SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We belatedly acknowledge a correct solution to #3340 by “Solver X”, dedicated to the memory of Jim Totten, which we had previously classified as incorrect. Our apologies.


For \( a \in \mathbb{R} \) define a sequence \( (x_n) \) by \( x_0 = a \) and \( x_{n+1} = 4x_n - x_n^2 \) for all \( n \geq 0 \). Prove that there exist infinitely many \( a \in \mathbb{R} \) such that the sequence \( (x_n) \) is periodic.

Similar solutions by George Apostolopoulos, Messolonghi, Greece and Michel Bataille, Rouen, France.

For each positive integer \( p \), let \( \theta_p = \frac{2\pi}{2p - 1} \) and \( a_p = 2(1 - \cos \theta_p) \).

Clearly, \( a_j \neq a_k \) for \( j \neq k \), so it suffices to show that the sequence \( (x_n) \) is periodic for \( a = a_p \). If \( x_0 = a_p \), then \( x_n = 2[1 - \cos(2^n\theta_p)] \) holds for \( n = 0 \). Moreover, if we assume \( x_n = 2[1 - \cos(2^n\theta_p)] \) to be true, then using the formula \( 2 \cos^2 y = 1 + \cos 2y \), we have

\[
x_{n+1} = 8[1 - \cos(2^n\theta_p)] - 4[1 - \cos(2^n\theta_p)]^2 = 4 - 4 \cos^2(2^n\theta_p) = 2[1 - \cos(2^{n+1}\theta_p)].
\]

Thus, for all nonnegative integers \( n \), we have \( x_n = 2[1 - \cos(2^n\theta_p)] \), and so

\[
x_{n+p} - x_n = 2[\cos(2^n\theta_p) - \cos(2^{n+p}\theta_p)] = 4 \sin(2^{n-1}\theta_p(2^p - 1)) \sin(2^{n-1}\theta_p(2^p + 1)) = 0,
\]

where the latter equality follows from \( \sin(2^{n-1}\theta_p(2^p - 1)) = \sin(2^n\pi) = 0 \). This shows that \( (x_n) \) is \( p \)-periodic.

Also solved by ARKADY ALI, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinnengymnasium, Innsbruck, Austria; MADHAV R. MODAK, formerly of Sir Parshurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Peter Y. Woo, Biola University, La Mirada, CA, USA sketched the iterates of the function \( f(x) = 4x - x^2 \) and indicated that periodic points are found by intersecting the graph of the line \( y = x \) with the graphs of these iterates, the \( n^{th} \) iterate having \( 2^{n-1} \) "bumps" on it. The function \( f(x) = 4x - x^2 \) is known as the logistic function, and its dynamics have been extensively studied. For example, see chapter 10 of the book by Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe, Chaos and Fractals: New Frontiers of Science, 2nd ed., Springer.
Proposed by Mihály Bencze, Brașov, Romania.

Prove that if $A$, $B$, $C$, and $D$ are the solutions of

$$X^2 = \begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix},$$

then $A^{2007} + B^{2007} + C^{2007} + D^{2007} = O$, where $O$ is the $2 \times 2$ zero matrix.

Solution by Oliver Geipel, Briühl, NRW, Germany, modified by the editor.

We generalize as follows: Let $M \in M_2(\mathbb{C})$ have two distinct nonzero eigenvalues in $\mathbb{C}$. Then the equation $X^2 = M$ has exactly four distinct solutions $A$, $B$, $C$, $D \in M_2(\mathbb{C})$ and moreover, if $m$ is any odd positive integer, then $A^m + B^m + C^m + D^m = 0$.

Proof. First we show that if $M$ is a diagonal matrix with distinct diagonal entries, then any solution to $X^2 = M$ is also a diagonal matrix. Indeed, let

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{pmatrix},$$

with $\lambda \neq \mu$. It follows immediately that $a^2 \neq d^2$, hence $a+d \neq 0$. Thus $b = c = 0$.

Secondly, if $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^2$, then $a^2 = \lambda$ and $d^2 = \mu$. Thus, if $\omega$ and $\sigma$ are choices of square roots of $\lambda$ and $\mu$, the equation $X^2 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ has exactly four distinct roots: $\begin{pmatrix} \pm \omega & 0 \\ 0 & \pm \sigma \end{pmatrix}$. [Ed: Recall that $\lambda$ and $\mu$ are nonzero.]

Now let $M$ be any $2 \times 2$ matrix with distinct nonzero eigenvalues $\lambda$, $\mu$. Then, there exists an invertible matrix $V$ such that $V^{-1}MV = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

It is easy to see that $X$ is a solution to $X^2 = M$ if and only if $Y = V^{-1}XV$ is a solution to $Y^2 = V^{-1}MV = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. By the first part it follows that $X^2 = M$ has exactly four solutions, namely

$$V \begin{pmatrix} \pm \omega & 0 \\ 0 & \pm \sigma \end{pmatrix} V^{-1}.$$

Then

$$A^m + B^m + C^m + D^m = V \begin{pmatrix} 2\omega^m - 2\omega^{m} & 0 \\ 0 & 2\sigma^m - 2\sigma^{m} \end{pmatrix} V^{-1} = 0.$$
ZARNOWSKI, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; REBECCA EVERDING, student, Southeast Missouri State University, Cape Girardeau, MO, USA; CODY GUINAN, student, Southeast Missouri State University, Cape Girardeau, MO, USA; WALTHER JANOS, Ursulinegymnasium, Innsbruck, Austria; WILLIAM McNEARY, Charleston, MO, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; NANCY MUELLER and SETH STAHLEBER, Southeast Missouri State University, Cape Girardeau, MO, USA; JENNIFER PAJDA, student, Southeast Missouri State University, Cape Girardeau, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incomplete solution submitted.

Some results observed that the problem is solved once it is known that $X^2 = M$ has exactly four roots, for then the roots can be grouped in pairs $\pm X$.

Barbara remarked that in general if $R$ is a unitary ring where 2 is not a zero divisor, $N$ is a positive odd integer, and $S = \{x \in R : x^2 = 0\}$ is finite for some $\theta \in R$, then $\sum_{x \in S} x^N = 0$.


Let $ABCD$ be a convex quadrilateral such that $AC$ and $BD$ intersect in right angles at $P$, and let $I$, $J$, $K$, and $L$ be the midpoints of $AB$, $BC$, $CD$, and $DA$, respectively. Show that the circles $(PIJ)$, $(PJK)$, $(PKL)$, and $(PLI)$ are congruent if and only if $ABCD$ is cyclic.

1. Solution by Václav Konečný, Big Rapids, MI, USA.

The midpoint quadrilateral $IJKL$ has sides parallel to the diagonals of $ABCD$, whence it is a rectangle. Because $ABCD$ is convex, the point $P$ lies inside the rectangle. Moreover, because $I$ and $J$ are the midpoints of $BA$ and $BC$, the line $IJ$ is the perpendicular bisector of $BP$ and, therefore, $\angle PIJ = \angle JIB$. Since $IJ \parallel AC$, we have $\angle JIB = \angle CAB$, whence

$$\angle PIJ = \angle CAB.$$  

Similarly, $JK$ is the perpendicular bisector of $CP$ and

$$\angle PKJ = \angle CDB,$$

because they both equal $\angle CKJ$. Furthermore, since $P$ is inside the rectangle (so that $\angle PIJ < \angle LJJ$ and $\angle PKJ < \angle LKJ$), both $\angle PIJ$ and $\angle PKJ$ are acute. Because two triangles with a common side that subtend acute angles have equal circumradii if and only if those angles are equal, we deduce that

$$\angle PIJ = \angle PKJ \iff (PIJ) \text{ and } (PJK) \text{ are congruent.}$$

Because $ABCD$ is convex, the vertices $A$ and $D$ lie on the same side of the line $BC$, whence

$$\angle CAB = \angle CDB \iff ABCD \text{ is cyclic.}$$
We conclude that if just the two circles \((PIJ)\) and \((PJK)\) are congruent, then \(ABCD\) must be cyclic. Because there was nothing special about the circumcircles \((PIJ)\) and \((PJK)\), we have as a converse, if \(ABCD\) is cyclic, then any two consecutive circumcircles of the chain \((PIJ)\), \((PJK)\), \((PKL)\), \((PLI)\) are congruent and, consequently, all four are congruent.

II. Solution by D.J. Smeenk, Zaltbommel, the Netherlands, expanded by the editor.

The circles \((PIJ)\), \((PJK)\), \((PKL)\), and \((PLI)\) are the nine-point (or Feuerbach) circles of triangles \(ABC\), \(BCD\), \(CDA\), and \(DAB\), respectively (because each contains the midpoints of two sides and the foot of an altitude); thus if \(A\), \(B\), \(C\), \(D\) are four points in any order on a circle for which \(AC\) and \(BD\) intersect orthogonally at \(P\), the radius of each of the nine-point circles equals half the radius of the large circle. The converse is not so obvious—without using the convexity of \(ABCD\), all we can conclude from the congruence of the circles \((PIJ)\), \((PJK)\), \((PKL)\), and \((PLI)\) is that the circles \((ABC)\), \((BCD)\), \((CDA)\), and \((DAB)\) are themselves congruent. Indeed, if two from the latter set of four circles are distinct, then these four circles are related as in TiTeica’s theorem: If three congruent circles pass through a common point, then their other three intersection points lie on a fourth circle of the same radius and, moreover, the four intersection points form an orthocentric quadrilateral (meaning each point is the orthocentre of the triangle formed by the other three). (See problem 3337 [2009: 191-192], where that theorem is discussed and references are provided.) Thus, either the points \(A\), \(B\), \(C\), \(D\) form an orthocentric quadrilateral, or they must lie on a circle. Since an orthocentric quadrilateral can never be convex (exactly one of the four triangles formed by three of the four points must be acute with the fourth point—the orthocentre—inside), we see that convexity forces \(ABCD\) to be cyclic.

Also solved by GEORGE APSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University; Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALther JANOUS, Ursulinen gymnasium, Innsbruck, Austria; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bay side, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănesti, Romania; and the proposer.


Let \(A\), \(B\), \(C\), \(D\), and \(E\) be concyclic with \(V\) and \(W\) on the lines \(AB\) and \(AD\), respectively. Show that if the line \(CE\), the parallel to \(CB\) through \(V\), and the parallel to \(CD\) through \(W\) are concurrent, then triangles \(EVB\) and \(EWD\) are similar. Does the converse hold?
Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

[Ed.: Note that the problem has been carefully worded to allow the points A through E to be any five distinct points on a circle (in any order). Because of this, we must use directed angles to avoid numerous special cases.] Let P be the point where the parallel to CB through V intersects the parallel to CD through W. We shall prove a modified version of the result: P lies on CE if and only if the triangles EVB and EWD are directly similar. Because V ∈ BA, W ∈ DA, and A, B, D, and E lie on a circle, we have

\[ \angle EBV = \angle EBA = \angle EDA = \angle EDW. \]  \hspace{1cm} (1)

Furthermore, because C lies on that same circle, \( \angle CBA = \angle CDA \). However, because \( PV \parallel CB \) and \( PW \parallel CD \), we have \( \angle PVA = \angle CBA \) and \( \angle PVA = \angle PWA \), whence \( \angle PVA = \angle PWA \) and, therefore, the points A, V, P, and W also lie on a circle. Note that “P lies on CE” means that E is on the transversal CP of the parallel lines WP and CD. Thus,

\[ P \in CE \iff \angle DCE = \angle WPE. \]

But, \( \angle DCE = \angle DAE = \angle WAE \), whence

\[ P \in CE \iff E \text{ lies on the circle containing } A, V, P, W. \]

This, in turn, is equivalent to \( \angle AWE = \angle AVE \). Since \( \angle DWE = \angle AWE \) and \( \angle BVE = \angle AVE \), we deduce finally that

\[ P \in CE \iff \angle DWE = \angle BVE. \]  \hspace{1cm} (2)

From (1) and (2) we now have two pairs of equal corresponding angles, which completes the proof that P lies on CE if and only if the triangles EVB and EWD are directly similar.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GÜPEL, Brühl, NRW, Germany; VACLAV KONEČNÝ, Big Rapids, MI, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer, who provided two solutions.

Note that in the statement of the problem the triangles are only required to be similar, not directly similar as all but one of the solvers assumed. The exception was Konečný, who joked that it was obvious that the converse does not hold because of the way the question was worded. He and Bataille both provided counterexamples that require a picture. For an easier counterexample, start by choosing the points A and E on opposite ends of a diameter, and complete the configuration with \( P \in CE \). Then, from the featured solution, we know that the triangles EVB and EWD are directly similar; moreover they have right angles at the corresponding vertices B and D. Reflect V in the line EB to the point \( V' \). Then \( EV'B \) and EVB are oppositely oriented congruent triangles, so that the triangles \( EV'B \) and EWBD are, indeed, similar triangles; however, the parallel to CB through \( V' \) is different from the parallel to CB through V so that it could not intersect the parallel to CD through W at a point of the line CE.

Let $ABC$ be a triangle with $a = BC$, $b = AC$, $c = AB$, and semiperimeter $s$. Prove that

$$
\frac{y + z}{x} \cdot \frac{A}{a(s - a)} + \frac{z + x}{y} \cdot \frac{B}{b(s - b)} + \frac{x + y}{z} \cdot \frac{C}{c(s - c)} \geq \frac{9\pi}{s^2},
$$

where the angles $A$, $B$, and $C$ are measured in radians and $x$, $y$, and $z$ are any positive real numbers.

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

By the AM–GM Inequality we have $\frac{y + z}{2} \geq \sqrt{y z}$, $\frac{z + x}{2} \geq \sqrt{z x}$, and $\frac{x + y}{2} \geq \sqrt{x y}$; hence $(y + z)(z + x)(x + y) \geq 8xyz$. Applying the AM–GM Inequality once more and then using the preceding inequality, we have

$$
\begin{align*}
\frac{y + z}{x} \cdot \frac{A}{a(s - a)} + \frac{z + x}{y} \cdot \frac{B}{b(s - b)} + \frac{x + y}{z} \cdot \frac{C}{c(s - c)} & \geq 3 \sqrt[3]{\frac{(y + z)(z + x)(x + y)}{xyz} \cdot \frac{ABC}{abc(s - a)(s - b)(s - c)}} \\
& \geq 6 \sqrt[6]{\frac{ABC}{abc(s - a)(s - b)(s - c)}}.
\end{align*}
$$

(1)

Now, using the inequality $\prod (\frac{3A}{\pi}) \geq \frac{2r}{R}$ (see 6.59, p. 188 of [1]) and $2s^2 \geq 27Rr$ (see 5.12, p. 52 of [2]); and also using the well-known relations $abc = 4RF$, $F = rs$, and $F^2 = s(s - a)(s - b)(s - c)$, where $F$ is the area of triangle $ABC$, we have

$$
\frac{ABC}{abc(s - a)(s - b)(s - c)} \geq \frac{(2r \pi^3)}{27R} \cdot \frac{r s^2}{s} = \frac{\pi^3}{54R^2 r^2 s^2}
$$

$$
\geq \frac{\pi^3}{54s^2} \cdot \frac{27^2}{48^2} = \frac{27\pi^3}{8s^6}.
$$

(2)

The desired inequality now follows directly from (1) and (2). Equality holds if and only if $A + B + C = \pi$ and $x = y = z$.

Also solved by WALTHER JANOUS, Ursulinsgymnasium, Innsbruck, Austria; and the proposer.

References


Let $ABCD$ be a tetrahedron with $h_A$ and $m_A$ the lengths of the altitude and the median from vertex $A$ to the opposite face $BCD$, respectively. If $V$ is the volume of the tetrahedron, prove that

$$(h_A + h_B + h_C + h_D)(m_A^2 + m_B^2 + m_C^2 + m_D^2) \geq \frac{128}{\sqrt{3}}V.$$ 

Solution by Oliver Geupel, Brihl, NRW, Germany.

First we consider the sum of altitudes. Let $[XYZ]$ denote the area of triangle $XYZ$. By the AM–HM inequality we obtain

$$h_A + h_B + h_C + h_D \geq \frac{16}{3V} \left( \frac{1}{[BCD]} + \frac{1}{[CDA]} + \frac{1}{[DAB]} + \frac{1}{[ABC]} \right).$$

Next we compute the median. By Weitzenböck’s inequality we have $BC^2 + CD^2 + DB^2 \geq 4\sqrt{3}[BCD]$, with similar inequalities for the other three faces, hence,

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 \geq 2\sqrt{3}([ABC] + [BCD] + [CDA] + [DAB]).$$

We also have

$$9m_A^2 = \left( \frac{\overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D} - 3\overrightarrow{A}}{3} \right)^2 = 9 \left( \frac{1}{3} \left( \overrightarrow{B} - \overrightarrow{A} \right)^2 + \frac{1}{3} \left( \overrightarrow{C} - \overrightarrow{A} \right)^2 + \frac{1}{3} \left( \overrightarrow{D} - \overrightarrow{A} \right)^2 \right) - \frac{1}{3} \left( \overrightarrow{C} - \overrightarrow{B} \right)^2 - \frac{1}{3} \left( \overrightarrow{D} - \overrightarrow{C} \right)^2 - \frac{1}{3} \left( \overrightarrow{B} - \overrightarrow{D} \right)^2 = 3 (AB^2 + AC^2 + AD^2) - BC^2 - CD^2 - DB^2,$$

with cyclic variants holding for the other three medians. Therefore,

$$9 \left( m_A^2 + m_B^2 + m_C^2 + m_D^2 \right) = 4 (AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2) \geq 8\sqrt{3}([ABC] + [BCD] + [CDA] + [DAB]).$$

Finally we put everything together:

$$(h_A + h_B + h_C + h_D)(m_A^2 + m_B^2 + m_C^2 + m_D^2) \geq 48V \cdot \frac{8\sqrt{3}}{9} = \frac{128}{\sqrt{3}}V.$$
This completes the proof. Equality holds if and only if the tetrahedron is regular, as can be seen from the condition for equality in Weitzenböck's inequality.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHALY BENZE, Brașov, Romania (two solutions); ÇAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLEH FAYNSHTEYN, Leipzig, Germany; WALTHE JANOUS, Ursulinegymnasium, Innsbruck, Austria; TITU ZVONARU, Comanesti, Romania; and the proposer. One incomplete solution was submitted.

By employing the Hadwiger-Finsler Inequality, Bence proved that

\[(h_A + h_B + h_C + h_D)(m_A^2 + m_B^2 + m_C^2 + m_D^2) \geq \frac{128}{\sqrt{3}} V + \frac{32r}{9} \sum (AB - AC)^2,\]

where \(r\) is the inradius of the tetrahedron and the sum is over the 12 pairs of edges of the tetrahedron that share a common vertex.

Janous proved that \((h_A h_B h_C h_D)^{1/4}(m_A^2 + m_B^2 + m_C^2 + m_D^2) \geq \frac{32}{\sqrt{3}} V\), from which the proposed inequality follows on account of the AM-GM Inequality.

3395. [2008 :484, 486] Proposed by Taichi Maekawa, Takatsuki City, Osaka, Japan.

Let triangle \(ABC\) have orthocentre \(H\) and circumradius \(R\). Prove that

\[4R^3 - (l^2 + m^2 + n^2)R - lmn = 0,\]

where \(AH = l, BH = m,\) and \(CH = n.\)

Solution by Michel Bataille, Rouen, France.

The relation \(4R^3 - (l^2 + m^2 + n^2)R - lmn = 0\) holds only for acute-angled or right-angled triangles. We show, more generally, that

\[4R^3 - (l^2 + m^2 + n^2)R - elmn = 0,\]  

(1)

where \(e = -1\) if one of the angles of \(\triangle ABC\) is obtuse and \(e = +1\) otherwise.

Let \(O\) be the circumcentre of \(\triangle ABC\) and \(A', B', C'\) the midpoints of the sides \(BC, CA, AB,\) respectively. It is well known and easy to prove that \(l = 2OA', m = 2OB',\) and \(n = 2OC'.\)

If \(A \leq 90^\circ,\) then \(\angle BOC = 2A\) and \(OA' = R \cos A;\) but if \(A > 90^\circ,\)

then we have \(\angle BOC = 360^\circ - 2A\) and \(OA' = R \cos(180^\circ - A) = -R \cos A.\)

Thus,

\[elmn = 8R^3 \cos A \cos B \cos C.\]

Using the well-known trigonometric identity for the angles of a triangle,

\[\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C,\]

we obtain

\[(l^2 + m^2 + n^2)R = 4R^3(\cos^2 A + \cos^2 B + \cos^2 C),\]

\[= 4R^3(1 - 2 \cos A \cos B \cos C).\]

The relation (1) follows.
Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SÉFRET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MIHÁLY BENČÉ, Brașov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, US; OLEH FAYNŠTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; JOEL SCHLOSSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comanesti, Romania; and the proposer.

Alt, Bataille, Geupe, Janous, Schlossberg, and Zvonaru each noted that the extra condition is required and either suggested adding a hypothesis or provided a counterexample for obtuse-angled triangles. The other solvers assumed the triangle is acute-angled.


Let \( n \) be a positive integer, and for \( i, j, \) and \( k \) in \( \{1, 2, \ldots, n\} \) let

\[
a_{ijk} = 1 + \text{mod}(k - i + j - 1, n) + n \text{mod}(i - j + k - 1, n) + n^2 \text{mod}(i + j + k - 2, n),
\]

where \( \text{mod}(a, n) \) is the residue of \( a \) modulo \( n \) in the range \( 0, 1, \ldots, n - 1 \). For which \( n \) is the cube with entries \( a_{ijk} \) a magic cube? (Here "magic" means that the sum of \( a_{ijk} \) is constant if both indices are fixed and the third index varies, and also the sums along the great diagonals of the cube are equal to this constant.)

Solution by Oliver Geupe, Brühl, NRW, Germany, modified by the editor.

We will prove that the cube is magic for all \( n \) and that the magic sum is

\[
S_n = \frac{n(n^3 + 1)}{2}.
\]

Let \( L_n = \{0, 1, 2, \ldots, n - 1\} \), and for convenience write

\[
b_{ijk} = k - i + j - 1, \quad c_{ijk} = i - j + k - 1, \quad \text{and} \quad d_{ijk} = i + j + k - 2.
\]

Then

\[
a_{ijk} = 1 + \text{mod}(b_{ijk}, n) + n \text{mod}(c_{ijk}, n) + n^2 \text{mod}(d_{ijk}, n).
\]

For fixed \( j \) and \( k \), the \( n \) numbers \( b_{ijk}, 1 \leq i \leq n \), are pairwise incongruent modulo \( n \), since \( k - i_1 + j - 1 \equiv k - i_2 + j - 1 \) (mod \( n \)) implies \( i_1 = i_2 \). Hence, \( \{\text{mod}(b_{ijk}, n) : 1 \leq i \leq n\} = L_n \). Similarly, \( \{\text{mod}(c_{ijk}, n) : 1 \leq i \leq n\} = L_n \).

It follows that

\[
\sum_{i=1}^{n} a_{ijk} = n + (1 + n + n^2) \sum_{l=0}^{n-1} l = n + (1 + n + n^2) \frac{n(n - 1)}{2} = n + \frac{n(n^3 - 1)}{2} = \frac{n(n^3 + 1)}{2} = S_n.
\]
Analogously, $\sum_{j=1}^{n} a_{ijk} = S_n$ and $\sum_{k=1}^{n} a_{ijk} = S_n$.

Now we compute the four diagonal sums. Since $b_{iii} = c_{iii} = i - 1$, we have $\{\text{mod}(b_{iii}, n) : 1 \leq i \leq n\} = \{\text{mod}(c_{iii}, n) : 1 \leq i \leq n\} = L_n$. Hence, $\sum_{i=1}^{n} \text{mod}(b_{iii}, n) = \sum_{i=1}^{n} \text{mod}(c_{iii}, n) = \sum_{i=1}^{n-1} l = \frac{n(n-1)}{2}$.

Next, we have $d_{iii} = 3i - 2$. If $3 \nmid n$, then $\{\text{mod}(d_{iii}, n) : 1 \leq i \leq n\} = L_n$ so $\sum_{i=1}^{n} \text{mod}(d_{iii}, n) = \frac{n(n-1)}{2}$.

If $3\mid n$, then $n = 3m$ for some positive integer $m$. Hence,

$$\sum_{i=1}^{n} \text{mod}(d_{iii}, n) = \sum_{i=1}^{n} \text{mod}(3i - 2, n) = 3 \sum_{i=1}^{m} (3i - 2) = 3 \left( \frac{3m(m+1)}{2} - 2m \right) = \frac{3m(3m-1)}{2} = \frac{n(n-1)}{2},$$

as before.

Therefore, the sum along the main diagonal is

$$\sum_{i=1}^{n} a_{iii} = \sum_{i=1}^{n} (1 + \text{mod}(b_{iii}, n) + n \text{mod}(c_{iii}, n) + n^2 \text{mod}(d_{iii}, n)) = n + (1 + n + n^2) \frac{n(n-1)}{2} = \frac{n(n^3+1)}{2} = S_n.$$

Next we consider the diagonal $\{(n+1-i, i, i) : 1 \leq i \leq n\}$. For notational convenience, let $i^* = n + 1 - i$. Then

$$b_{i^*ii} = i - (n+1-i) + i - 1 = 3i - n - 2,$$
$$c_{i^*ii} = (n+1-i) - i + i - 1 = -i + n,$$
$$d_{i^*ii} = (n+1-i) + i + i - 2 = i + n - 1.$$

Hence,

$$\sum_{i=1}^{n} a_{i^*ii} = \sum_{i=1}^{n} (1 + \text{mod}(3i - n - 2, n) + n \text{mod}(-i + n, n) + n^2 \text{mod}(i + n - 1, n))
= n + (1 + n + n^2) \sum_{i=0}^{n-1} l = \frac{n(n^3+1)}{2} = S_n.$$
Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA, and the proposer. There was one partly incorrect solution. The proposer and the editor had tacitly assumed that the entries of the cube must be 1, 2, \ldots, n^3 in which case the cube would be magic if and only if n is odd. This was proved by the proposer. Apparently, neither Curtis nor Geupel made this assumption in their solutions.


Evaluate

$$\lim_{n \to \infty} \frac{1}{n^2} \int_0^n \frac{\sqrt{n^2 - x^2}}{2 + x^{-a}} \, dx.$$  

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let I denote the above integral. Making the substitution \(x = nt\), we have

$$\frac{1}{n^2} I = \int_0^1 \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}} \, dt \tag{1}$$

Observe that for \(0 < t < 1\), we have

$$0 < \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}} < \frac{1}{2}, \tag{2}$$

and for \(\sqrt{n} < t < 1\) we have \(\sqrt{n} < nt\) and \(t^n < 1\), so that

$$\frac{1}{2 + \sqrt{n}(\sqrt{n})} < \frac{1}{2 + (nt)^{-nt}} < \frac{1}{2 + n^{-n}}. \tag{3}$$

From (2) we obtain

$$0 < \int_0^{\sqrt{n}} \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}} \, dt < \frac{1}{2\sqrt{n}} \tag{4}$$

and from (3) we obtain

$$\frac{1}{2 + \sqrt{n}(\sqrt{n})} \int_0^{\sqrt{n}} \sqrt{1-t^2} \, dt \quad < \quad \int_0^{\sqrt{n}} \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}} \, dt \quad < \quad \frac{1}{2 + n^{-n}} \int_0^{\sqrt{n}} \sqrt{1-t^2} \, dt. \tag{5}$$
Letting \( n \) tend to infinity in (5), we see that the integral in the middle approaches \( \frac{1}{2} \int_0^1 \sqrt{1-t^2} \, dt \). Since \( \int_0^1 \sqrt{1-t^2} \, dt = \frac{\pi}{4} \), we obtain from (1), (4), and (5) that
\[
\lim_{n \to \infty} \frac{1}{n^2} \int_0^n \frac{\sqrt{n^2 - x^2}}{2 + x^{-x}} \, dx = 0 + \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}.
\]

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One solution was submitted with an incomplete justification of the calculations.

3398. [2008: 485, 487] Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Given the equation
\[
\left[ \frac{n}{10} \right] + \left( n - 10 \left[ \frac{n}{10} \right] \right) \cdot 10^{\log_{10} n} = \frac{2n}{3},
\]
(a) show that \( n = 5294117647058823 \) is a solution,

(b) \( \star \) find all other positive integer solutions of the equation.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We will prove that all solutions are given by
\[
n = \frac{3b(10^{16k} - 1)}{17},
\]
where \( k \) is a positive integer and \( b \) is any integer with \( 1 \leq b \leq 5 \).

The solution in (a) is obtained by setting \( k = 1 \) and \( b = 3 \).

Each positive integer \( n \) has an unique representation \( n = 10a + b \), where \( a \) and \( b \) are nonnegative integers and \( b \leq 9 \). Let \( l + 1 \) be the number of decimal digits of \( n \). It is easy to check that there is no solution with \( n \leq 9 \), hence we can assume that \( l \geq 1 \). Then, the number \( n \) is a solution if and only if
\[
a + 10^l b = \frac{2}{3} (10a + b),
\]
or equivalently
\[
a = \frac{(3 \cdot 10^l - 2)b}{17},
\]
where
\[
10^{l-1} \leq a < 10^l \quad \text{and} \quad 0 \leq b \leq 9.
\]

For \( l = 1 \) we get \( 28b = 17a \), which is impossible because 17 is not a divisor of \( 28b \) unless \( b \) is zero. Therefore, we have \( l \geq 2 \). The inequality involving \( a \) will now be satisfied if and only if \( 1 \leq b \leq 5 \).
\[ \text{Ed: } 10^l - 1 \leq a < 10^l \iff 10^{l-1} \leq (3 \cdot 10^l - 2)b \leq 10^l - 1 \iff 17 \cdot 10^{l-1} - (3 \cdot 10^l - 2)b \leq 17 \cdot (10^l - 1). \text{ Now } 17 \cdot 10^{l-1} < 30 \cdot 10^{l-1} - 2 \text{ because } l \geq 2, \text{ and since } b \text{ is a nonnegative integer the first inequality is equivalent to } b \geq 1. \]

The second inequality becomes \( b \leq \frac{17 \cdot (10^l - 1)}{3 \cdot 10^l - 2} = 5 + \left( \frac{2 \cdot 10^l - 7}{3 \cdot 10^l - 2} \right). \]

Since \( b \) is an integer, this is equivalent to \( b \leq 5. \]

Thus, \( n \) is a solution to the given equation if and only if

\[ 3 \cdot 10^l \equiv 2 \pmod{17} \tag{1} \]

and \( 1 \leq b \leq 5. \]

By Fermat's Little Theorem, \( 10^{16+l} \equiv 1 \pmod{17} \), and an easy computation shows that the only solution \( l \) of (1) with \( 1 \leq l \leq 16 \) is \( l = 15. \]

Thus, (1) is equivalent to \( l \equiv 1 \pmod{16} \), or \( l + 1 = 16k. \]

Putting everything together, we have that \( n \) is a solution if and only if

\[ n = 10a + b = 10 \cdot \left( \frac{3 \cdot 10^l - 2}{17} \right)b + b = \frac{3 \cdot (10^{l+1} - 3)}{17}b = \frac{3b(10^{16k} - 1)}{17}, \]

where \( \lceil \frac{3b}{17} \rceil \geq 1 \) and \( 1 \leq b \leq 5. \]

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and ALBERT STADLER, Hertrish, Switzerland.

Part (a) was also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENZKE, Brașov, Romania; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBŁAW, Walla Walla, WA, USA; WALther Janous, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete solution submitted.

The proposer remarked that \( \frac{3b}{17} \) is a decimal digit of the first solution to the equation are rotated five places to the right upon multiplication by \( \frac{3}{17}. \]

\[ \text{3399. [2008 : 485, 487]} \text{ Proposed by Vincentiu Rădulescu, University of Craiova, Craiova, Romania.} \]

Prove that there does not exist a positive, twice differentiable function \( f : [0, \infty) \to \mathbb{R} \) such that \( f(x)f''(x) + 1 \leq 0 \) for all \( x \geq 0. \]

Comments by Michel Bataille, Rouen, France; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Bataille indicated this problem was posed in the College Mathematics Journal by the same proposer in January, 2008. A solution and a generalization appeared in the January, 2009 issue of that journal (problem 869, pp. 60-61).
Janous indicated that this problem was also posed in the Swiss journal *Elemente der Mathematik* in the problem section of Heft 4, 2008, by the same proposer.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; APOSTOLIS K. DEMIS, Varavakesi High School, Athens, Greece; ALBERT STADLER, Herisberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.

Barbara showed that if \( f \) is negative and satisfies the given inequality, then \( -f \) is positive and also satisfies the inequality. Hence, the condition "\( f \) positive" can be omitted. On the other hand, by \( m \)-scaling, the number \( 1 \) can be replaced by \( e \). The result can be reformulated as: Let \( \epsilon > 0 \). There does not exist a twice differentiable function \( f : [0, \infty) \to \mathbb{R} \) such that \( f(x) f''(x) + \epsilon \leq 0 \) for all \( x > 0 \).

Barbara further observed that \( \epsilon \) cannot be omitted. The function \( f(x) = \log(x + 2) \) satisfies \( f(x) f''(x) < 0 \) for all \( x > 0 \). This leads to another reformulation of this result: let \( f : [0, \infty) \to \mathbb{R} \) be a twice differentiable function such that \( f(x) f''(x) < 0 \) (or \( \leq 0 \)) for all \( x > 0 \). Then, \( f(x) f''(x) \) tends to zero as \( x \) tends to infinity.

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**3400.** [2008: 485, 487] Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.

For positive integers \( m \) and \( k \) let \( (m)_k = m(1 + 10 + 10^2 + \cdots + 10^{k-1}) \), for example, \((1)_2 = 11\) and \((3)_4 = 3333\). Find all real numbers \( \alpha \) such that

\[
\left[ 10^n \sqrt{(1)_{2n}} + \alpha \right] = 33 - \left[ \frac{5 - 9\alpha}{6} \right]
\]

holds for each positive integer \( n \), where \([x]\) is the greatest integer not exceeding \( x \).

**Solution by the proposer.**

The answer for \( \alpha \) is the union of 16 closed and disjoint intervals:

\[
\alpha \in \bigcup_{m=-8}^{7} \left[ \frac{m^2 - 66m - 11}{100}, \frac{5 - 6m}{9} \right]. \quad (1)
\]

To prove this, we first let \( n = 1 \). Then

\[
\left[ 10\sqrt{11} + \alpha \right] = 33 - \left[ \frac{5 - 9\alpha}{6} \right]. \quad (2)
\]

Set \( m = \left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor \). Then

\[
m \leq \frac{5 - 9\alpha}{6} < m + 1,
\]

which is equivalent to

\[
\frac{6m + 1}{9} < \alpha \leq \frac{5 - 6m}{9}. \quad (3)
\]
From (2) we obtain $33 - m \leq 10\sqrt{11 + \alpha} < 34 - m$. Solving for $\alpha$ yields

$$\frac{m^2 - 66m - 11}{100} \leq \alpha < \frac{m^2 - 68m + 56}{100}. \quad (4)$$

It is easy to check that

$$\frac{5 - 6m}{9} < \frac{m^2 - 68m + 56}{100}$$

and

$$\frac{6m + 1}{9} < \frac{m^2 - 66m - 11}{100}.$$

Then, from (3) and (4), it follows that

$$\frac{m^2 - 66m - 11}{100} \leq \alpha \leq \frac{5 - 6m}{9}. \quad (5)$$

From (5) we obtain successively

$$\frac{m^2 - 66m - 11}{100} \leq \frac{5 - 6m}{9}, \quad (6)$$

$$\frac{(3m + 1)^2}{600}, \quad |3m + 1| \leq \left\lfloor \frac{10\sqrt{6}}{24} \right\rfloor = 24,$$

from which we see that inequality (6) is strict for $m = -8, -7, \ldots, 7$. Thus, the equality (2) may only hold for values of $\alpha$ satisfying the inequality (5) for $m = -8, -7, \ldots, 7$; these are precisely the values of $\alpha$ covered in (1).

We shall now prove that the given equality does indeed hold for these values of $\alpha$. We note that the given equality is equivalent to each of the following double inequalities:

$$(3)_{2n} - m \leq 10^n \sqrt{(1)_{2n} + \alpha} < (3)_{2n} - m + 1,$$

$$-m \leq 10^n \sqrt{(1)_{2n} + \alpha} - (3)_{2n} < -m + 1,$$

$$-m \leq 10^n \sqrt{\frac{10^{2n} - 1}{9} + \alpha} - \frac{10^{2n} - 1}{3} < -m + 1,$$

$$-1 - 3m \leq 10^n \sqrt{10^{2n} + 9\alpha - 1} - 10^{2n} < 2 - 3m,$$

$$-1 - 3m \leq \frac{10^n (9\alpha - 1)}{\sqrt{10^{2n} + 9\alpha - 1} + 10^n} < 2 - 3m,$$

$$-1 - 3 \left[ \frac{5 - 9\alpha}{6} \right] \leq \frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^{2n} + 1}} < 2 - 3 \left[ \frac{5 - 9\alpha}{6} \right]. \quad (7)$$
Note that the double inequality (7) holds for \( \alpha = \frac{1}{9} \), and that

\[
\frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^2}} + 1} \leq \frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^{2n}}} + 1} < \frac{9\alpha - 1}{2}
\]

holds (consider separately the cases \( \alpha > \frac{1}{9} \) and \( \alpha < \frac{1}{9} \)). Therefore, in order to prove (7), it suffices to show that

\[
-1 - 3 \left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor \leq \frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^2}} + 1}
\]

and

\[
\frac{9\alpha - 1}{2} \leq 2 - 3 \left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor .
\]

Inequality (8) can be written as

\[
m \geq -\frac{30\alpha + \sqrt{11 + \alpha}}{10 + 3\sqrt{11 + \alpha}},
\]

which simplifies to \( m \geq 33 - 10\sqrt{11 + \alpha} \). This last inequality is equivalent to the left inequality of (5). Inequality (9) simplifies to

\[
\left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor \leq \frac{5 - 9\alpha}{6},
\]

which is obvious.

Remark. The given equality holds asymptotically for any value of \( \alpha \). This can be easily proved by fixing \( \alpha \) and letting \( n \rightarrow \infty \) in inequality (7).

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany. One incomplete solution and two incorrect solutions were submitted.

The proposer mentioned that some problems on equalities involving the integer part function and a parameter had appeared earlier [1], [2].

References