Several Symmetric Inequalities of Exponential Kind

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In this article we suggest a general approach for proving certain symmetric inequalities of exponential kind in three variables which have appeared in print at various times.

**Theorem 1** Let \( n, m, p, \) and \( q \) be arbitrary nonnegative real numbers, such that \( n \geq m \) and \( p \geq q \). Then for any positive real numbers \( a, b, c \) the following inequality holds

\[
\frac{a^{n+p} + b^{n+p} + c^{n+p}}{a^{m+q} + b^{m+q} + c^{m+q}} \geq \frac{a^n + b^n + c^n}{a^m + b^m + c^m} \cdot \frac{a^p + b^p + c^p}{a^q + b^q + c^q}.
\]

**Proof:** Let \( \sigma(x) = \sigma(x; a, b, c) = \sum_{\text{cyclic}} a^x; \) the inequality then becomes

\[
\frac{\sigma(n + p)}{\sigma(m + q)} \geq \frac{\sigma(n)}{\sigma(m)} \cdot \frac{\sigma(p)}{\sigma(q)}.
\]

The inequality is essentially the same upon switching \( n \) and \( p \) or \( m \) and \( q \), so we may suppose that \( n \geq p \) and \( m \geq q \). Then \( q = \min\{n, m, p, q\} \).

Since the inequality to be proved is equivalent to \( \sigma(n + p) \sigma(m) \sigma(q) \geq \sigma(m + q) \sigma(n) \sigma(p) \) and we also have

\[
\sigma(n + p) \sigma(m) \sigma(q)
\]

\[
= \sum_{\text{cyclic}} a^{n+p} \cdot \left( \sum_{\text{cyclic}} a^{m+q} + \sum_{\text{cyclic}} (a^m b^q + b^m a^q) \right)
\]

\[
= \left( \sum_{\text{cyclic}} a^{n+p} \right) \left( \sum_{\text{cyclic}} a^{m+q} \right) + \sum_{\text{cyclic}} \left( a^{n+p} + b^{n+p} \right) (a^m b^q + b^m a^q)
\]

\[
+ \sum_{\text{cyclic}} c^{n+p} (a^m b^q + b^m a^q),
\]

with the analogous equality holding for \( \sigma(m + q) \sigma(n) \sigma(p) \), it therefore suffices to prove the following two inequalities:

\[
\sum_{\text{cyclic}} (a^{n+p} + b^{n+p}) (a^m b^q + b^m a^q) \geq \sum_{\text{cyclic}} (a^{m+q} + b^{m+q}) (a^n b^p + b^n a^p),
\]

\[
\sum_{\text{cyclic}} c^{n+p} (a^m b^q + b^m a^q) \geq \sum_{\text{cyclic}} c^{m+q} (a^n b^p + b^n a^p).
\]
The first inequality above is settled by the following calculation:
\[
\begin{align*}
&\sum_{\text{cyclic}} (a^{n+p} + b^{n+p})(a^{m}b^{q} + b^{m}a^{q}) \\
&- \sum_{\text{cyclic}} (a^{m+q} + b^{m+q})(a^{n}b^{p} + b^{n}a^{p}) \\
&= \sum_{\text{cyclic}} (a^{n+p+m}b^{q} + b^{n+p+m}a^{q} + a^{m}b^{n+p+q} + b^{m}a^{n+p+q} \\
&- a^{n+m+q}b^{p} - b^{n+m+q}a^{p} - a^{n}b^{m+p+q} - b^{n}a^{m+p+q}) \\
&= \sum_{\text{cyclic}} a^{q}b^{q}(a^{n+m+p-q} + b^{n+m+p-q} - a^{n+m}b^{p-q} - b^{n+m}a^{p-q}) \\
&+ \sum_{\text{cyclic}} a^{m}b^{m}(a^{n+p+q-m} + b^{n+p+q-m} - a^{p+q}b^{n-m} - b^{p+q}a^{n-m}) \\
&= \sum_{\text{cyclic}} a^{q}b^{q}(a^{n+m} - b^{n+m}) (a^{p-q} - b^{p-q}) \\
&+ \sum_{\text{cyclic}} a^{m}b^{m}(a^{p+q} - b^{p+q}) (a^{n-m} - b^{n-m}) \geq 0.
\end{align*}
\]
Lastly, since
\[
\begin{align*}
&\sum_{\text{cyclic}} c^{n+p}(a^{m}b^{q} + b^{m}a^{q}) = \sum_{\text{cyclic}} c^{q}(a^{n+p}b^{m} + b^{n+p}a^{m}); \\
&\sum_{\text{cyclic}} c^{m+q}(a^{n}b^{p} + b^{n}a^{p}) = \sum_{\text{cyclic}} c^{q}(a^{m+p}b^{n} + b^{m+p}a^{n}),
\end{align*}
\]
the second inequality that remains to be proved now follows immediately from
\[
\begin{align*}
&\sum_{\text{cyclic}} c^{q}(a^{n+p}b^{m} + b^{n+p}a^{m} - a^{m+p}b^{n} - b^{m+p}a^{n}) \\
&= \sum_{\text{cyclic}} a^{m}b^{m}c^{q}(a^{n-m+p} + b^{n-m+p} - a^{p}b^{n-m} - b^{p}a^{n-m}) \\
&= \sum_{\text{cyclic}} a^{m}b^{m}c^{q}(a^{p} - b^{p})(a^{n-m} - b^{n-m}) \geq 0. \hspace{1cm} \blacksquare
\end{align*}
\]

**Corollary 1** Let \( k \) be a nonnegative integer and let \( p \geq q \geq 0 \). Then for any positive real numbers \( a, b, \) and \( c \) the following inequality holds
\[
\frac{a^{kp} + b^{kp} + c^{kp}}{a^{kq} + b^{kq} + c^{kq}} \geq \left( \frac{a^{p} + b^{p} + c^{p}}{a^{q} + b^{q} + c^{q}} \right)^{k}.
\]

**Proof:** We set \( n = kp, m = kq \) in Theorem 1 to obtain
\[
\frac{\sigma(kp + p)}{\sigma(kq + q)} \geq \frac{\sigma(kp)}{\sigma(kq)} \cdot \frac{\sigma(p)}{\sigma(q)}
\]
and that yields the inequality
\[
\frac{\sigma((k+1)p)}{\sigma((k+1)q)} \left(\frac{\sigma(p)}{\sigma(q)}\right)^{-(k+1)} \geq \frac{\sigma(kp)}{\sigma(kq)} \left(\frac{\sigma(p)}{\sigma(q)}\right)^{-k},
\]
which implies that
\[
\frac{\sigma(kp)}{\sigma(kq)} \left(\frac{\sigma(p)}{\sigma(q)}\right)^{-k} \geq \frac{\sigma(1 \cdot p)}{\sigma(1 \cdot q)} \left(\frac{\sigma(p)}{\sigma(q)}\right)^{-1} = 1,
\]
and the inequality to be proved now follows. □

**Theorem 2** Let \(a, b, c\) be positive real numbers. Then for any positive integer \(n\) the function
\[
L_n(x) = L_n(x; a, b, c) = \frac{a^n + b^n + c^n}{a^{nx} + b^{nx} + c^{nx}} \sum_{\text{cyclic}} \left(\frac{a^x}{b + c}\right)^n
\]
is increasing in \(x\) on \((0, \infty)\).

**Proof:** Let \(p, q \in (0, \infty)\) and \(q < p\). Due to the homogeneity of \(L_n(x; a, b, c)\) with respect to \(a, b, c\), it suffices to prove the assertion when \(a+b+c = 1\).

Using the expansion \(\frac{1}{(1-t)^n} = \sum_{k=0}^{\infty} (\frac{k+n-1}{n-1})t^k\) we obtain
\[
\frac{\sigma(np)\sigma(nq)}{\sigma(n)} (L_n(p) - L_n(q))
\]
\[
= \sigma(nq) \sum_{\text{cyclic}} \frac{a^{np}}{(1-a)^n} - \sigma(np) \sum_{\text{cyclic}} \frac{a^{nq}}{(1-a)^n}
\]
\[
= \sigma(nq) \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \left(\frac{k+n-1}{n-1}\right)a^{k+np} - \sigma(np) \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \left(\frac{k+n-1}{n-1}\right)a^{k+nq}
\]
\[
= \sum_{k=0}^{\infty} \left(\frac{k+n-1}{n-1}\right) (\sigma(nq)\sigma(k+np) - \sigma(np)\sigma(k+nq))
\]
\[
= \sum_{k=0}^{\infty} \left(\frac{k+n-1}{n-1}\right) \sum_{\text{cyclic}} \left(a^{k+np}b^{nq} + a^{nq}b^{k+np} - a^{k+np}b^{nq} - a^{np}b^{k+nq}\right)
\]
\[
= \sum_{k=0}^{\infty} \left(\frac{k+n-1}{n-1}\right) \sum_{\text{cyclic}} a^{nq}b^{nq} (a^{n(p-q)} - b^{n(p-q)}) (a^k - b^k) \geq 0,
\]
since \((a^{n(p-q)} - b^{n(p-q)}) (a^k - b^k) \geq 0\) for any nonnegative integer \(k\). □

**Corollary 2** For any positive real numbers \(a, b, c, r\) and any positive numbers \(p, q\) such that \(q < r < p\) the following inequality holds
\[
\frac{1}{\sigma(nq)} \sum_{\text{cyclic}} \left(\frac{a^q}{b^r + c^r}\right)^n \leq \frac{1}{\sigma(nr)} \sum_{\text{cyclic}} \left(\frac{a^r}{b^r + c^r}\right)^n \leq \frac{1}{\sigma(np)} \sum_{\text{cyclic}} \left(\frac{a^p}{b^r + c^r}\right)^n.
\]
Proof: Since $L_n(x; a^r, b^r, c^r)$ is increasing in $x$ and $q < r < p$, we have

$$L_n \left( \frac{q}{r}; a^r, b^r, c^r \right) \leq L_n \left( 1; a^r, b^r, c^r \right) \leq L_n \left( \frac{p}{r}; a^r, b^r, c^r \right),$$

which is equivalent to the inequality to be proved.

By the results of Corollary 1 and Corollary 2 we obtain successively

$$\frac{1}{\sigma(nq)} \sum_{\text{cyclic}} \left( \frac{a^q}{b^r + c^r} \right)^n \leq \frac{1}{\sigma(nr)} \sum_{\text{cyclic}} \left( \frac{a^r}{b^r + c^r} \right)^n;$$

$$\sum_{\text{cyclic}} \left( \frac{a^q}{b^r + c^r} \right)^n \geq \frac{\sigma(nr)}{\sigma(nq)} \geq \left( \frac{\sigma(nr)}{\sigma(nq)} \right)^n;$$

and similarly we obtain

$$\sum_{\text{cyclic}} \left( \frac{\sigma(p)}{\sigma(nr)} \right)^n \geq \left( \frac{\sigma(p)}{\sigma(r)} \right)^n.$$

It follows that for any positive real numbers $a$, $b$, $c$, $r$ and any positive real numbers $p$, $q$ such that $q < r < p$, the following inequality holds

$$\frac{1}{\sigma^n(q)} \sum_{\text{cyclic}} \left( \frac{a^q}{b^r + c^r} \right)^n \leq \frac{1}{\sigma^n(r)} \sum_{\text{cyclic}} \left( \frac{a^r}{b^r + c^r} \right)^n \leq \frac{1}{\sigma^n(p)} \sum_{\text{cyclic}} \left( \frac{a^p}{b^r + c^r} \right)^n.$$

Corollary 3 Let $a$, $b$, $c$ be positive real numbers and let

$$F(x) = F(x; a, b, c) = \frac{a + b + c}{a^x + b^x + c^x} \sum_{\text{cyclic}} \frac{a^2 + b^2}{a + b},$$

$$E(x) = E(x; a, b, c) = \frac{1}{a^x + b^x + c^x} \sum_{\text{cyclic}} \frac{a (b^x + c^x)}{b + c}.$$

Then $F(x)$ and $E(x)$ are each decreasing on $(0, \infty)$.

Proof: We have

$$L_1(x) = \frac{\sigma(1)}{\sigma(x)} \sum_{\text{cyclic}} \frac{\sigma(x)}{b + c} - \frac{\sigma(1)}{\sigma(x)} \sum_{\text{cyclic}} \frac{b^x + c^x}{b + c} = \sum_{\text{cyclic}} \frac{a + b + c}{b + c} - F(x),$$

hence, $F(x)$ is decreasing on $(0, \infty)$ because $L_1(x)$ is increasing on $(0, \infty)$ by Theorem 2. Straightforward calculations show that $E(x) = F(x) - 2$, hence $E(x)$ is also decreasing on $(0, \infty)$.
We now apply the preceding results to obtain some generalizations of various problems.

**Problem** For any positive real numbers \(a, b, c, r\) and any positive real numbers \(p, q\) such that \(q < r < p\) prove the following inequalities:

\[
\frac{1}{\sigma(p)} \sum_{\text{cyclic}} \frac{a^p + b^p}{a^r + b^r} \leq \frac{3}{\sigma(r)} \sum_{\text{cyclic}} \frac{a^q + b^q}{a^r + b^r}; \quad (4)
\]

\[
\frac{1}{\sigma(p)} \sum_{\text{cyclic}} \frac{a^r (b^p + c^p)}{b^r + c^r} \leq 1 \leq \frac{1}{\sigma(q)} \sum_{\text{cyclic}} \frac{a^r (a^q + b^q)}{a^r + b^r}. \quad (5)
\]

**Solution:** We have \(F(P; a^r, b^r, c^r) \leq F(1; a^r, b^r, c^r) \leq F(P; a^r, b^r, c^r)\) by Corollary 2, and since \(F(1; a^r, b^r, c^r) = 3\) the first inequality follows.

Similarly, \(E(P; a^r, b^r, c^r) \leq E(1; a^r, b^r, c^r) \leq E(P; a^r, b^r, c^r)\) and since \(E(1; a^r, b^r, c^r) = 1\) the second inequality follows.

Inequality (4) is a generalization of the inequality \(\sum_{\text{cyclic}} \frac{a^2 + b^2}{a + b} \leq \frac{3\sigma(2)}{\sigma(1)}\) in [2], and also a generalization of the inequality in [3].

Inequality (5) generalizes the inequality \(\sum_{\text{cyclic}} \frac{x^p (y + z)}{y^p + z^p} \geq x + y + z\), for positive \(x, y, z,\) and \(p > 1\), which is Peter Wao’s generalization of the inequality in [4] (see the commentary on p. 180). Furthermore, by using the rightmost relation of Inequality (5) we can obtain a generalization of the inequality \(\sum_{\text{cyclic}} \frac{a^{\lambda+1}}{b^\lambda + c^\lambda} \geq \frac{a + b + c}{2}\), for \(\lambda \geq 0\), suggested by Walther Janous in [4] (again, see the commentary on p. 180). Namely: for any positive real numbers \(a, b, c, p,\) and \(q\) the following inequality holds

\[
\sum_{\text{cyclic}} \frac{a^{p+q}}{b^p + c^p} \geq \frac{a^q + b^q + c^q}{2}. \quad (6)
\]

**Proof:** The inequality \(\sum_{\text{cyclic}} \frac{a^{p+q}}{b^p + c^p} \frac{(b^q + c^q)}{b^{p+q} + c^{p+q}} \leq \frac{2a^{p+q}}{b^p + c^p}\) holds since simple manipulations show that it is equivalent to \((b^q - c^q)(b^p - c^p) \geq 0\), and from inequality (5) it follows that \(\sum_{\text{cyclic}} \frac{a^{p+q}}{b^p + c^p} \geq \frac{a^q + b^q + c^q}{2}\), hence,

\[
\sum_{\text{cyclic}} \frac{a^{p+q}}{b^p + c^p} \geq \frac{1}{2} \sum_{\text{cyclic}} \frac{a^{p+q}}{b^p + c^p} \geq \frac{a^q + b^q + c^q}{2},
\]

which proves inequality (6).

In [1] the inequality \(\sum_{\text{cyclic}} \left(\frac{c^2}{a^2 + b^2}\right)^n \geq \sum_{\text{cyclic}} \left(\frac{c}{a + b}\right)^n\) was suggested. The next theorem offers a generalization.
Theorem 3 Let \( n \) be a positive integer and \( a, b, c \) be positive real numbers. Then \( G(x) = G_n(x; a, b, c) = \sum_{\text{cyclic}} \left( \frac{c^x}{a^x + b^x} \right)^n \) is increasing on \((0, \infty)\).

Proof: Let \( p > q > 0 \) and let \( A_x = \frac{a^x}{\sigma(x)}, B_x = \frac{b^x}{\sigma(x)}, \) and \( C_x = \frac{c^x}{\sigma(x)} \). Then we obtain

\[
G_n(p) \geq G_n(q) \iff \sum_{\text{cyclic}} \frac{A^n_p}{(1 - A_p)^n} \geq \sum_{\text{cyclic}} \frac{A^n_q}{(1 - A_q)^n}
\]

\[
\iff \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \binom{k + n - 1}{n - 1} A^{k+n}_p \geq \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \binom{k + n - 1}{n - 1} A^{k+n}_q
\]

\[
\iff \sum_{k=1}^{\infty} \binom{k + n - 1}{n - 1} \sum_{\text{cyclic}} A^{k+n}_p \geq \sum_{k=1}^{\infty} \binom{k + n - 1}{n - 1} \sum_{\text{cyclic}} A^{k+n}_q
\]

\[
\iff \sum_{k=1}^{\infty} \binom{k + n - 1}{n - 1} \frac{\sigma((k+n)p)}{\sigma^{k+n}(p)} \geq \sum_{k=1}^{\infty} \binom{k + n - 1}{n - 1} \frac{\sigma((k+n)q)}{\sigma^{k+n}(q)},
\]

and the last inequality above holds termwise by the result of Corollary 1.

By applying the result of Theorem 3 to the terms of an infinite series we obtain the following corollary.

Corollary 4 Let \( h(t) = \sum_{n=0}^{\infty} h_n t^n \), where each \( h_n \) is nonnegative and the series converges for \( t \geq 0 \). Then for any positive real numbers \( a, b, c \) the function \( G_h(x; a, b, c) = \sum_{\text{cyclic}} h \left( \frac{c^x}{a^x + b^x} \right) \) is increasing in \( x \) on \((0, \infty)\).

References


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