Problem of the Month

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This month, we'll look at another problem that is accessible to talented younger students, but that can still give pause to those with more experience.

Problem (2009 Gauss Contest, Grade 8) A list of six positive integers $p, q, r, s, t, u$ satisfies $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. The sums of these 15 pairs of numbers are:

$$25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117.$$ 

Which sum equals $r + s$?

(A) 52    (B) 54    (C) 63    (D) 68    (E) 76

Now, we can do this the hard way, or we can do this the easy way. Of course, the hard way is actually easier than the easy way (which is pretty hard). Got that? Great – let’s begin with the easy way.

Solution 1 The Big Idea: Because $p, q, r, s, t,$ and $u$, are given to us in order of size, we should actually be able to deduce which sum corresponds with which pair in at least a few cases; if we’re lucky, we’ll be able match enough sums to pairs to determine which sum equals $r + s$. (Of course, since $r$ and $s$ are the “middle” numbers, then we’d expect their sum to be pretty much in the middle, so 63 is probably a good guess; since this is a multiple choice problem, 63 is almost certain to be the wrong answer.)

Let’s get started. Can we tell which of the pairs should give the smallest sum? Yes! Since $p$ and $q$ are the smallest two numbers in the list, their sum must be the smallest of all the sums. In other words, $p + q = 25$.

Can we tell which of the pairs should give the largest sum? Yes! Since $t$ and $u$ are the largest two numbers in the list, then their sum should be the largest of all the sums. In other words, $t + u = 117$.

Next, we’ll use the facts that $q = 25 - p$ and $t = 117 - u$ to do a little housekeeping and reduce the number of unknowns in our list to four. We thus rewrite our list as $p, 25 - p, r, s, 117 - u, u$.

Are there other pairs that we can track down easily? We had some success starting at the ends of the list of pairs, so let’s keep trying to work inwards from the ends of the list of sums. Can we tell which of the pairs should give the second smallest sum? It’s not $p + q$, since we’ve already used this one. We might guess $q + r$ or $p + r$. In fact, $p + r$ is smaller than
\( q + r \), which we can actually deduce by using a chart:

\[
\begin{array}{ccc}
  p + q & q + r \\
  p + r & q + s & r + s \\
  p + s & q + t & r + t & s + t \\
  p + u & q + u & r + u & s + u & t + u \\
\end{array}
\]

In the chart, when we move down a fixed column, the sums increase since the first summand stays the same and the second increases; when we move to the right along a fixed row, the sums increase because the first summand increases and the second stays the same. Therefore, \( p + r \) is smaller than \( q + r \) (since it's immediately to the left) and is in fact smaller than any other sum except \( p + q \). Therefore, \( p + r \) is the second smallest sum, so \( p + r = 30 \).

Similarly, the second largest sum must be \( s + u \), so \( s + u = 103 \).

We can now do some more housekeeping, writing \( r = 30 - p \) and \( s = 103 - u \) to make our list \( p, 25 - p, 30 - p, 103 - u, 117 - u, u \).

Now we come to a fork in the road. The third smallest total, according to our snazzy chart, is either \( p + s \) or \( q + r \). (Can you see why?) So either \( p + s = p + 103 - u = 38 \) (that is, \( p = u - 65 \)) or \( q + r = 25 - p + 30 - p = 38 \).

In the second case, \( 55 - 2p = 38 \) or \( 2p = 17 \). But \( p \) is an integer, so this is not the case. Therefore, \( p = u - 65 \), and our list becomes

\[
\begin{align*}
  &p = u - 65, \\
  &q = 90 - u, \\
  &r = 95 - u, \\
  &s = 103 - u, \\
  &t = 117 - u, \\
  &u.
\end{align*}
\]

We have written each unknown in terms of one variable, which is undoubtedly a good thing.

Do you feel like we’re zeroing in on the answer? (I’m not so sure myself!) Let’s consolidate. We can at this point calculate some of the pairs, since we’ve got each unknown in terms of \( u \). We can match up any unknown whose expression includes a “-u” with any unknown whose expression includes a “-u” to actually get a numerical sum:

\[
\begin{align*}
  &p + q = (u - 65) + (90 - u) = 25; \\
  &q + u = (90 - u) + u = 90; \\
  &p + r = (u - 65) + (95 - u) = 30; \\
  &r + u = (95 - u) + u = 95; \\
  &p + s = (u - 65) + (103 - u) = 38; \\
  &s + u = (103 - u) + u = 103; \\
  &p + t = (u - 65) + (117 - u) = 52; \\
  &t + u = (117 - u) + u = 117.
\end{align*}
\]

Let’s redraw our chart, but this time expressing the remaining pairs in terms of \( u \) only:

\[
\begin{array}{cccc}
  25 & 185 - 2u & 193 - 2u & 198 - 2u \\
  30 & 207 - 2u & 212 - 2u & 220 - 2u \\
  38 & 90 & 95 & 103 \\
  52 & 2u - 65 & 117
\end{array}
\]

Phew! The remaining numerical sums that are not yet visible in the chart are 41, 49, 54, 63, 68, 76, and 79.
Using our "over and down" idea from above, we see that the smallest remaining numerical sum, 41, cannot occur in the first column, because it is smaller than 52.

Therefore, 41 occurs at the top of the second column, and this yields $q + r = 185 - 2u = 41$, or $2u = 144$, or $u = 72$. It then follows that $r + s = 198 - 2u = 198 - 144 = 54$. The correct answer is (B). □

At this point, you may feel like taking a holiday after this "easy way" before proceeding to the hard way! A couple of quick notes to cleanse the palate.

First, since $u = 72$, we could determine the values of the six original unknown integers to be 7, 18, 23, 31, 45, and 72. (We could then double check that the list of sums of pairs is correct.)

Second, we could have skipped the last two paragraphs of Solution 1 by making the following observation:

Writing $r + s$ in terms of $u$, we obtain $r + s = 198 - 2u$. Is this even or odd? Since $u$ is a positive integer, $r + s$ is even. Which of the remaining "unknown" sums are even? We can actually tell by looking at their representations in terms of $u$. The even ones are $r + s = 198 - 2u$, $r + t = 212 - 2u$, and $s + t = 220 - 2u$. Of these, $r + s$ is the smallest. (Why?) Thus, $r + s$ must equal the smallest remaining even sum, so $r + s = 54$.

Enough of the easy way! On to the hard way, but please bear with me – the hard way is actually pretty easy.

**Solution 2** We easily deduced in Solution 1 that $p + q = 25$ and $t + u = 117$. We also made a chart that listed all of the expressions for the sums of each of the pairs. We also have a list of all of the numerical values the sums (expressions) can have. Let's add up both of these lists.

When we add up the 15 expressions in the chart in Solution 1, we get $5p + 5q + 5r + 5s + 5t + 5u$. First question: is it surprising that each unknown appears the same number of times? Second question: could you have figured out this sum without having to write out all of the pairs?

When we add up the known numerical values for the 15 sums, we get 980. Therefore, $5(p + q + r + s + t + u) = 980$ and $p + q + r + s + t + u = 196$. Thus, $r + s = 196 - (p + q) - (t + u) = 196 - 25 - 117 = 54$, as before. □

A couple of thoughts about all of this to wrap up. First, we notice in the second solution that we never had to figure out the values of any of the unknowns. (We could avoid this in the first solution too by using the addendum.) Second, there's the issue of "easy" and "hard". The first solution, to me, requires more persistence than insight. We basically ground away until things fell apart. We didn't have to do anything hard, but it took a lot of effort. The second solution, to my mind, does require a couple of nice insights. Coming up with these insights is not easy, but once we have the right idea, the solution is actually pretty straightforward. Of course, this is what problem solving is all about!