

M410. *Proposé par Matthew Babbitt, étudiant, Albany Area Math Circle, Fort Edward, NY, É-U.*

Un cube d'arête a , un tétraèdre régulier d'arête b et un octaèdre régulier d'arête c ont tous la même surface. Trouver la valeur de $\frac{\sqrt{bc}}{a}$.

M411. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit a , b et c les côtés du triangle ABC . Si

$$2a + 3b + 4c = 4\sqrt{2a-2} + 6\sqrt{3b-3} + 8\sqrt{4c-4} - 20,$$

montrer que ABC est un triangle rectangle.

M412. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Si x est un nombre réel, on désigne par $\lfloor x \rfloor$ le plus grand entier plus petit ou égal à x , et par $\{x\} = x - \lfloor x \rfloor$ la partie fractionnaire de x . Trouver tous les nombres réels x pour lesquels $\lfloor x \rfloor \cdot \{x\} = x$.

Mayhem Solutions

M376. *Proposed by the Mayhem Staff.*

Determine the value of x if $(10^{2009} + 25)^2 - (10^{2009} - 25)^2 = 10^x$.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

Expanding, we have

$$\begin{aligned} & (10^{2009} + 25)^2 - (10^{2009} - 25)^2 \\ &= \left[(10^{2009})^2 + 2 \cdot 25 \cdot 10^{2009} + 25^2 \right] \\ &\quad - \left[(10^{2009})^2 - 2 \cdot 25 \cdot 10^{2009} + 25^2 \right] \\ &= 4 \cdot 25 \cdot 10^{2009} \\ &= 100 \cdot 10^{2009} \\ &= 10^{2011}, \end{aligned}$$

and so $x = 2011$.

Alternatively, factoring the left side as a difference of squares, we have

$$\begin{aligned} & (10^{2009} + 25)^2 - (10^{2009} - 25)^2 \\ &= [(10^{2009} + 25) - (10^{2009} - 25)] \cdot [(10^{2009} + 25) + (10^{2009} - 25)] \\ &= 50 \cdot (2 \cdot 10^{2009}) \\ &= 100 \cdot 10^{2009} = 10^{2011}, \end{aligned}$$

and so $x = 2011$, as above.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; KATHERINE JANELL EYRE and EMILY HENDRYX, students, Angelo State University, San Angelo, TX, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M377. Proposed by the Mayhem Staff.

An arithmetic sequence consists of 9 positive integers. The sum of the terms in the sequence is greater than 200 and less than 220. If the second term in the sequence is 12, determine the sequence.

Solution by Katherine Janell Eyre and Emily Hendryx, students, Angelo State University, San Angelo, TX, USA.

Suppose that a is the first term in the sequence and d is the difference between successive terms. Then using formulae for arithmetic sequences and series, the n^{th} term can be written as $a_n = a + (n - 1)d$ and the sum of the first n terms can be written as $S_n = \frac{n}{2}(2a + (n - 1)d)$. Hence,

$$S_9 = \frac{9}{2}(2a + 8d) = 9(a + 4d).$$

Also, we know that $a_2 = 12 = a + d$, from which we obtain $d = 12 - a$. Combining these, we get

$$S_9 = 9(a + 4d) = 9(a + 4(12 - a)) = 9(48 - 3a) = 27(16 - a).$$

Since $200 < S_9 < 220$, we get $200 < 27(16 - a) < 220$ which reduces to $7\frac{11}{27} < 16 - a < 8\frac{4}{27}$, or $7\frac{23}{27} < a < 8\frac{16}{27}$. Since a has to be a positive integer, we have $a = 8$, and so $d = 12 - a = 4$. We can then check that the resulting sequence satisfies all the requirements. Therefore, the sequence is 8, 12, 16, 20, 24, 28, 32, 36, 40.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT,

student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incomplete solution submitted.

M378. Proposed by the Mayhem Staff.

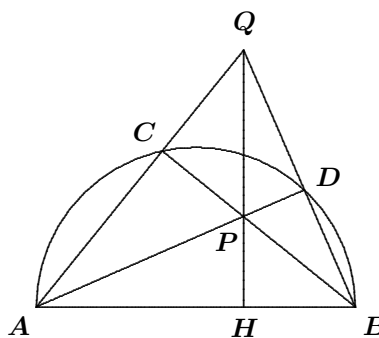
Points C and D are chosen on the semicircle with diameter AB so that C is closer to A . Segments CB and DA intersect at P ; segments AC and BD extended intersect at Q . Prove that QP extended is perpendicular to AB .

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since AB is a diameter, then $\angle ACB = \angle ADB = 90^\circ$. Thus, $\angle QCP = \angle QDP = 90^\circ$, so quadrilateral $CPDQ$ is cyclic since a pair of opposite angles adds to 180° . Note that $ACDB$ is also cyclic.

Let H be the intersection of PQ extended and AB . Then we have $\angle ABC = \angle ADC$, since each is subtended by the same arc. As well, $\angle PDC = \angle CQP$ since $CPDQ$ is a cyclic quadrilateral. Therefore, $\angle ADC = \angle PDC = \angle CQP = \angle AQH$. Hence, $\angle ABC = \angle AQH$.

But $\angle CAB = \angle QAH$ is a common angle in $\triangle ABC$ and $\triangle QAH$, and so $\triangle AQH$ is similar to $\triangle ABC$. Therefore, $\angle AHQ = \angle ACB = 90^\circ$, and so QP extended is perpendicular to AB .



Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania.

Most solvers noted that AD and BC are altitudes of $\triangle ABQ$. Thus, their point of intersection, P , is the orthocentre of the triangle and so the line segment through Q and P must be the third altitude of the triangle, and so is perpendicular to AB , as required.

M379. Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

The integers $27 + C$, $555 + C$, and $1371 + C$ are all perfect squares, the square roots of which form an arithmetic sequence. Determine all possible values of C .

Solution by Matthew Babbitt, student, Albany Area Math Circle, Fort Edward, NY, USA.

Since the three given expressions have square roots in arithmetic sequence, then we may suppose that the square roots of $27 + C$, $555 + C$, and $1371 + C$ are $a - d$, a , and $a + d$, respectively, for some a and d . Thus, $(a - d)^2 = 27 + C$, $a^2 = 555 + C$, and $(a + d)^2 = 1371 + C$.

Note that

$$\begin{aligned}(a + d)^2 + (a - d)^2 - 2a^2 &= (a^2 + 2ad + d^2) + (a^2 - 2ad + d^2) - 2a^2 \\ &= 2d^2,\end{aligned}$$

and so

$$(1371 + C) + (27 + C) - 2(555 + C) = 2d^2,$$

or $288 = 2d^2$, or $d^2 = 144$, which yields $d = \pm 12$.

If $d = 12$, then $(a + d)^2 = 1371 + C$ and $(a + d)^2 = a^2 + 2ad + d^2 = 555 + C + 24a + 144$. Combining these equations yields $24a = 672$, or $a = 28$. Thus, $C = a^2 - 555 = 28^2 - 555 = 229$.

If $d = -12$, then $(a + d)^2 = 1371 + C$ and $(a + d)^2 = a^2 - 2ad + d^2 = 555 + C - 24a + 144$. Combining these equations yields $-24a = 672$, or $a = -28$. Thus, $C = a^2 - 555 = (-28)^2 - 555 = 229$.

In either case, $C = 229$. We can check by substitution that $C = 229$ has the desired properties.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M380. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle ABC is right-angled at C and has $BC = a$ and $CA = b$, with $a \geq b$. Squares $ABDE$, $BCFG$, and $CAHI$ are drawn externally to triangle ABC . The lines through FI and EH intersect at P , the lines through FI and DG intersect at Q , and the lines through DG and EH intersect at R . If triangle PQR is right-angled, determine the value of $\frac{b}{a}$.

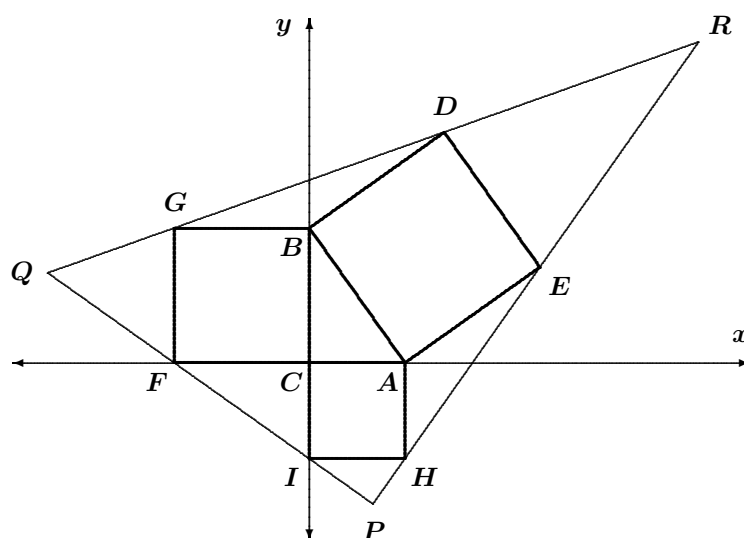
Composite of solutions by George Apostolopoulos, Messolonghi, Greece and Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.

We coordinatize the diagram on the following page by placing C at $(0, 0)$, B at $(0, a)$ and A at $(b, 0)$. Note that $a \geq b > 0$.

Since $BCFG$ is a square external to $\triangle ABC$, then it has side length a and has sides parallel to the axes, so the coordinates of G are $(-a, a)$ and the coordinates of F are $(-a, 0)$.

Similarly, since $CAHI$ is a square external to $\triangle ABC$, the coordinates of H are $(b, -b)$ and the coordinates of I are $(0, -b)$.

Since $ABDE$ is a square external to $\triangle ABC$ and the vector \overrightarrow{AB} equals $(b, -a)$, then the vectors \overrightarrow{BD} and \overrightarrow{AE} are both equal to (a, b) (because they are perpendicular to AB and equal in length). Thus, the coordinates of D are $(a, a + b)$ and the coordinates of E are $(a + b, b)$.



The slope of QP equals the slope of FI . The coordinates of F are $(-a, 0)$ and the coordinates of I are $(0, -b)$, so the slope of QP is $-\frac{b}{a}$.

The slope of QR equals the slope of GD . The coordinates of G are $(-a, a)$ and the coordinates of D are $(a, a + b)$, so the slope of QR is $\frac{b}{2a}$.

The slope of PR equals the slope of HE . The coordinates of H are $(b, -b)$ and the coordinates of E are $(a + b, b)$, so the slope of PR is $\frac{2b}{a}$.

For two line segments to be perpendicular, the product of their slopes must equal -1 . Since the slopes of QR and PR are both positive, then these segments are not perpendicular and the angle at R is not a right angle.

If there was a right angle at Q , then QR and QP would be perpendicular, or $-\frac{b}{a} \cdot \frac{b}{2a} = -1$, or $b^2 = 2a^2$, contradicting the fact that $a \geq b > 0$.

Thus, if $\triangle PQR$ has a right angle, then it can only occur at P . In that case, the segments QP and PR are perpendicular, so $-\frac{b}{a} \cdot \frac{2b}{a} = -1$, or $\frac{b^2}{a^2} = \frac{1}{2}$. Since $a \geq b > 0$, then $\frac{b}{a} = \frac{1}{\sqrt{2}}$.

Also solved by RICARD PEIRÓ, IES "Abastos", Valencia, Spain.