SKOLIAD No. 120
Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by 1 May, 2010. A copy of Crux will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Maritime Mathematics Competition, 2009. Our thanks go to David Horrocks, University of Prince Edward Island, for providing us with this contest and for permission to publish it.

Maritime Mathematics Competition 2009
2 hours allowed

1. Two cars leave city A at the same time. The first car drives to city B at 40 km/hr and then immediately returns to city A at the same speed. The second car drives to city B at 60 km/hr and then returns to city A at a constant speed, arriving at the same time as the first car. What was the second car’s speed on its return trip?

2. The perimeter of a regular hexagon H is identical to that of an equilateral triangle T. Find the ratio of the area of H to the area of T.

3. Some integers may be expressed as the sum of consecutive odd positive integers. For example, 64 = 13 + 15 + 17 + 19. Is it possible to express 2009 as the sum of consecutive odd positive integers? If so, find all such expressions for 2009.

4. The diagram shows three squares and angles x, y, and z. Find the sum of the angles x, y, and z.

5. Suppose that $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$ are real numbers satisfying the following equations.

\[
\begin{align*}
x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 &= 1, \\
4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 &= 8, \\
9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 &= 23.
\end{align*}
\]

Find the value of $x_1 + x_2 + x_3 + x_4 + x_5$. 
6. A math teacher writes the equation \( x^2 - Ax + B = 0 \) on the blackboard where \( A \) and \( B \) are positive integers and \( B \) has two digits. Suppose that a student erroneously copies the equation by transposing the two digits of \( B \) as well as the plus and minus signs. However, the student finds that her equation shares a root, \( r \), with the original equation. Determine all possible values of \( A \), \( B \), and \( r \).

**Concours de mathématiques des Maritimes 2009**

**Durée : 2 heures**

1. Deux voitures quittent simultanément la ville \( A \) pour se rendre à la ville \( B \) et revenir ensuite à la ville \( A \). La première voiture roule à 40 km/h pendant tout le trajet. La seconde roule à 60 km/h à l’aller et à une autre vitesse constante au retour. Si les deux voitures reviennent au point de départ en même temps, quelle fut la vitesse de la seconde voiture au retour?

2. Le périmètre d’un hexagone régulier \( H \) est identique à celui d’un triangle équilatéral \( T \). Trouver le rapport de l’aire de \( H \) à celle de \( T \).

3. Certains entiers s’expriment comme la somme d’entiers positifs impairs consécutifs. Par exemple, \( 64 = 13 + 15 + 17 + 19 \). Le nombre 2009 s’exprime-t-il sous cette forme? Si oui, trouver toutes les expressions possibles de 2009 sous cette forme.

4. Dans le diagramme ci-dessous on trouve trois carrés et trois angles \( x \), \( y \), et \( z \). Trouver la somme des angles \( x \), \( y \), et \( z \).

![Diagramme des angles](image)

5. Soient \( x_1 \), \( x_2 \), \( x_3 \), \( x_4 \), et \( x_5 \) des nombres réels satisfaits aux équations suivantes.

\[
\begin{align*}
    x_1 &+ 4x_2 + 9x_3 + 16x_4 + 25x_5 = 1, \\
    4x_1 &+ 9x_2 + 16x_3 + 25x_4 + 36x_5 = 8, \\
    9x_1 &+ 16x_2 + 25x_3 + 36x_4 + 49x_5 = 23.
\end{align*}
\]

Évaluer \( x_1 + x_2 + x_3 + x_4 + x_5 \).

6. Un professeur écrit au tableau l’équation \( x^2 - Ax + B = 0 \) où \( A \) et \( B \) sont des entiers positifs et \( B \) est un nombre de deux chiffres. Une étudiante, en copiant l’équation, transpose les deux chiffres de \( B \) et transpose également les signes plus et moins. Malgré ces erreurs, elle trouve que son équation possède une racine, \( r \), en commun avec l’équation correcte. Trouver toutes les valeurs possibles de \( A \), \( B \), et \( r \).
Next we give solutions to the questions of the Math Kangaroo Contest Practice Set given at [2009 : 1-6].

1. (Grades 3-4) In the addition example, each letter represents a digit. Equal digits are represented by the same letter. Different digits are represented by different letters. Which digit does the letter $K$ represent?

<table>
<thead>
<tr>
<th></th>
<th>$O$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+$</td>
<td>$K$</td>
<td>$O$</td>
</tr>
<tr>
<td>$W$</td>
<td>$O$</td>
<td>$W$</td>
</tr>
</tbody>
</table>

(A) 0  (B) 1  (C) 2  (D) 8  (E) 9

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The sum must be less than 200 since $99 + 99 = 198$. Thus, $W = 1$. Since the sum in the unit column is different from the sum in the tens column, the former must yield a carry. Thus $K + O = 11$. Trial and error now quickly reveals that $O = 2$ and $K = 9$.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School; Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; IAN CHEN, student, Centennial Secondary School, Coquitlam, BC; and LENA CHOI, student, Ecole Baniting Middle School, Coquitlam, BC.

Trial and error can be avoided: Once $K + O = 11$ is known, the sum in the tens column is then $1 + O + K = 12$, so $O = 2$ and, thus, $K = 9$.

2. (Grades 5-6) Ten caterpillars, arranged in a row one behind another, walked in the park. The length of each caterpillar was equal to 8 cm, and the distance any two adjacent caterpillars kept for safety reasons was 2 cm. What is the total length of their row?

(A) 100 cm  (B) 98 cm  (C) 82 cm  (D) 102 cm  (E) 96 cm

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Ten caterpillars and nine spaces yields $(10 \times 8) + (9 \times 2) = 98$ cm.

Also solved by ALISON TAM, student, Burnaby South Secondary School; Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School; Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; IAN CHEN, student, Centennial Secondary School, Coquitlam, BC; KEVIN LI, student, Pinetree Secondary School, Coquitlam, BC; and LENA CHOI, student, Ecole Baniting Middle School, Coquitlam, BC.

3. (Grades 7-8) An ant is running along a ruler of length 10 cm with a constant speed of 1 cm per second (see the figure). Any time when the ant reaches one of the ends of the ruler, it turns back and runs in the opposite direction. It takes the ant exactly 1 second to make a turn. The ant starts from the left end of the ruler. Nearest which number will it be after 2009 seconds?

(A) 1 cm  (B) 2 cm  (C) 3 cm  (D) 4 cm  (E) 5 cm
Solution by Cindy Chen, student, Burnaby North Secondary School, Burnaby, BC.

Walking the length of the ruler and turning around takes the ant 11 seconds. Now 2009 = 182 · 11 + 7. After 182 · 11 seconds, the ant will again be at the left end of the ruler since 182 is even. Thus, after 2009 seconds, the ant will be at the 7 cm mark. The closest given number is then 5 cm.

Also solved by IAN CHEN, student, Centennial Secondary School, Coquitlam, BC.

4. (Grades 9-10) Which of the numbers 2⁶, 3⁵, 4⁴, 5³, 6² is the greatest?
   (A) 2⁶  (B) 3⁵  (C) 4⁴  (D) 5³  (E) 6²

Solution by Kevin Li, student, Pinetree Secondary School, Coquitlam, BC.

Since 2⁶ = 64, 3⁵ = 243, 4⁴ = 256, 5³ = 125, and 6² = 36, the largest clearly is 4⁴.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; and LENA CHOI, student, Ecole Banting Middle School, Coquitlam, BC.

5. (Grades 11-12) A decorator has prepared a mixed paint, in which the volumes of red and yellow colours were in the ratio 2 : 3. The resulting colour seemed too light to him, so he added 2 L of red paint. This way, the ratio of the volumes of the red and yellow colours changed to 3 : 2. How many litres of paint did the decorator use?
   (A) 5 L  (B) 6 L  (C) 7 L  (D) 8 L  (E) 9 L

Solution by the editors.

Say the original amounts of paint were 2x and 3x. Then the amount of red in the final product is 2x + 2 while the amount of yellow is 3x. Thus \(\frac{2x + 2}{3x} = \frac{3}{2}\). Solving the equation yields \(x = \frac{4}{5}\), so the total amount of paint is \(2x + 2 + 3x = 5x + 2 = 6\).

6. (Grades 3-6) Two boys are playing tennis until one of them wins four times. A tennis match cannot end in a draw. What is the greatest number of games they can play?
   (A) 8  (B) 7  (C) 6  (D) 5  (E) 9

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

They can play at most seven games; for example ABABABA.

Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; KEVIN LI, student, Pinetree Secondary School, Coquitlam, BC; LENA CHOI, student, Ecole Banting Middle School, Coquitlam, BC.
7. (Grades 5-6) In two years, my son will be twice as old as he was two years ago. In three years, my daughter will be three times as old as she was three years ago. Which of the following best describes the ages of the daughter and the son?

(A) The son is older;   (B) The daughter is older;   (C) They are twins;   
(D) The son is twice as old as the daughter;   
(E) The daughter is twice as old as the son.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

One easily finds that both children are six years old and therefore twins.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; and LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

To find the children's ages without resorting to guess-and-check, say the son is now \(x\) years old. Then two years ago he was \(x - 2\) years old, and in two years he will be \(x + 2\) years old. Thus \(2(x - 2) = x + 2\), which yields that \(x = 6\). Likewise for the daughter.

8. (Grades 7-8) Some points are marked on a straight line so that all distances 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm, 7 cm, and 9 cm are among the distances between these points. At least how many points are marked on the line?

(A) 4   (B) 5   (C) 6   (D) 7   (E) 8

Solution by the editors.

Only \(\binom{4}{2} = \frac{4!}{2!2!} = 6\) distances are possible with four points. You need eight different distances, so at least five points are needed. The diagram shows that five points are sufficient.

9. (Grades 9-10) Eva, Betty, Linda, and Cathy went to the cinema. Since it was not possible to buy four seats next to each other, they bought tickets for seats number 7 and 8 in the 10th row and tickets for seats number 3 and 4 in the 12th row. How many seating arrangements can they choose from, if Cathy does not want to sit next to Betty?

(A) 24   (B) 20   (C) 16   (D) 12   (E) 8

Solution Lena Choi, student, École Banting Middle School, Coquitlam, BC.

Cathy can sit with Linda and Eva with Betty in eight ways: CLEB, CLBE, LCEB, LCBE, EBLC, EBCL, BECL, and BELC. Likewise, Cathy can sit with Betty and Eva with Linda in eight ways. That makes 16 arrangements in all.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; and CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC.
10. (Grades 11-12) Triangle $ABC$ is isosceles with $BC = AC$. The segments $DE, FG, HI, KL, MN, OP$, and $XY$ divide the sides $AC$ and $CB$ into equal parts. Find $XY$, if $AB = 40$ cm.

(A) 38 cm  (B) 35 cm  
(C) 33 cm  (D) 30 cm  (E) 27 cm

Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

From $C$ and each of the points on $AC$ and $BC$ drop a line orthogonally onto $AB$. Since the points on $AC$ and $BC$ are equally spaced, the intersection points on $AB$ are also equally spaced. Thus

$$|AQ| = \frac{1}{16}|AB| = \frac{5}{2}.$$  Therefore,  

$$|XY| = |QR| = 40 - \frac{5}{2} \cdot 2 = 35.$$  

Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; and LENA CHOI, student, Ecole Banting Middle School, Coquitlam, BC.

11. (Grades 3-4) Matt and Nick constructed two buildings, shown in the figures, using identical cubes. Matt's building weighs 200 g, and Nick's building weighs 600 g. How many cubes from Nick's building are hidden and cannot be seen in the figure?

(A) 1  (B) 2  (C) 3  
(D) 4  (E) 5  Nick's building  Matt's building

Solution by Eliza Kuan, student, Meadowridge School, Maple Ridge, BC.

Matt's building is built from exactly five cubes, so each cube weighs \(\frac{1}{2}(200\, \text{g}) = 40\, \text{g}\). Therefore, Nick's building must consist of 600 g/40 g = 15 cubes. Of these eleven are visible, so four are hidden.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; IAN CHEN, student, Centennial Secondary School, Coquitlam, BC; KEVIN LI, student, Pinetree Secondary School, Coquitlam, BC; and LENA CHOI, student, Ecole Banting Middle School, Coquitlam, BC.

12. (Grades 5-6) Consider all four-digit numbers divisible by 6 whose digits are in increasing order, from left to right. What is the hundreds digit of the largest such number?

(A) 7  (B) 6  (C) 5  (D) 4  (E) 3
Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

For the number to be divisible by 6, its unit digit must be even so at most 8. Therefore the thousands digit is at most 5. If the thousands digit is 5, the number must be 5678 which is not divisible by 3 and so not by 6. If the thousands digit is 4, the unit digit must still be 8 for the number to be even. Only the numbers 4568, 4578, and 4678 are like this. Of these only 4578 is divisible by 3 (and so 6), so 4578 is the desired number.

13. (Grades 7-8) A square of side length 3 is divided by several segments into polygons as shown in the figure. What percent of the area of the original square is the area of the shaded figure?

(A) 30%  (B) 33 1/3%  (C) 35%
(D) 40%  (E) 50%

Solution by the editors.

Consider the portion of the diagram on the left. By the Pythagorean Theorem, we have $|AC| = \sqrt{10}$.

Since $\triangle ABC$ and $\triangle ADB$ are similar, $\frac{|BD|}{3} = \frac{1}{\sqrt{10}}$, so $|BD| = \frac{3}{\sqrt{10}}$. Using the Pythagorean Theorem again, $|AD| = \sqrt{3^2 - \left(\frac{3}{\sqrt{10}}\right)^2} = \sqrt{\frac{81}{10}} = \frac{9}{\sqrt{10}}$. It follows that $\triangle ABD$ has area $\frac{1}{2} |AD| |BD| = \frac{27}{20}$.

Note that the white region in the original diagram is exactly four copies of $\triangle ABD$. Therefore, the area of the white region is $\frac{27}{5}$ and the area of the shaded square is $3^2 - \frac{27}{5} = \frac{18}{5}$, which is 40% of the large square.

14. (Grades 9-10) A boy always tells the truth on Thursdays and Fridays, always tells lies on Tuesdays, and tells either truth or lies on the rest of the days of the week. Every day he was asked what his name was and six times in a row he gave the following answers: John, Bob, John, Bob, Pit, Bob. What did he answer on the seventh day?

(A) John  (B) Bob  (C) Pit  (D) Kate  (E) Not enough information to decide

Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

Since the boy tells the truth on Thursdays and Fridays, he must give
the same answer twice in a row. So far the boy has never answered the same
twice in a row, so his seventh answer must either be “Bob” to match the sixth
answer or “John” to match the first answer. If his seventh answer is “Bob”,
then the seventh day is a Friday, so the fourth day was a Tuesday on which
he lied and said “Bob.” This is a contradiction. Thus the seventh answer is
“John”. The table below demonstrates that this is indeed possible.

<table>
<thead>
<tr>
<th>John</th>
<th>Bob</th>
<th>John</th>
<th>Bob</th>
<th>Pit</th>
<th>Bob</th>
<th>John</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friday</td>
<td>Saturday</td>
<td>Sunday</td>
<td>Monday</td>
<td>Tuesday</td>
<td>Wednesday</td>
<td>Thursday</td>
</tr>
<tr>
<td>the truth</td>
<td>either</td>
<td>either</td>
<td>a lie</td>
<td>either</td>
<td>the truth</td>
<td></td>
</tr>
</tbody>
</table>

Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC.

15. (Grades 11-12) An equilateral triangle and a circle
M are inscribed in a circle K, as shown in the figure.
What is the ratio of the area of K to the area of M?

(A) 8 : 1  (B) 10 : 1  (C) 12 : 1
(D) 14 : 1  (E) 16 : 1

Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

Let O be the centre of the triangle and sup-
pose that the circle K has radius 1 and, thus,
area π. Since the triangle is equilateral,
∠DAO = 30°, so |OD| = |AO| sin 30° = \frac{1}{2} and
hence the diameter of circle M is \frac{1}{2}. There-
fore, the area of the circle M is \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{16}
and the ratio of areas is 16 : 1.

This issue’s prize for the best solutions goes to Gesine Geupel, student,
Max Ernst Gymnasium, Brühl, NRW, Germany. We hope that all our readers
enjoy the challenge of solving our problems and presenting their solutions to
others.