THE OLYMPIAD CORNER

No. 281

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We begin with the 24th Iranian Mathematical Olympiad 2006–2007 and the First Round problems. Thanks go to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them for our use.

24th IRANIAN MATHEMATICAL OLYMPIAD
First Round

1. Given integers \( m > 2 \) and \( n > 2 \), prove there is a sequence of integers \( a_0, a_1, \ldots, a_k \) such that \( a_0 = m, a_k = n \), and \( (a_i + a_{i+1}) | (a_i a_{i+1} + 1) \) for each \( i = 0, 1, \ldots, k - 1 \).

2. Let \( I_1, I_2, \ldots, I_n \) be \( n \) closed intervals of \( \mathbb{R} \) such that among any \( k \) of them there are two with nonempty intersection. Prove that one can choose \( k - 1 \) points in \( \mathbb{R} \) such that each of the intervals contains at least one of the chosen points.

3. Let \( A, B, C, \) and \( D \) be four points on a circle \( \omega \) and occurring on the circumference in that order. Prove that there exist four points \( M_1, M_2, M_3, \) and \( M_4 \) on \( \omega \) which form a quadrilateral with perpendicular diagonals and are such that for each \( i \in \{1, 2, 3, 4\} \)

\[
\frac{AM_i}{BM_i} = \frac{DM_i}{CM_i}.
\]

4. Find all two-variable polynomials \( p(x, y) \) with real coefficients such that 

\[
p(x + y, x - y) = 2p(x, y)
\]

for all real numbers \( x \) and \( y \).

5. Let \( \omega_1 \) and \( \omega_2 \) be two circles such that the centre of \( \omega_1 \) is located on \( \omega_2 \). If the circles intersect at \( M \) and \( N \), \( AB \) is an arbitrary diameter of \( \omega_1 \), and \( A_1 \) and \( B_1 \) are the second intersections of \( AM \) and \( BN \) with the circle \( \omega_2 \) (respectively), prove that \( A_1B_1 \) is equal to the radius of \( \omega_1 \).

6. A stack of \( n \) books, numbered \( 1, 2, \ldots, n \) is given. In a single round we make \( n \) moves, where the the \( i \)th move in a round consists of turning over the \( i \) books at the top, taking them as a single block in the course of turning them over. After each round we start a new round similar to the previous one. Show that the initial arrangement will appear again after some number of rounds.
Continuing with the 24th Iranian Mathematical Olympiad, we give the Second Round. Thanks again go to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them.

24th IRANIAN MATHEMATICAL OLYMPIAD
Second Round

1. A regular polyhedron is a polyhedron which is convex and all of its faces are regular polygons. We call a regular polyhedron a TLP if and only if none of its faces is a triangle.

(a) Prove that each TLP can be inscribed in a sphere.

(b) Prove that the faces of each TLP are polygons of at most 3 different kinds. (That is, there are \(m\), \(n\), and \(p\) such that each face of the TLP is a regular \(n\)-gon, \(m\)-gon, or \(p\)-gon.)

(c) Prove that there is exactly one TLP with only pentagonal and hexagonal faces (the soccer ball).

(d) For \(n > 3\), a prism which has 2 regular \(n\)-gons and \(n\) squares as its faces, is a TLP. Prove that except for these TLP's, there are only finitely many other TLP's.

2. A fluid flows in an infinite, line-shaped pipe. If a molecule of the fluid is at the point with coordinate \(x\), then after \(t\) seconds it will be at the point with coordinate \(P(t, x)\). Prove that if \(P(t, x)\) is a polynomial in \(t\) and \(x\), then all molecules are moving with a unique and constant speed.

3. Let \(C\) be a convex subset of \(\mathbb{R}^3\) with positive volume. Suppose that \(C_1, C_2, \ldots, C_n\) are \(n\) translated (but not rotated) copies of \(C\) such that \(C_i \cap C \neq \emptyset\) for each \(i\), and such that \(C_i\) and \(C_j\) intersect at most on the boundary whenever \(i \neq j\).

(a) Prove that if \(C\) is symmetric, then \(n \leq 27\) and this is best possible.

(b) Prove the same thing for an arbitrary convex subset \(C\).

4. A finite number of disjoint shapes in the plane are given. A convex partition of the plane is a collection of convex parts (subsets) such that the parts can intersect at most on their boundaries, the parts cover the plane, and each part contains exactly one of the shapes.

For which of the following sets of shapes does a convex partition exist?

(a) A finite number of distinct points.

(b) A finite number of disjoint line segments.

(c) A finite number of disjoint circular disks.
5. For $A \subseteq \mathbb{Z}$ and $a, b \in \mathbb{Z}$, let $aA + b = \{ax + b : x \in A\}$. If $a \neq 0$, then we say that $aA + b$ is similar to $A$. The Cantor Set, $C$, is the set of all nonnegative integers which have no digit 1 in their base 3 representation.

A representation of $C$ is a partition of $C$ into a finite number of two or more sets similar to $C$. That is, $C = \bigcup_{i=1}^{n} C_i$, where $n > 1$, the set $C_i = a_iC + b_i$ is similar to $C$ for each $i$, and $C_i \cap C_j = \emptyset$ whenever $i \neq j$.

Note that $C = (3C) \cup (3C + 2)$ is a representation of $C$, and so is $C = (3C) \cup (9C + 2) \cup (9C + 6)$. A representation of $C$ is a primitive representation if and only if the union of fewer than $n$ of the $C_i$'s is not similar to nor equal to $C$.

Prove that

(a) In a primitive representation of $C$, we have $a_i > 1$ for each $i$.

(b) In a primitive representation of $C$, each $a_i$ is a power of 3.

(c) In a primitive representation of $C$, we have $a_i > b_i$ for each $i$.

(d) The only primitive representation of $C$ is $C = (3C) \cup (3C + 2)$.

6. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients and let $P(x)$ be monic (that is, $P(x)$ has leading coefficient 1). Prove that there exists a monic polynomial $R(x)$ with integer coefficients such that $P(x) \mid Q(R(x))$.

To complete the 24th Iranian Mathematical Olympiad, we now give the Third Round, with thanks to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them.

**24th IRANIAN MATHEMATICAL OLYMPIAD**

**Third Round**

1. Let $A$ be a largest subset of $\{1, 2, \ldots, n\}$ such that each element of $A$ divides at most one other element of $A$. Prove that

\[
\frac{2n}{3} \leq |A| \leq 3 \left\lfloor \frac{n}{4} \right\rfloor .
\]

2. Does there exist a sequence of positive integers $a_0, a_1, a_2, \ldots$ such that $\gcd(a_i, a_j) = 1$ whenever $i \neq j$, and for all $n$ the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?

3. Triangle $ABC$ is isosceles with $AB = AC$. The line $\ell$ passes through $A$ and is parallel to $BC$. The points $P$ and $Q$ are on the perpendicular bisectors of $AB$ and $AC$, respectively, and such that $PQ \perp BC$. The points $M$ and $N$ are on $\ell$ and such that $\angle APM$ and $\angle AQN$ are right angles. Prove that

\[
\frac{1}{AM} + \frac{1}{AN} \leq \frac{2}{AB}.
\]
4. Suppose that \( n \) lines are placed in the plane, such that no two are parallel and no three are concurrent. For each two lines let the angle between them be the smallest angle produced at their intersection. Find the largest value of the sum of the \( \binom{n}{2} \) angles between the lines.

5. The point \( O \) is inside triangle \( ABC \) and such that \( OA = OB + OC \). Let \( B' \) and \( C' \) be the midpoints of the arcs \( AOC \) and \( AOB \), respectively. Prove that the circumcircles of \( COC' \) and \( BOB' \) are tangent to each other.

6. Find all polynomials \( p(x) \) of degree 3 such that for all nonnegative real numbers \( x \) and \( y \)
\[
p(x + y) \geq p(x) + p(y).
\]

Next we give the two days of the 11th Mathematical Olympiad of Bosnia and Herzegovina, May 2006. Thanks go to Šefket Arslanagić, University of Sarajevo, Bosnia and Herzegovina, for sending them for our use.

11\textsuperscript{th} MATHEMATICAL OLYMPIAD OF BOSNIA AND HERZEGOVINA
May, 2006
First Day

1. A \( Z \)-tile is any tile congruent to the one shown at right. What is the least number of \( Z \)-tiles needed to cover an \( 8 \times 8 \) grid, if every square of a \( Z \)-tile coincides with a square of the grid or is outside the grid. (The \( Z \)-tiles can overlap.)

2. Triangle \( ABC \) is given. Determine the set of the centres of all rectangles inscribed in the triangle \( ABC \) so that one side of the rectangle lies on the side \( AB \) of the triangle \( ABC \).

3. Prove that for every positive integer \( n \) the inequality \( \{n\sqrt{7}\} > \frac{3\sqrt{7}}{14n} \) holds, where \( \{x\} \) is the fractional part of the real number \( x \). (If \( \lfloor x \rfloor \) is the greatest integer not exceeding \( x \), then \( \{x\} = x - \lfloor x \rfloor \).)

Second Day

4. For any two positive integers \( a \) and \( d \) prove that the infinite arithmetic progression
\[ a, a + d, a + 2d, \ldots, a + nd, \ldots \]
contains an infinite geometric progression of the form
\[ b, bq, bq^2, \ldots, bq^n, \ldots, \]
where \( b \) and \( q \) are also positive integers.
5. The acute triangle $ABC$ is inscribed in a circle with centre $O$. Let $P$ be a point on the arc $AB$, where $C \not\in AB$. The perpendicular from the point $P$ to the line $BO$ cuts the side $AB$ at point $S$ and the side $BC$ at point $T$. The perpendicular from the point $P$ to the line $AO$ cuts the side $AB$ at point $Q$ and the side $AC$ at point $R$. Prove that:

(a) The triangle $PQS$ is isosceles.

(b) $PQ^2 = QR \cdot ST$.

6. Let $a_1, a_2, \ldots, a_n$ be real constants and for each real number $x$ let

$$f(x) = \cos(a_1 + x) + \frac{\cos(a_2 + x)}{2} + \frac{\cos(a_3 + x)}{2^2} + \cdots + \frac{\cos(a_n + x)}{2^{n-1}}.$$  

If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2 = m\pi$, where $m$ is an integer.

We finish this number with problems of the Vietnamese Mathematical Olympiad 2006–2007. Thanks go to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them.

**VIETNAMESE MATHEMATICAL OLYMPIAD 2006–2007**

**February 8, 2007**

Time: 3 hours

1. Solve the system of equations

$$1 - \frac{12}{y + 3x} = \frac{2}{\sqrt{x}},$$

$$1 + \frac{12}{y + 3x} = \frac{6}{\sqrt{y}}.$$

2. Let $x$ and $y$ be integers with $x \neq -1$ and $y \neq -1$, and such that

$$\frac{x^4 - 1}{y + 1} + \frac{y^4 - 1}{x + 1}$$

is also an integer. Prove that $x^4y^{44} - 1$ is divisible by $x + 1$.

3. Triangle $ABC$ has two fixed vertices, $B$ and $C$, while the third vertex $A$ is allowed to vary. Let $H$ and $G$ be the orthocentre and the centroid of $ABC$, respectively. Find the locus of $A$ such that the midpoint $K$ of the segment $HG$ lies on the line $BC$. 

4. Given a regular 2007-gon, find the smallest positive integer $k$ satisfying the following property: In every set of $k$ vertices there are 4 vertices which form a quadrilateral with three edges of the given 2007-gon.

5. Let $b$ be a positive real number. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) = f(x) \cdot 3^{b^y + f(y) - 1} + b^x (3^{b^y + f(y) - 1} - b^y)$$

for all real numbers $x$ and $y$.

6. A trapezoid $ABCD$ with $BC \parallel AC$ and $BC > AD$ is inscribed in a circle $k$ with centre $O$. The point $P$ varies on the line $BC$ outside the segment $BC$ such that $PA$ is not tangent to $k$. The circle with diameter $PD$ intersects $k$ at $E \neq D$. The lines $BC$ and $DE$ meet at $M$, and $PA$ intersects $k$ at $N \neq A$. Prove that the line $MN$ passes through a fixed point.

7. Let $a > 2$ be a real number and

$$f_n(x) = a^{10}x^{n+10} + x^n + x^{n-1} + \cdots + x + 1$$

for each positive integer $n$. Prove that for each $n$ the equation $f_n(x) = a$ has exactly one real root $x_n \in (0, \infty)$, and that the sequence $\{x_n\}_{n=1}^\infty$ has a finite limit as $n$ approaches infinity.

We begin the solutions section of this number of the Corner with a continuation of solutions from our readers to problems of the Olimpiada Matemática Española 2005 given at [2008 : 341–342].

2. A triangle is said to be multiplicative if the product of the lengths of two of its sides equals the length of the third side.

Let $AB ... Z$ be a regular polygon with $n$ sides, each of length 1. The $n - 3$ diagonals from the vertex $A$ divide the triangle $ZAB$ into $n - 2$ smaller triangles. Prove that all of these triangles are multiplicative.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{triangle.png}
\caption{A triangle divided by diagonals}
\end{figure}

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the version of Zvonaru.

Let $\alpha = \frac{\pi}{n}$, $T_0 = Z$, and $T_{n-2} = B$. As we traverse the segment $ZB$ from $Z$ to $B$, let the intersection points of the $n - 3$ diagonals with the segment $ZB$ be $T_1, T_2, \ldots, T_{n-3}$. In $\triangle AZT_k$ we then have $\angle AZT_k = \alpha$, $\angle ZAT_k = k\alpha$, and $\angle ZT_kA = \pi - (k + 1)\alpha.$
By the Law of Sines in $\triangle AZT_k$, we obtain

$$\frac{ZT_k}{\sin k\alpha} = \frac{AT_k}{\sin \alpha} = \frac{1}{\sin(k+1)\alpha},$$

hence, $AT_k = \frac{\sin \alpha}{\sin(k+1)\alpha}$ and $ZT_k = \frac{\sin k\alpha}{\sin(k+1)\alpha}$ for each $k$. We will prove that $\triangle AT_{k-1}T_k$ is multiplicative by proving that $AT_{k-1} \cdot AT_k = T_{k-1}T_k$.

Since $\sin^2 a = \frac{1 - \cos 2a}{2}$ and $\sin a \sin b = \frac{1}{2}(\cos(a - b) - \cos(a + b))$, we obtain

$$\begin{align*}
&\iff AT_{k-1} \cdot AT_k = T_{k-1}T_k \\
&\iff AT_{k-1}AT_k = ZT_k - ZT_{k-1} \\
&\iff \frac{\sin \alpha}{\sin k\alpha} \cdot \frac{\sin \alpha}{\sin(k+1)\alpha} = \frac{\sin(k+1)\alpha}{\sin k\alpha} - \frac{\sin k\alpha}{\sin(k+1)\alpha} \\
&\iff \sin^2 \alpha = \sin^2 k\alpha - \sin(k - 1)\alpha \cdot \sin(k + 1) \alpha \\
&\iff 1 - \cos 2\alpha = 1 - \cos 2k\alpha - \cos 2\alpha + \cos 2k\alpha,
\end{align*}$$

and the last line is certainly true.

4. In triangle $ABC$ the sides $BC$, $AC$, and $AB$ have lengths $a$, $b$, and $c$, respectively, and $a$ is the arithmetic mean of $b$ and $c$. Let $r$ and $R$ be the radius of the incircle and circumcircle of $ABC$, respectively. Prove that:

(a) $0^\circ \leq \angle BAC \leq 60^\circ$.

(b) The altitude from $A$ is three times the inradius $r$.

(c) The distance from the circumcentre of $ABC$ to the side $BC$ is $R - r$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille’s version.

(a) From the Law of Cosines and $2a = b + c$ we have

$$\cos(\angle BAC) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{3b^2 + 3c^2 - 2bc}{8bc},$$

so it suffices to show that $\frac{3b^2 + 3c^2 - 2bc}{8bc} \geq \frac{1}{2}$. This is readily rewritten as $3b^2 + 3c^2 - 6bc \geq 0$, or $3(b - c)^2 \geq 0$, which clearly holds. Thus, $0^\circ \leq \angle BAC \leq 60^\circ$.

(b) Let $h$ be the altitude from $A$. Twice the area of $\triangle ABC$ is equal to both $ah$ and $r(a + b + c) = r(3a)$. Hence, $h = 3r$.

(c) Let $O$ be the circumcentre of $\triangle ABC$. Since $\angle BAC = A$ is acute, $\angle BOC = 2A$, and so the distance from $O$ to $BC$ is $R \cos A$. Thus, we wish to prove that $r = R(1 - \cos A)$, that is, $r = 2R \sin^2 \frac{A}{2}$. Now, we have
\[ r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \] (a well-known formula), and from \(2a = b + c\) and the Law of Sines we obtain \(2 \sin A = \sin B + \sin C\), which easily transforms into \(2 \sin \frac{A}{2} = \cos \frac{B - C}{2}\). As a result

\[
\begin{align*}
r &= 2R \sin \frac{A}{2} \left( 2 \sin \frac{B}{2} \sin \frac{C}{2} \right) \\
&= 2R \sin \frac{A}{2} \left( \cos \left( \frac{B - C}{2} \right) - \cos \left( \frac{B + C}{2} \right) \right) \\
&= 2R \sin \frac{A}{2} \left( \cos \left( \frac{B - C}{2} \right) - \sin \frac{A}{2} \right) \\
&= 2R \sin^2 \frac{A}{2},
\end{align*}
\]

as desired.

5. In triangle \(ABC\) we have \(\angle BAC = 45^\circ\) and \(\angle ACB = 30^\circ\). Let \(M\) be the midpoint of the side \(BC\). Prove that \(\angle AMB = 45^\circ\) and that \(BC \cdot AC = 2AM \cdot AB\).

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give Apostolopoulos' write-up.

If \(BD\) is perpendicular to \(AC\), then \(\angle ABD = 45^\circ\) and hence triangle \(ABD\) is isosceles and \(AD = BD\). Note that the circumcircle of triangle \(BDC\) has \(M\) as its centre and \(BC\) as a diameter. So \(\angle MDC = 30^\circ\) and \(30^\circ = 2\angle MAD\), whence \(\angle MAD = \angle AMD = 15^\circ\); \(\angle AMB = 60^\circ - 15^\circ = 45^\circ\).

By the Law of Sines

\[
\frac{AB}{\sin 30^\circ} = \frac{BC}{\sin 45^\circ} \quad \text{and} \quad \frac{AC}{\sin 135^\circ} = \frac{AM}{\sin 30^\circ}.
\]

Therefore,

\[
\frac{AB}{BC} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{AM}{AC} = \frac{1}{\sqrt{2}},
\]

so that

\[
\frac{AB \cdot AM}{BC \cdot AC} = \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2},
\]

and \(BC \cdot AC = 2AM \cdot AB\), as desired.

7. Prove that the equation \(x^2 + y^2 - z^2 - x - 3y - z - 4 = 0\) has infinitely many integer solutions.
Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give Zvonaru’s write-up.

The given equation is successively equivalent to

\[ x^2 - x - 6 = z^2 + z + \frac{1}{4} - y^2 + 3y - \frac{9}{4}; \]
\[ (x - 3)(x + 2) = \left( z + \frac{1}{2} \right)^2 - \left( y - \frac{3}{2} \right)^2; \]
\[ (x - 3)(x + 2) = (z + y - 1)(z - y + 2). \]

Taking \( x = \alpha \) where \( \alpha \) is an integer and equating

\[
\begin{cases}
    z + y - 1 = \alpha - 3, \\
    z - y + 2 = \alpha + 2,
\end{cases}
\]

we obtain infinitely many solutions \( (x, y, z) = (\alpha, -1, \alpha - 1) \), where \( \alpha \) is an arbitrary integer.

On the other hand, choosing

\[
\begin{cases}
    z + y - 1 = \alpha + 2, \\
    z - y + 2 = \alpha - 3,
\end{cases}
\]

we obtain the solutions \( (x, y, z) = (\alpha, 4, \alpha - 1) \), with \( \alpha \) an integer.

Now we return to readers’ solutions to problems of the 54th Czech Mathematical Olympiad 2004/2005, Category B, 10th Class [2008: 342–344].

D1. Find all pairs \((a, b)\) of real numbers such that each of the equations

\[
\begin{align*}
    x^2 + ax + b &= 0, \\
    x^2 + (2a + 1)x + 2b + 1 &= 0,
\end{align*}
\]

has two distinct real roots and the roots of the second equation are reciprocals of the roots of the first equation.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Apostolopoulos, modified by the editor.

Note that \( b \neq 0 \), because the roots of the first equation are nonzero. If \( x^2 + ax + b = 0 \) has roots \( x_1 \) and \( x_2 \), then since each constant term of each quadratic is the product of the roots we have

\[
\frac{1}{b} = \frac{1}{x_1 x_2} = \frac{1}{x_1} \cdot \frac{1}{x_2} = 2b + 1,
\]

hence \( 2b^2 + b - 1 = 0 \) and we deduce that \( b = -1 \) or \( b = \frac{1}{2} \).
On the other hand, the roots of each quadratic sum to minus the coefficient of \( x \), so that
\[
-2a + 1 = \frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1 x_2} = \frac{-a}{b},
\]
hence \( a = b(2a + 1) \).

Now, if \( b = \frac{1}{2} \), then \( a = b(2a + 1) \) leads to the contradiction \( 0 = \frac{1}{2} \).
However, taking \( b = -1 \) in the equation \( a = b(2a + 1) \) and then solving for \( a \) yields \( (a, b) = \left( -\frac{1}{3}, -1 \right) \), which one can check yields quadratics with the required properties.

**D2.** Let \( ABCD \) be a parallelogram. A line through \( D \) meets the segment \( AC \) in \( G \), the side \( BC \) in \( F \), and the line \( AB \) in \( E \). The triangles \( BEF \) and \( CGF \) have the same area. Determine the ratio \(|AG| : |GC|\).

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We present the solution of Amengual Covas.

From similar triangles \( AGD \), \( CGF \) and \( FBE \), \( FCD \) we have
\[
\frac{AG}{GC} = \frac{AD}{FC} = \frac{BC}{FC} = \frac{BF + FC}{FC} = \frac{BF}{FC} + 1 = \frac{BE}{CD} + 1 = \frac{BE}{AB} + 1. \tag{1}
\]

Since the triangles \( BEF \) and \( CGF \) have the same area, it follows that \( BF \cdot FE = GF \cdot FC \), that is, \( \frac{BF}{GF} = \frac{FC}{FE} \).

Thus, in triangles \( BFG \) and \( CFE \), the sides about equal angles \( BFG \) and \( CFE \) are proportional, implying the triangles are similar and the equal corresponding angles \( FGB \) and \( FEC \) imply that \( BG \) and \( EC \) are parallel. Hence,
\[
\frac{AG}{GC} = \frac{AB}{BE}. \tag{2}
\]

From (1) and (2), it follows that \( \frac{AG}{GC} \) satisfies the equation \( \phi = \frac{1}{\phi} + 1 \).
Therefore, the required ratio is the golden ratio:
\[
\frac{AG}{GC} = \frac{1 + \sqrt{5}}{2}.
\]
D3. Let \( k \geq 3 \) be an integer. We have \( k \) piles of stones with (respectively) 1, 2, \ldots, \( k \) stones in them. At each turn we choose three piles, merge them together, and add one stone (not already in a pile) to the resulting pile. Prove that if after some number of turns only one pile remains, then the number of stones in that pile is not divisible by 3.

Solution by Titu Zvonaru, Comănești, Romania.

Denote the total number of stones at the start by \( S = \frac{k(k+1)}{2} \). After the first turn we will have \( k - 2 \) piles and \( S + 1 \) stones. After the second turn we will have \( k - 2 \cdot 2 \) piles and \( S + 2 \) stones. Continuing in this manner, after the \( p \)-th turn we will have \( k - 2p \) piles and \( S + p \) stones.

If after \( p \) turns only one pile remains, then we must have \( k - 2p = 1 \), that is, \( k = 2p + 1 \) and the one pile will have

\[
S + p = \frac{(2p+1)(2p+2)}{2} + p = (2p+1)(p+1) + p
\]

stones in it. Thus, the one pile has \( 2p^2 + 4p + 1 \) stones, and we have to prove that this number is not divisible by 3.

Write

\[
2p^2 + 4p + 1 = 2(p-1)(p+1) + p + 3(p+1).
\]

If \( p \) is divisible by 3, then \( p - 1 \) and \( p + 1 \) are not divisible by 3, hence \( 2(p-1)(p+1) \) is not divisible by 3 while \( p + 3(p+1) \) is divisible by 3. Therefore, in this first case, \( 2p^2 + 4p + 1 \) is not divisible by 3.

If \( p \) is not divisible by 3, then one of \( p - 1 \) or \( p + 1 \) is divisible by 3, hence \( 2(p-1)(p+1) \) is divisible by 3, hence \( 2(p-1)(p+1) + 3(p+1) \) is divisible by 3 and adding \( p \) to this last quantity gives a result not divisible by 3. Therefore, in this second case, \( 2p^2 + 4p + 1 \) is not divisible by 3.

In all cases, the number of stones in the last pile is not divisible by 3.

D4. Let \( ABC \) be a scalene triangle with orthocentre \( H \) and circumcentre \( O \). Prove that if \( \angle ACB = 60^\circ \), then the bisector of \( \angle ACB \) is the perpendicular bisector of \( OH \).

Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain; and Titu Zvonaru, Comănești, Romania. We give the solution of Barroso Campos.

We have \( \angle CBA = 60^\circ - \alpha \) and \( \angle CAB = 60^\circ + \alpha \).

Furthermore, \( \angle HAH_a = 30^\circ \) (since \( AH \perp BC \)), \( \angle AHH_a = 60^\circ \), \( \angle HCH_a = 30^\circ - \alpha \) (since \( CH \perp AB \)), and \( \angle H_aHC = 60^\circ + \alpha \).
Also, $\angle COB = 2\angle CAB = 120^\circ + 2\alpha$ and $\angle AOB = 2\angle ACB = 120^\circ$.

Since $\angle AOB = \angle AHB = 120^\circ$ the quadrilateral $AHOB$ is cyclic, and $\angle BHO = \angle BAO = 30^\circ$.

We now have

$$\angle CHO = 360^\circ - \angle CHA - \angle AHB - \angle BHO = 360^\circ - (60^\circ + \alpha) - 60^\circ - 120^\circ - 30^\circ = 90^\circ - \alpha$$

$$\angle HCO = \angle ACO - \angle ACH = (90^\circ - \angle ABC) - (30^\circ - \alpha) = 90^\circ - (60^\circ - \alpha) - 30^\circ + \alpha = 2\alpha,$$

hence, $\angle COH = 180^\circ - (90^\circ - \alpha) - 2\alpha = 90^\circ - \alpha$.

Therefore, triangle $HOC$ is isosceles with $CH = CO$, and the angle bisector at $C$ is the perpendicular bisector of $OH$.

**D5.** Find all real numbers $x$ such that

$$\frac{x}{x + 4} = \frac{5|x| - 7}{7|x| - 5},$$

where $|x|$ denotes the greatest integer not exceeding $x$.

Solved by Titu Zvonaru, Comănești, Romania.

Writing $\alpha = |x|$ and solving for $x$, we obtain

$$x = \frac{10\alpha - 14}{\alpha + 1}.$$

Since $\alpha \leq x < \alpha + 1$, we have

$$\alpha \leq x \iff \frac{\alpha^2 + \alpha - 10\alpha + 14}{\alpha + 1} \leq 0$$

$$\quad \iff \frac{(\alpha - 2)(\alpha - 7)}{\alpha + 1} \leq 0;$$

$$\alpha + 1 > x \iff \frac{\alpha^2 + 2\alpha + 1 - 10\alpha + 14}{\alpha + 1} > 0$$

$$\quad \iff \frac{(\alpha - 3)(\alpha - 5)}{\alpha + 1} > 0;$$

From the first set of deductions, $\alpha \in (-\infty, -1) \cup [2, 7]$; from the second set of deductions, $\alpha \in (-1, 3) \cup (5, \infty)$; hence $\alpha \in [2, 3) \cup (5, 7]$. Since $\alpha$ is an integer, $\alpha = 2$ and $x = 2$; or $\alpha = 6$ and $x = \frac{46}{7}$; or $\alpha = 7$ and $x = 7$.

Therefore, all solutions are given by $x \in \left\{2, \frac{46}{7}, 7\right\}$. 
D6. In a circle $\Gamma$ with radius $r$ are inscribed two mutually tangent circles, $\Gamma_1$ and $\Gamma_2$, each with radius $r/2$. Circle $\Gamma_3$ is tangent to $\Gamma_1$ and $\Gamma_2$ externally and to $\Gamma$ internally. Circle $\Gamma_4$ is tangent to $\Gamma_2$ and $\Gamma_3$ externally and to $\Gamma$ internally. Determine the radii of the circles $\Gamma_3$ and $\Gamma_4$.

Solution by Titu Zvonaru, Comănești, Romania.

(i) Let $\Gamma$ have centre $O$ and $\Gamma_i$ have centre $O_i$ for each $i$. Let $\Gamma_3$ have radius $x$.

We have $OO_1 = \frac{r}{2}$, $O_1O_3 = \frac{r}{2} + x$, and $OO_3 = r - x$. By the Pythagorean Theorem, $O_1O_3^2 = OO_1^2 + OO_3^2$, or

$$\frac{1}{4}r^2 + rx + x^2 = \frac{5}{4}r^2 + r^2 - 2rx + x^2,$$

hence, $3rx = r^2$ and $\Gamma_3$ has radius $\frac{1}{3}r$.

(ii) Denote by $y$ the radius of $\Gamma_4$. We have

$$OO_2 = \frac{r}{2}; \quad O_2O_4 = \frac{r}{2} + y;$$

$$O_4O_3 = \frac{r}{3} + y; \quad OO_3 = \frac{2r}{3};$$

$$OO_4 = r - y.$$

Let $M$ be the projection of $O_4$ onto $OO_2$ and let $N$ be the projection of $O_4$ onto $OO_3$. By the Pythagorean Theorem,

$$OO_4^2 - OM^2 = O_4O_3^2 - (OO_2 - OM)^2,$$

which upon substituting the above yields

$$r^2 + y^2 - 2ry - OM^2 = \frac{r^2}{4} + y^2 + ry - \frac{r^2}{4} + r \cdot OM - OM^2,$$

hence, $OM = r - 3y$.

Heron’s formula $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$ for the area $[ABC]$ of $\triangle ABC$ (where $a = BC$, $b = AC$, $c = AB$, and $s = \frac{1}{2}(a + b + c)$) yields

$$O_4N = \frac{2[OO_3O_4]}{OO_3} = \frac{2\sqrt{r \cdot \frac{r}{3} \left( \frac{2r}{3} - y \right) \cdot y}}{\left( \frac{2r}{3} \right)} = \sqrt{(2r - 3y)}.$$

We now have $(r - 3y)^2 = OM^2 = O_4N^2 = y(2r - 3y)$, since $ONO_4M$ is a rectangle. Hence, $(6y - r)(2y - r) = 0$, which yields $y \in \left\{ \frac{r}{6}, \frac{r}{2} \right\}$.

However, $y < \frac{r}{2}$, therefore $\Gamma_4$ has radius $\frac{r}{6}$. 
S2. Let $ABC$ be a right triangle with $a = |BC|$, $b = |AC|$, and $c = |BC|$ and such that $a < b < c$. Let $Q$ be the midpoint of $BC$ and let $S$ be the midpoint of $AB$. The line $CA$ meets the perpendicular bisector of $AB$ at $R$. Prove that $|RQ| = |RS|$ if and only if $a^2 : b^2 : c^2 = 1 : 2 : 3$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Ricardo Barroso Campos, University of Seville, Seville, Spain; and Titu Zvonaru, Comănești, Romania. We give the solution and comment of Amengual Covas.

Triangles $ABC$ and $ARS$ are similar, so we obtain

\[
AR = \frac{AS}{AC} \cdot AB = \frac{c}{2b} AB = \frac{c^2}{2b};
\]

\[
CR = \frac{CA - AR}{b - \frac{c^2}{2b}} = \frac{2b^2 - (a^2 + b^2)}{2b} = \frac{b^2 - a^2}{2b},
\]

since $c^2 = a^2 + b^2$ holds by the Pythagorean Theorem in $\triangle ABC$.

Thus,

\[
|RQ| = |RS| \iff RQ^2 = RS^2
\]

\[
\iff QC^2 + CR^2 = AR^2 - AS^2 \quad \text{(Pyth. Thm. in $\triangle QCR, \triangle ASR$)}
\]

\[
\iff AR^2 - CR^2 = QC^2 + AS^2
\]

\[
\iff (AR + CR)(AR - CR) = QC^2 + AS^2
\]

\[
\iff AC(AR - CR) = QC^2 + AS^2
\]

\[
\iff AR - CR = \frac{QC^2 + AS^2}{AC} = \frac{(\frac{a}{2})^2 + (\frac{c}{2})^2}{b} = \frac{a^2 + c^2}{4b}
\]

\[
\iff AR = \frac{AR + CR}{2} + \frac{AR - CR}{2} = \frac{b}{2} + \frac{2}{2} = \frac{b + (c^2 - b^2) + c^2}{8b} = \frac{2b^2 + 2c^2}{8b}
\]

and

\[
CR = \frac{AR + CR}{2} - \frac{AR - CR}{2} = \frac{CA}{2} - \frac{AR - CR}{2} = \frac{b^2 - a^2}{8b}
\]

\[
\iff c^2 = \frac{3b^2 + 2c^2}{8b} \quad \text{and} \quad \frac{b^2 - a^2}{8b} = \frac{3b^2 - 2a^2}{8b}
\]

(substituting $AR = \frac{c^2}{2b}$ and $CR = \frac{b^2 - a^2}{2b}$ by (1) and (2), resp.)

\[
\iff 3b^2 = 2c^2 \quad \text{and} \quad 2a^2 = b^2 \iff a^2 : b^2 : c^2 = 1 : 2 : 3.
\]

Comment. For other properties of triangles with $a^2 : b^2 : c^2 = 1 : 2 : 3$, see Problem 3 of the 31st Spanish Mathematical Olympiad, First Round, given in [1998 : 452–453], with solution at [2000 : 143–144].
S3. Find all real numbers \( x \) such that

\[ \left| \frac{x}{1 - x} \right| = \left| \frac{x}{1 - \lfloor x \rfloor} \right|, \]

where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \).

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write-up.

Note first that

\[ \frac{1}{1 - \lfloor x \rfloor} - 1 = \frac{x}{1 - \lfloor x \rfloor} = \left| \frac{x}{1 - x} \right| = \left| \frac{1}{1 - x} - 1 \right| = \left| \frac{1}{1 - x} \right| - 1. \]

Hence, the given equation is equivalent to

\[ \left| \frac{1}{1 - x} \right| = \frac{1}{1 - \lfloor x \rfloor}. \]  

Since the left side of (1) is an integer, we must have \( 1 - \lfloor x \rfloor = \pm 1 \), that is, \( \lfloor x \rfloor = 0 \) or 2. If \( \lfloor x \rfloor = 0 \), then \( x \in [0, 1) \) while if \( \lfloor x \rfloor = 2 \), then \( x \in [2, 3) \).

However, if \( x \in \left[ \frac{1}{2}, 1 \right) \), then \( \frac{1}{1 - \lfloor x \rfloor} = 1 \), while \( 0 < 1 - x \leq \frac{1}{2} \) implies that \( \frac{1}{1 - x} \geq 2 \), and so \( \left| \frac{1}{1 - x} \right| \geq 2 \). Hence, in this case, (1) cannot hold.

If \( x \in [0, \frac{1}{2}) \), then \( \lfloor x \rfloor = 0 \). Also, \( 1 - x > \frac{1}{2} \) implies \( 1 \leq \frac{1}{1 - x} < 2 \).

Hence, (1) holds with a value of 1 on each side.

If \( x \in [2, 3) \), then \( \lfloor x \rfloor = 2 \). Also, \( -2 < 1 - x \leq -1 \) implies that \( -1 \leq \frac{1}{1 - x} < -\frac{1}{2} \). Hence, (1) holds with a value of \(-1\) on each side.

By these three cases, the solution set is \([0, \frac{1}{2}) \cup [2, 3)\).

K1. Circle \( \Gamma_1 \) with radius 1 is externally tangent to circle \( \Gamma_2 \) with radius 2. Each of the circles \( \Gamma_1 \) and \( \Gamma_2 \) is internally tangent to circle \( \Gamma_3 \) with radius 3. Determine the radius of the circle \( \Gamma \), which is tangent externally to the circles \( \Gamma_1 \) and \( \Gamma_2 \) and internally to the circle \( \Gamma_3 \).

Solution by Titu Zvonaru, Comănești, Romania.

Let \( \Gamma \) have centre \( O \) and \( \Gamma_i \) have centre \( O_i \) for each \( i \). Let \( \Gamma \) have radius \( x \).

Note that

\[ O_1 O_3 = 1 + x; \quad O_2 O_2 = 2 + x; \quad O_2 O_3 = 1; \]
\[ O_2 O_3 = 3 - x; \quad O_1 O_3 = 2. \]

By Stewart's Theorem we obtain

\[ O_2^2 \cdot O_3 O_2 - O_3^2 \cdot O_1 O_2 + O_2^2 \cdot O_1 O_3 = O_1 O_3 \cdot O_3 O_2 \cdot O_1 O_2, \]

which upon substituting yields 
\[ 1 + 2x + x^2 - 27 - 3x^2 + 18x + 8 + 2x^2 + 8x = 6. \]
Upon solving for \( x \), we obtain \( x = \frac{24}{28} = \frac{6}{7}. \)
On a public website participants vote for the world’s best hockey player of the last decade. The percentage of votes a player receives is rounded off to the nearest percent and displayed on the website. After Jožko votes for Miroslav Šatan, the hockey player’s score of 7% remains unchanged. What is the minimum number of people (including Jožko) who voted? (Each participant votes exactly once and for a single player only.)

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give the solution of Geupel.

Let $N$ and $m$ denote the total number of participants and the number of people who voted for Šatan, respectively. Jožko is counted in both $N$ and $m$, so the given information yields

$$\frac{13}{200} \leq \frac{m-1}{N-1} < \frac{m}{N} \leq \frac{15}{200},$$

$$13N + 187 \leq 200m \leq 15N,$$

thus, $187 \leq 2N$, and hence $N \geq 94$. Writing $N = 94 + k$, where $k$ is a nonnegative integer, and substituting this into (1), we obtain

$$1409 + 13k \leq 200m \leq 1410 + 15k.$$ 

It follows that $m \geq 8$ and $15k \geq 1600 − 1410 = 190$, so that $k \geq 13$.

We now check the case $k = 13$, $N = 94 + k = 107$, $m = 8$. By direct calculation of $\frac{m-1}{N-1} = \frac{7}{106} = 0.0666\ldots$ and $\frac{m}{N} = \frac{8}{107} = 0.074\ldots$, we readily see that $N = 107$ is indeed the minimum number of participants (where eight people voted for Šatan).

K4. Find all triples of real numbers $x, y, z$ such that

$$[x] − y = 2[y] − z = 3[z] − x = \frac{2004}{2005},$$

where $[x]$ denotes the greatest integer not exceeding $x$.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution by Apostolopoulos.

For convenience, let $k = \frac{2004}{2005}$. We then have

$$x = 3[z] − k; \quad y = [x] − k; \quad z = 2[y] − k. \quad (1)$$

Since $0 < k < 1$, the preceding equations imply

$$[x] = 3[z] − 1; \quad [y] = [x] − 1; \quad [z] = 2[y] − 1.$$

Thus, $[z] = 2([x] − 1) − 1 = 2[x] − 3$ and $[x] = 3(2[x]−3)−1$, and solving for $[x]$ yields $[x] = 2$. Then $[y] = 2−1 = 1$ and $[z] = 2·1 − 1 = 1$.

Finally, substituting the values for $[x], [y], \text{ and } [z]$ back into (1) gives the unique solution $(x, y, z) = \left(\frac{4011}{2005}, \frac{2006}{2005}, \frac{2006}{2005}\right)$. 
Next we turn to readers’ solutions to problems of the First Round, 23rd Iranian Mathematical Olympiad given in the Corner at [2008: 344–345].

5. The segment $BC$ is the diameter of a circle and $XY$ is a chord perpendicular to $BC$. The points $P$ and $M$ are chosen on $XY$ and $CY$, respectively, such that $CY \parallel PB$ and $CX \parallel MP$. Let $K$ be the intersection of the lines $CX$ and $PB$. Prove that $PB \perp MK$.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution by Zvonaru.

Let $O$ and $S$ be the midpoints of $BC$ and $XY$, respectively. It is easy to see that $CX = CY$.

Since $CY \parallel PB$, we have

$$\angle CBK = \angle BCK = \angle SCY = \angle SCX = \angle BCK.$$  

It follows that $BK = CK$ and $K$ lies on the perpendicular bisector of $BC$. Since $\triangle CXS$ and $\triangle CKO$ are similar,

$$CK = \frac{CX \cdot CO}{CS}. \quad (1)$$

Also, $\triangle CSY$ and $\triangle PSB$ are similar, so it follows that

$$SP = \frac{BS \cdot YS}{CS}. \quad (2)$$

By (2) we have $YP = SP + YS = \frac{YS(BS + CS)}{CS}$, hence $YP = 2 \cdot \frac{YS \cdot OC}{CS}$.

Since $\triangle YCX$ is similar to $\triangle YMP$, we deduce that

$$YM = \frac{YC \cdot YP}{YX} \quad \Rightarrow \quad YM = \frac{YC \cdot YP}{2 \cdot YS \cdot 2 \cdot \frac{YS \cdot OC}{CS}},$$

hence,

$$YM = \frac{CY \cdot OC}{CS}. \quad (3)$$

By (1) and (3), we have $BK = YM$, hence quadrilateral $BKMY$ is a parallelogram as $BK \parallel YM$. Since $\angle BYM = \angle BYC = 90^\circ$, the parallelogram $BKMY$ is a rectangle and $MK \perp PB$.

That completes this number of the Corner. Send me your nice solutions and generalizations.