^2 \cdot DE = EJ \cdot DJ \cdot DE
\]

\[
\implies A_1E^2 \cdot \frac{h\alpha}{a} = (s - c)^2 \cdot \frac{h^2}{a^2} \cdot \frac{ah}{\alpha} + h^2 \cdot \frac{h^3}{a\alpha} - \frac{h^5}{a\alpha}
\]

\[
\implies A_1E^2 = \frac{(s - c)^2 \cdot h^2}{a^2 + h^2}
\]

(1)

\[
BD^2 \cdot EJ - BE^2 \cdot EJ + BJ^2 \cdot DE = EJ \cdot DJ \cdot DE
\]

\[
\implies BE^2 \cdot \frac{h\alpha}{a} = (s - c)^2 \cdot \frac{c^2}{4a^2} \cdot \frac{ah}{\alpha} + [(s - c)^2 + h^2] \cdot \frac{h^3}{a\alpha} - \frac{h^5}{a\alpha}
\]

\[
\implies BE^2 = \frac{(s - c)^2 \cdot a^2}{a^2 + h^2}
\]

(2)
\[
AD^2 \cdot DE - AE^2 \cdot DJ + AJ^2 \cdot DE = DE \cdot EJ \cdot DJ
\]
\[
\implies AE^2 \cdot \frac{h\alpha}{a} = \frac{s^2}{a^2} \cdot \frac{c^2}{4\alpha^2} \cdot \frac{ah}{\alpha} + (s^2 + h^2) \cdot \frac{h^3}{a\alpha} - \frac{h^5}{a\alpha}
\]
\[
\implies AE^2 = \frac{s^2 a^2}{a^2 + h^2}
\]
(3)

\[
B_1 D^2 \cdot DE - B_1 E^2 \cdot DJ + B_1 J^2 \cdot DE = DE \cdot EJ \cdot DJ
\]
\[
\implies B_1 E^2 \cdot \frac{h\alpha}{a} = \frac{s^2}{a^2} \cdot \frac{h^2}{\alpha^2} \cdot \frac{ah}{\alpha} + h^2 \cdot \frac{h^3}{a\alpha} - \frac{h^5}{a\alpha}
\]
\[
\implies B_1 E^2 = \frac{s^2 h^2}{a^2 + h^2}
\]
(4)

It follows from (1)-(4) that

\[
A_1 E^2 + B E^2 = BA_1^2 \quad \text{and} \quad A E^2 + B_1 E^2 = AB_1^2
\]

so by the converse of the Pythagorean Theorem, \( \angle BEA_1 = \angle AEB_1 = 90^\circ \).

G5. Circles \( \omega_1 \) and \( \omega_2 \) with centres \( O_1 \) and \( O_2 \) are externally tangent at point \( D \) and internally tangent to a circle \( \omega \) at points \( E \) and \( F \), respectively. Line \( t \) is the common tangent of \( \omega_1 \) and \( \omega_2 \) at \( D \). Let \( AB \) be the diameter of \( \omega \) perpendicular to \( t \), so that \( A, E, \) and \( O_1 \) are on the same side of \( t \). Prove that the lines \( AO_1, BO_2, EF, \) and \( t \) are concurrent.

Solved by Michel Bataille, Rouen, France; Oliver Geipel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We first give the solution of Geipel.

Let \( O \) be the centre of \( \omega \). The triangles \( O_1 ED \) and \( OEB \) are isosceles with \( O_1 E || OE \) and \( O_1 D || OB \); hence \( ED || EB \); which means that points \( B, D, \) and \( E \) are collinear. Similarly, \( A, D, \) and \( F \) are collinear. Denote by \( G \) the intersection of \( AO_1 \) with \( BO_2 \). We apply Pappus' theorem to the collinear points \( A, B, O, \) and the collinear points \( O_2, O_1, D, \) thus obtaining that the points \( E = BD \cap OO_1, F = AD \cap OO_2, \) and \( G = AO_1 \cap BO_2 \) are collinear.

It therefore remains to prove that \( G \) lies on the line \( t \).

The line \( O_1 D \) intersects \( \omega_1 \) and \( \omega_2 \) again at points \( H \) and \( I \), respectively. The homothety with centre \( E \) that maps \( O_1 \) to \( O \) also maps \( \omega_1 \) to \( \omega \) and thus \( H \) to \( A \). Therefore, \( H \) lies on the line \( AE \). Similarly, \( I \) lies on \( BF \). Denote by \( C \) the intersection of \( AE \) and \( BF \). Since \( AF \perp BC \) and
$BE \perp AC$ it follows that $D = AF \cap BE$ is the orthocentre of $\triangle ABC$ and $t$ is the third altitude of $\triangle ABC$, which shows that $C$ lies on $t$.

Let $J$ denote the intersection of $AB$ and $t$. Because the triangles $HIC$ and $ABC$ are homothetic,

$$\frac{DO_1}{DO_2} = \frac{DH}{DI} = \frac{JA}{JB}.$$ 

Consequently, the point $G$ at the intersection of the lines $AO_1$ and $BO_2$ also lies on the line $t$, which completes the proof.

Next we give the solution of Bataille.

Since $AB$ is a diameter of $\omega$, we have $AF \perp FB$ and $AE \perp EB$. It follows that the point $H$ of intersection of $AE$ and $BF$ is the orthocentre of $\triangle ADB$. This said, we shall make use of the inversion $I$ with pole $D$ such that $I(\omega) = \omega$. We clearly have $I(E) = B$ and $I(F) = A$ so that the line $EF$ is transformed into the circumcircle $\Gamma$ of $\triangle ADB$. We denote by $U$ the point of intersection of $t$ and $EF$, $W$ the centre of $\Gamma$, $O$ the midpoint of $AB$, and $U' = I(U)$. Note that $U'$ is on $t$ and $\Gamma$ with $U$, $U'$ on either side of $D$ on $t$ and that $\overline{DH} = 2\overline{WO}$.

Let $t_B$ be the tangent to $\omega$ at $B$ and $O_1'$ be the reflection of $D$ in $t_B$. From $I(O_1) = O_1'$ (because $I(\omega_1) = t_B$), we deduce that the image of the line $AO_1$ under $I$ is the circle $(DFO_1')$. Let $V$ be the centre of this circle. Clearly, $V$ is on $t_B$ and also on the perpendicular bisector of $DF$.

Since the latter is parallel to $BH$ and passes through the midpoint $I$ of $HD$ (note that the circle with diameter $DH$ passes through $E$ and $F$), it follows
that HIVB is a parallelogram. Thus $\overrightarrow{BV} = \overrightarrow{HI} = \overrightarrow{OW}$ and $V$ is on the perpendicular to $DU'$ through $W$, that is, on the perpendicular bisector of $DU'$. As a result, the circle $(DFO')$ passes through $U'$ and its inverse, and the line $AO_1$ passes through $U$. Similarly, the line $BO_2$ passes through $U$ and we are done.

G6. In a triangle $ABC$, let $M_a$, $M_b$, $M_c$ be the respective mid-points of the sides $BC$, $CA$, $AB$ and let $T_a$, $T_b$, $T_c$ be the mid-points of the arcs $BC$, $CA$, $AB$ of the circumcircle of $ABC$ not containing $A$, $B$, $C$, respectively. For each $i \in \{a, b, c\}$, let $\omega_i$ be the circle with diameter $M_iT_i$. Let $p_i$ be the common external tangent to $\omega_j$, $\omega_k$ such that $\{i, j, k\} = \{a, b, c\}$ and such that $\omega_i$ lies on one side of $p_i$ while $\omega_j$, $\omega_k$ lie on the other side. Prove that the lines $p_a$, $p_b$, $p_c$ form a triangle similar to $ABC$ and find the ratio of similitude.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $a = BC$, $b = CA$, $c = AB$, and $2s = a + b + c$. Let $F = [ABC]$ be the area of $\triangle ABC$ with circumradius $R$, circumcentre $O$, and incentre $I$.

Lemma. Lines $p_a$ and $M_bM_c$ are parallel, and their distance is $\frac{F(s-a)}{2sa}$.

Proof. For each $i \in \{a, b, c\}$, let $O_i$, $r_i$ be the centre and the radius of $\omega_i$. The perpendicular from $O_b$ to $BC$ meets $M_bM_c$ and $\omega_b$ at $N_b$ and $P_b$, respectively, with $N_c$, $P_c$ defined similarly. Then $r_b = O_bT_b = \frac{R - OM_b}{2}$ and $O_bN_b = O_bM_b \cos C = r_b \cos C$; hence

$$N_bP_b = r_b - O_bN_b = \frac{R(1 - \cos B)}{2} \cdot \frac{ (s - a)(s - c) \cdot (s - a)(s - b) }{ac \cdot ab} = \frac{F(s-a)}{2sa}.$$

Similarly, $N_cP_c = \frac{F(s-a)}{2sa}$. We conclude that $p_a = P_bP_c$. \hfill \blacksquare

Corollary. The triangles $\triangle(p_a, p_b, p_c)$ and $M_aM_bM_c$ are homothetic with centre $I$ and ratio 2. Thus, $\triangle(p_a, p_b, p_c) \sim \triangle ABC$ with ratio 4.

Proof. Denoting by $\overrightarrow{P}$ the position vector of point $P$, we have

$$\overrightarrow{I} = \frac{a\overrightarrow{A} + b\overrightarrow{B} + c\overrightarrow{C}}{a + b + c} = \frac{(s-a)\overrightarrow{M_a} + (s-b)\overrightarrow{M_b} + (s-c)\overrightarrow{M_c}}{s}.$$
Thus, \( d(I, M_b M_c) = \frac{2[M_b M_c]}{M_b M_c} = \frac{4(a - a)[M_a M_b M_c]}{sa} = 2d(p_a, M_b M_c) \), and by entirely similar calculations we have \( d(I, M_c M_a) = 2d(p_b, M_c M_a) \) and \( d(I, M_a M_b) = 2d(p_c, M_a M_b) \), where \( d \) denotes the Euclidean distance.

**G8.** Points \( A_1, B_1, C_1 \) are on the sides \( BC, CA, AB \) of a triangle \( ABC \), respectively. The circumcircles of triangles \( AB_1C_1, BC_1A_1, CA_1B_1 \) intersect the circumcircle of triangle \( ABC \) again at points \( A_2, B_2, C_2 \), respectively (that is, \( A_2 \neq A, B_2 \neq B, C_2 \neq C \)). Points \( A_3, B_3, C_3 \) are symmetric to \( A_1, B_1, C_1 \) with respect to the midpoints of the sides \( BC, CA, AB \) respectively. Prove that the triangles \( A_2B_2C_2 \) and \( A_3B_3C_3 \) are similar.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let us consider the situation in the plane of complex numbers, and let \( a, b, c, \ldots \) denote the coordinates of the points \( A, B, C, \ldots \). It is well-known (see Titu Andreescu, Dorin Andrica, *Complex numbers from A to Z*, Birkhäuser, Boston, 2006, page 68) that triangles \( PQR \) and \( STU \) are similar (with the same orientation) if and only if \( \frac{p - r}{q - r} = \frac{s - u}{t - u} \).

First we recognize inscribed angles on the circumcircles of \( \triangle ABC \) and \( \triangle AB_1C_1 \). We have

\[
\angle A_2BC_1 = \angle A_2BA = \angle A_2CA = \angle A_2CB_1
\]

and

\[
\angle A_2C_1B = 180^\circ - \angle A_2C_1A \\
= 180^\circ - \angle A_2B_1A = \angle A_2B_1C.
\]

Therefore, the (likewise) oriented triangles \( A_2BC_1 \) and \( A_2CB_1 \) are similar, which implies that

\[
\frac{a_2 - c_1}{b - c_1} = \frac{a_2 - b_1}{c - b_1}.
\]
We obtain $a_2 = \frac{bb_1 - cc_1}{b + b_1 - c - c_1}$. Similarly,

$$b_2 = \frac{cc_a a_1}{c + c_1 - a - a_1}, \quad \text{and} \quad c_2 = \frac{aa_1 - bb_1}{a + a_1 - b - b_1}.$$ 

Since $A_3$ is symmetric to $A_1$ with respect to the midpoint of $BC$, we have $a_3 - c = b - a_1$, and hence $a_3 = b + c - a_1$. Similarly $b_3 = c + a - b_1$ and $c_3 = a + b - c_1$.

By the characterization given above, it suffices to prove that

$$(a_2 - c_2)(b_3 - c_3) = (b_2 - c_2)(a_3 - c_3).$$

But this can easily be verified by employing the relations above and clearing denominators. This completes the proof.

**N1.** Given $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose $n^{th}$ digit after the decimal point is the $(2^{n})^{th}$ digit after the decimal point of $x$. Prove that if $x$ is a rational number, then $y$ is a rational number.

*Solution by Oliver Geupel, Brühl, NRW, Germany.*

For each positive integer $n$, let $x_n$ be the $n^{th}$ digit of $x$ after the decimal point. Because $x$ is rational, there exists a positive integer $N$ such that the sequence $(x_n)_{n \geq N}$ is periodic with some period $q$. By the Pigeonhole Principle, there exist positive integers $m$ and $n$ such that $2^m > 2^n \geq N$ and $2^m \equiv 2^n \pmod q$. Then the sequence of the least nonnegative remainders $(2^k \mod q)_{k \geq n}$ is periodic with a period not greater than $m - n$. Consequently, the sequence $(y_k)_{k \geq n} = (x_{2^k})_{k \geq n}$ is periodic; hence the decimal expansion of the number $y$ is eventually periodic and therefore $y$ is rational.

**N2.** For each positive integer $n$ let

$$f(n) = \frac{1}{n} \left( \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor \right),$$

where $\left\lfloor x \right\rfloor$ is the greatest integer not exceeding $x$.

(a) Prove that $f(n + 1) > f(n)$ for infinitely many $n$.

(b) Prove that $f(n + 1) < f(n)$ for infinitely many $n$.

*Solved by Oliver Geupel, Brühl, NRW, Germany.*

Let $d(n)$ denote the number of positive divisors of the integer $n$. Observe that for each positive integer $k$, there are exactly $\left\lfloor \frac{n}{k} \right\rfloor$ numbers in the set \{1, 2, \ldots, n\} which are divisible by $k$. Therefore,

$$\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor = d(1) + d(2) + \cdots + d(n).$$
This implies that \( f(n) \) is the arithmetic mean of the numbers of divisors of the first \( n \) positive integers.

Because the function \( d \) is unbounded, there are infinitely many numbers \( n \), such that \( d(n + 1) > \max\{d(k) : 1 \leq k \leq n\} \). By rewriting this as

\[
f(n + 1) = \frac{d(1) + d(2) + \ldots + d(n + 1)}{n + 1} > \frac{d(1) + d(2) + \ldots + d(n)}{n} = f(n)
\]

we have completed part (a).

For part (b), on the one hand since \( f(6) = \frac{7}{3} > 2 \) and \( d(k) \geq 2 \) for each \( k > 1 \), it follows that \( f(n) > 2 \) whenever \( n \geq 5 \). On the other hand, for each prime number \( n + 1 \) we have \( d(n + 1) = 2 \). Consequently, for any prime number \( n + 1 \geq 7 \) we have \( f(n + 1) < f(n) \), and we have completed part (b).

\( \textbf{N4.} \) Let \( a \) and \( b \) be relatively prime integers with \( 1 < b < a \). Define the weight of an integer \( c \), denoted by \( w(c) \), to be the minimum possible value of \( |x| + |y| \) taken over all pairs of integers \( x \) and \( y \) such that

\[
ax + by = c.
\]

An integer \( c \) is called a local champion if \( w(c) \geq \max\{w(c \pm a), w(c \pm b)\} \). Find all local champions and determine their number.

\textit{Solution by Oliver Geipel, Brühl, NRW, Germany.}

We prove a series of lemmas.

\textbf{Lemma 1.} Let \( x, y, \) and \( c \) be integers such that \( y \geq 0 \) and

\[
ax + by = c.
\] (1)

Then the condition

\[
w(c) = |x| + |y|
\] (2)

is equivalent to

\[
|x + b| + |y - a| \geq |x| + y.
\] (3)

\textit{Proof:} If (2) holds, then (3) follows immediately from the definition of the weight. Conversely, assume (3). Note that \( y < a \). We are to show that for each integer \( n \),

\[
|x + nb| + |y - na| \geq |x| + y.
\] (4)

If \( n > 0 \), then (4) follows from

\[
|x + nb| + |y - na|
\]

\[
= |x + nb| + na - y \geq |x + nb| + (n - 1)b + a - y
\]

\[
= |x + nb| + |(1 - n)b| + |y - a| \geq |x + b| + |y - a| \geq |x| + y.
\]
If \( n \leq 0 \), then we derive (4) as follows:

\[
|x + nb| + |y - na| = |x + nb| - na + y \\
\geq |x + nb| + | - nb| + y \geq |x| + y.
\]

Lemma 2. Let \( d \) and \( x \) be integers such that \(-b + 1 \leq d \leq b\). Then the conditions

\[
|x + b| \geq |x| + d
\]

and

\[
2x + b \geq d
\]

are equivalent. Also the conditions \(|x + b| \leq |x| + d\) and \(2x + b \leq d\) are equivalent.

**Proof:** If \( x \leq -b \), then both (5) and (6) are false. If \(-b < x < 0\), then \(|x + b| = x + b\) and \(|x| = -x\); hence (5) is equivalent to \(x + b \geq -x + d\); that is to (6). Finally, if \( x \geq 0 \), then both (5) and (6) are satisfied. The proof of the second claim is similar.

Lemma 3. Let \( x \), \( y \), and \( c \) be integers satisfying (1). Then, (2) is satisfied if and only if one of the following holds:

(a) \( -\left(\frac{a + b}{2}\right) \leq y < -\left(\frac{a - b}{2}\right) \) and \( x \leq y + \frac{a + b}{2} \);

(b) \( -\left(\frac{a - b}{2}\right) \leq y \leq \frac{a - b}{2} \);

(c) \( \frac{a - b}{2} < y \leq \frac{a + b}{2} \) and \( y \leq x + \frac{a + b}{2} \).

**Proof:** We can assume without loss of generality that \( y > 0 \), since we can switch from \( x \), \( y \), \( c \) to \(-x\), \(-y\), \(-c\), whenever \( y < 0 \). Now, (2) is equivalent to (3) by Lemma 1. Again \( y < a \) holds; hence (3) is equivalent to

\[
|x + b| \geq |x| + 2y - a. \tag{7}
\]

If \( y \leq \frac{a - b}{2} \), then \( 2y - a \leq -b \) and (7) is true. Next, if \( \frac{a - b}{2} < y \leq \frac{a + b}{2} \), then by Lemma 2 inequality (7) is equivalent to \( 2x + b \geq 2y - a \). Finally, for \( y > \frac{a + b}{2} \) no solution exists.

Lemma 4. Let \( x \), \( y \), \( c \) be integers satisfying (1), (2). Let \( \frac{a - b}{2} < y \leq \frac{a + b}{2} \). Then \( c \) is a local champion if and only if \( x < 0 \) and

\[
x + \left[\frac{a + b}{2}\right] = y. \tag{8}
\]

**Proof:** If \( c \) is a local champion, then by Lemma 3

\[
y \leq x + \frac{a + b}{2} < x + 1 + \frac{a + b}{2}.
\]
Again by Lemma 3, we obtain \( w(c + a) = |x + 1| + y \). Because \( c \) is a local champion, it follows that \( |x + 1| \leq |x| \); hence \( x < 0 \). If \( y \leq x - 1 + \frac{a+b}{2} \), then we obtain from Lemma 3 that \( w(c - a) = |x - 1| + y = w(c) + 1 \), which contradicts the hypothesis that \( c \) is a local champion. Therefore,

\[
\frac{x + \frac{a + b}{2} - 1}{2} < y \leq \frac{x + \frac{a + b}{2}}{2},
\]

that is (8).

Conversely, it remains to prove that \( c \) is indeed a local champion. We show that \( w(c \pm a) \leq w(c) \) and \( w(c \pm b) \leq w(c) \).

First, \( w(c + a) = |x + 1| + y \leq |x| + y = w(c) \) holds.

Second, we have \( c - a = a(x + b - 1) + b(y - a) \), as well as

\[
-\left( \frac{a+b}{2} \right) < y - a < -\left( \frac{a-b}{2} \right)
\]

and \( x + b - 1 \leq y - a + \frac{a+b}{2} \). By Lemma 3, it follows \( w(c - a) = |x + b - 1| + a - y \). From \( x + \frac{a+b-2}{2} \leq y \), we obtain \( 2x + b - 1 \leq 2y - a \); thus

\[
w(c - a) = |x + b - 1| + a - y \leq |x| + y = w(c).
\]

Third, \( c + b = a(x + b) + b(y + 1 - a) \); hence

\[
w(c + b) \leq b + a - y - 1 = a + b - 1 - \left\lfloor \frac{a+b}{2} \right\rfloor \leq \frac{a+b}{2} = w(c).
\]

Fourth, \( c - b = ax + b(y - 1) \); thus

\[
w(c-b) \leq |x|+y-1 = w(c)-1.
\]

**Lemma 5.** Let \( x \), \( y \), and \( c \) be integers satisfying inequalities (1) and (2). If \( -\left( \frac{a+b}{2} \right) \leq y < \frac{a-b}{2} \), then \( c \) is a local champion if and only if \( x > 0 \) and \( x = y + \left\lfloor \frac{a+b}{2} \right\rfloor \). Moreover, there are no local champions \( c \) with \( |y| \leq \frac{a-b}{2} \).

**Proof:** The first part follows from Lemma 4 by replacing \( x \) by \( -x \), \( -y \). Let \( |y| \leq \frac{a-b}{2} \). We conclude by Lemma 3 that \( w(c - a) = |x - 1| + |y| \) and \( w(c + a) = |x + 1| + |y| \). Clearly, one of the numbers \( |x - 1| \) and \( |x + 1| \) is greater than \( |x| \), which completes the proof of the second part.

**Corollary 6.** There are \( b - 1 \) local champions if \( ab \) is odd and \( 2(b - 1) \) local champions if \( ab \) is even.

**Proof:** We can describe the local champions explicitly. If \( ab \) is odd, then by Lemma 4 and Lemma 5 the set of local champions is

\[
\left\{ \frac{(a+b)(2-b)}{2} + n(a+b) : 0 \leq n \leq b-2 \right\}.
\]
If \(ab\) is even, then Lemma 4 and Lemma 5 yield the disjoint sets

\[
M = \left\{ \frac{2a - ab + b - b^2}{2} + n(a + b) : 0 \leq n \leq b - 2 \right\},
\]

\[
N = \left\{ \frac{2a - ab + b - b^2}{2} + n(a + b) + b : 0 \leq n \leq b - 2 \right\},
\]

respectively. The set \(M\) is an arithmetic progression with difference \(a + b\), and the set \(N\) is the same progression with offset \(b\).


1. Find all solutions in integers \(x\) and \(y\) of the equation

\[
(x + y^2)(x^2 + y) = (x + y)^3.
\]

_Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania._ We give the solution of Amengual Covas.

The equation is equivalent to \(xy + x^2y^2 = 3x^2y + 3xy^2\), which can be rewritten as \(xy(1 + xy - 3x - 3y) = 0\). We observe that \((x, y) = (0, m), (m, 0)\), where \(m\) is an integer, are solutions.

Now suppose that \(xy \neq 0\). Then \(1 + xy - 3x - 3y = 0\). Since no pairs \((x, y)\) with \(x = 3\) can satisfy this last equation, we rewrite it in the form

\[
y = \frac{3x - 1}{x - 3}, \text{ or } y = 3 + \frac{8}{x - 3}.
\]

It follows that \(y\) is an integer if and only if \(x - 3\) divides 8. Thus, \(x - 3 \in \{\pm 1, \pm 2, \pm 4, \pm 8\}\), so that \(x \in \{-5, -1, 1, 2, 4, 5, 7, 11\}\).

We conclude that the complete solution set for \((x, y)\) is

\[
\{(0, m), (m, 0) : m \in \mathbb{Z}\} \cup \{(2, -5), (-5, 2), (-1, 1), (1, -1), (4, 11), (11, 4), (5, 7), (7, 5)\}.
\]

2. A queue in front of a counter consists of 12 persons. The counter is then closed because of a technical problem and the 12 people are redirected to another one. In how many different ways can the new queue be formed if each person maintains the same position as before, or is one step closer to the front, or is one step farther from the front?

_Solution by Oliver Geulp, Brühl, NRW, Germany._

Let \(Q_n\) be the number of distinct permutations \(\pi\) of \(\{1, 2, \ldots, n\}\) such that for \(1 \leq k \leq n\) we have \(k - 1 \leq \pi(k) \leq k + 1\). For \(n \geq 3\), if \(\pi\) has the desired property, then we have \(\pi(n) \in \{n - 1, n\}\). First consider the
case \( \pi(n) = n - 1 \). Then \( \pi(n - 1) = n \), and the numbers \( 1, 2, \ldots, n - 2 \) can be arranged in \( Q_{n-2} \) ways. Second, if \( \pi(n) = n \), then the numbers \( 1, 2, \ldots, n - 1 \) can be arranged in \( Q_{n-1} \) ways. Hence, for \( n \geq 3 \), we have the recursion \( Q_n = Q_{n-2} + Q_{n-1} \) with \( Q_1 = 1 \) and \( Q_2 = 2 \).

Therefore, \( Q_n \) is the \((n + 1)^{st}\) Fibonacci number \( F_{n+1} \). (specifically, the solution of our exercise is \( Q_{12} = F_{13} = 233 \).

3. In the triangle \( ABC \) the angle bisector from \( A \) intersects the side \( BC \) in the point \( D \) and the angle bisector from \( C \) intersects the side \( AB \) in the point \( E \). The angle at \( B \) is greater than \( 60^\circ \). Prove that \( AE + CD < AC \).

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give the version of Apostolopoulos.

We have \( \cos B < \frac{1}{2} \). By the Law of Cosines, \( b^2 = a^2 + c^2 - 2ac \cdot \cos B \), so \( \cos B = \frac{a^2 + c^2 - b^2}{2ac} < \frac{1}{2} \). Then

\[
\begin{align*}
& a^2 + c^2 < 2ac + b^2, \\
& bc + a^2 + c^2 + ab < ab + ac + b^2 + bc, \\
& c(b + c) + a(a + b) < (a + b)(b + c), \\
& \frac{bc}{a + b} + \frac{ab}{b + c} < b.
\end{align*}
\]

From the Bisector Theorem, we have \( AE = \frac{bc}{a + b} \) and \( CD = \frac{ab}{b + c} \); hence the conclusion \( AE + CD < AC \).

Comment by Zvonaru. This problem is part (c) of Problem 3 of the contest Trentième Olympiade Mathématique Belge, [2008 : 80].

4. The polynomial \( f(x) \) is of degree four. The zeroes of \( f \) are real and form an arithmetic progression, that is, the zeroes are \( a, a + d, a + 2d, \) and \( a + 3d \) where \( a \) and \( d \) are real numbers. Prove that the three zeroes of \( f'(x) \) also form an arithmetic progression.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's version.

First, note that the zeroes of \( f'(x) \) are also real numbers. This is obvious if \( d = 0 \) and follows from Rolle’s theorem otherwise.

Let \( x_k = a + kd \) (\( k = 0, 1, 2, 3 \)) and set \( \lambda = \frac{x_0 + x_3}{2} = \frac{x_1 + x_2}{2} \).

Then,

\[
f(x) = m \prod_{k=0}^{3} (x - x_k) = m \left( x^2 - 2\lambda x + p \right) \left( x^2 - 2\lambda x + q \right),
\]
where $m$ is a nonzero real number, $p = x_0 x_3$, and $q = x_1 x_2$. It readily follows that

$$f'(x) = 4m(x - \lambda) \left( x^2 - 2\lambda x + \frac{p + q}{2} \right).$$

The zeroes of $f'(x)$ are $\lambda$ and real numbers $y_1, y_2$ such that $y_1 + y_2 = 2\lambda$. It follows that $y_1$, $\lambda$, $y_2$ is an arithmetic progression, and the proof is complete.

5. Each square on a $2005 \times 2005$ chessboard is painted either black or white. This is done in such a way that each $2 \times 2$ “sub-chessboard” contains an odd number of black squares. Show that the number of black squares among the four corner squares is even. In how many different ways can the chessboard be painted so that the above condition is satisfied?

Solution by Oliver Geupel, Brihl, NRW, Germany.

Consider more generally integers $m \geq 2$, $n \geq 2$, and an $m \times n$ board represented by a matrix

$$
\begin{pmatrix}
   s_{11} & \cdots & s_{1n} \\
   \vdots & \ddots & \vdots \\
   s_{m1} & \cdots & s_{mn}
\end{pmatrix}
$$

where $s_{ij} = \begin{cases} 1 & \text{if square } (i, j) \text{ is painted black,} \\ 0 & \text{if square } (i, j) \text{ is painted white.} \end{cases}$

Call the board odd if for each $2 \times 2$ sub-board

$$B_{ij} = \begin{pmatrix}
   s_{ij} & s_{i,j+1} \\
   s_{i+1,j} & s_{i+1,j+1}
\end{pmatrix}
$$

the sum of its entries $b_{ij} = s_{ij} + s_{i,j+1} + s_{i+1,j} + s_{i+1,j+1}$ is odd, where $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$.

**Proposition 1.** For each odd $m \times n$ board, if $2 \mid (m - 1)(n - 1)$, then the sum of its corner entries is even.

**Proof:** By hypothesis, the number

$$\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} b_{ij} = 4 \sum_{i=2}^{m-1} s_{i,j} + 2 \sum_{i=2}^{m-1} (s_{i1} + s_{in})$$

$$+ 2 \sum_{j=2}^{n-1} (s_{1j} + s_{mj}) + (s_{11} + s_{in} + s_{m1} + s_{mn})$$

is even. Hence, the sum $s_{11} + s_{in} + s_{m1} + s_{mn}$ is also even.  

**Proposition 2.** For $m \geq 2$ and $n \geq 2$, the number of odd $m \times n$ boards is $2^{m+n-1}$.
Proof: By induction on $n$.
   
   For $n = 2$, there are eight ways to paint the topmost $2 \times 2$ sub-board. 
   The rows 3, 4, ..., $m$ can be successively painted, where we have two choices 
   for each row. We obtain $8 \cdot 2^{m-2} = 2^{m+1}$ different painted boards. 
   
   In an $m \times n$ board the leftmost $n - 1$ columns form an $m \times (n - 1)$ 
   board:
   $$
   \begin{pmatrix}
   s_{11} & \cdots & s_{1,n-1} & s_{1n} \\
   \vdots & \ddots & \vdots & \vdots \\
   s_{m1} & \cdots & s_{m,n-1} & s_{mn}
   \end{pmatrix},
   $$
   which can be painted in $2^{m+n-2}$ ways by the induction hypothesis. We now 
   have two choices for $s_{1n}$ and $s_{2n}$. Each further entry $s_{in}$ in the last column 
   is uniquely determined by the elements $s_{i-1,n-1}$, $s_{i-1,n}$, and $s_{i,n-1}$. We 
   therefore obtain $2^{m+n-2} \cdot 2 = 2^{m+n-1}$ painted boards. 
   
   Remark: For $m = n = 2005$ we obtain $2^{2009}$ painted boards.

That brings us to the start of the problems given in numbers of the 
Corner for 2009, and solutions from our readers to problems of the German 
Mathematical Olympiad, Final Round, Grades 12–13, Munich, April 29–May 
2, 2006 given at [2009 : 21].

1. Determine all positive integers $n$ for which the number
   $$
   z_n = \underbrace{101 \cdots 101}_{2n+1 \text{ digits}}
   $$
   is a prime.

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; 
Alex Song, student, Elizabeth Ziegler Public School, Waterloo, ON; Edward 
T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Konstantine 
Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the write-

up of Babbitt, modified by the editor.

We have
$$
z_n = \sum_{i=0}^{n} 100^i = \frac{100^{n+1} - 1}{100 - 1} = \frac{10^{2n+2} - 1}{99} = \frac{(10^{n+1} + 1)(10^{n+1} - 1)}{99}.
$$

It is obvious that $9 \mid (10^{n+1} - 1)$ for all positive integers $n$. Note that $99$ 
divides $(10^{n+1} + 1)(10^{n+1} - 1)$ because $z_n$ is an integer, and $11$ divides $99$, 

hence $11$ divides $(10^{n+1} + 1)(10^{n+1} - 1)$. 


When 11 does not divide $10^{n+1} - 1$, then it divides $10^{n+1} + 1$, which is always greater than 11. Also, 9 divides $10^{n+1} - 1$, which is always greater than 9. Therefore, $z_n$ is not prime in this case.

When 11 does divide $10^{n+1} - 1$, then we have that $10^{n+1} - 1$ is a multiple of 99. Since $10^{n+1} + 1$ is always greater than 1, $z_n$ can only be prime when $10^{n+1} - 1 = 99$, or $n = 1$.

Therefore, $z_n$ is not prime for $n > 1$, and $z_1 = 101$ is prime.

5. Let $x$ be a nonzero real number satisfying the equation $ax^2 + bx + c = 0$. Furthermore, let $a$, $b$, and $c$ be integers satisfying $|a| + |b| + |c| > 1$. Prove that

$$|x| \geq \frac{1}{|a| + |b| + |c| - 1}.$$  

_Solved by Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Bataille._

Since $a$, $b$, and $c$ are integers, $|a| + |b| + |c| \geq 2$. We rewrite the inequality as

$$|(a| + |b| + |c|)|x| \geq 1 + |x|,$$

which certainly holds if $|x| \geq 1$, since then

$$|(a| + |b| + |c|)|x| \geq 2|x| \geq 1 + |x|.$$  

Therefore, we suppose that $0 < |x| < 1$ in what follows. Now, $|x| > |x|^2$, so we have $|a||x| + |b| \cdot |x| \geq |a||x|^2 + |b| \cdot |x| \geq |ax^2 + bx| = |c| = |c|$ and it follows that

$$|(a| + |b| + |c|)|x| \geq |c|(1 + |x|).$$

Thus, (1) certainly holds if $|c| \geq 1$, that is, if $c \neq 0$.

Finally, if $c = 0$, then $ax + b = 0$ (since $x$ is nonzero) and $b \neq 0$ since otherwise $a = 0$, as well and $|a| + |b| + |c| > 1$ would not be true. The left-hand side of (1) then becomes $(|a| + |b|)|x|$ and

$$|(a| + |b|)|x| = |ax| + |b| \cdot |x|$$

$$= | - b| + |b| \cdot |x|$$

$$= |b|(1 + |x|) \geq 1 + |x|,$$

since $b$ is a nonzero integer. Thus, (1) continues to hold in that case and the proof is complete.

6. Let a circle through $B$ and $C$ of a triangle $ABC$ intersect $AB$ and $AC$ in $Y$ and $Z$, respectively. Let $P$ be the intersection of $BZ$ and $CY$, and let $X$ be the intersection of $AP$ and $BC$. Let $M$ be the point that is distinct from $X$ and on the intersection of the circumcircle of the triangle $XYZ$ with $BC$. Prove that $M$ is the midpoint of $BC$. 
Solution by Oliver Geupel, Brühl, NRW, Germany.

We reduce the geometric problem to an algebraic one, which can be solved by direct computation. Let \( a, b, c, x, y, z \) be the coordinates of \( A, B, C, X, Y, Z \) in the complex plane, respectively. Without loss of generality let \( b = -1 \) and \( c = 1 \). It suffices to prove that \( 0, x, y, z \) are concyclic. It is well known that distinct noncollinear points \( p_1, p_2, p_3, p_4 \) are concyclic if \( \frac{(p_3 - p_2)(p_1 - p_4)}{p_1 - p_2}(p_3 - p_4) \) is a real number (see Titu Andreescu, Dorin Andrica, *Complex numbers from A to… Z*, Birkhäuser, Boston, 2006, page 68). Therefore, it suffices to show that

\[
\frac{(y - x)z}{x(y - z)} \in \mathbb{R}. \tag{1}
\]

Let \( AY = \lambda \cdot AB \); that is \( y = a - \lambda(a + 1) \), where \( \lambda \in \mathbb{R} \). The fact that \( B, C, Y, Z \) are concyclic implies that \( \triangle ABC \sim \triangle AZY \), hence

\[ AZ = \frac{AB^2}{AC^2} \cdot \frac{AY}{AB} \cdot AC; \]

that is, \( z = a - \frac{|a+1|^2}{|a-1|^2} \lambda(a-1) \). By Ceva’s theorem we see that

\[ \frac{BY}{AY} \cdot \frac{AZ}{CZ} = \frac{XZ}{XC}, \]

that is \( x + 1 = \frac{(y + 1)(z - a)}{(y - a)(z - 1)} \cdot (1 - x) \), and thus

\[ x = \frac{|a + 1|^2 - |a - 1|^2}{|a + 1|^2 + |a - 1|^2 - 2 \lambda |a + 1|^2}. \]

Substituting these expressions into the fraction in (1) and clearing real factors yields the expression

\[
(2 \lambda^2 |a + 1|^2(a + 1) - \lambda (|a + 1|^2(3a + 1) + |a - 1|^2(a + 1)) + |a + 1|^2(a - 1) + |a - 1|^2(a + 1)) \\
\cdot (\lambda |a + 1|^2(a - 1) - |a - 1|^2a) \cdot (\overline{\lambda} - 1) \cdot \left( \frac{\lambda - 1}{a + 1} \right).
\]

We rewrite this last expression as \( f_3(a) \lambda^3 + f_2(a) \lambda^2 + f_1(a) \lambda + f_0(a) \), where

\[
\begin{align*}
    f_3(a) &= 2 |a + 1|^4 |a - 1|^2, \\
    f_2(a) &= |a + 1|^2 |a - 1|^2((a - 1)^2 - 5|a|^2 - a - a) - 2a - 2\overline{a} - 1 + |a + 1|^2 |a - 1|^2, \\
    f_1(a) &= |a - 1|^2(5|a|^2 - 2a - a - a - 2\overline{a} - 2a - 2\overline{a} - 1 + |a + 1|^2 |a - 1|^2), \\
    f_0(a) &= -2 |a|^2 |a + 1|^2 |a - 1|^2. 
\end{align*}
\]

Each of the four coefficients above is a real number, and so is \( \lambda \). This completes the proof of the relation (1).

That completes the Corner for this issue. Send me your nice solutions and generalizations.