Note that if \((x, y, z)\) is a solution, then so are \((x, -y, -z)\), \((-x, y, -z)\), and \((-x, -y, z)\). (No other combination of minus signs will work, since an even number of minus signs is necessary to make the sign of \(xyz\) correct.) Hence, we may assume that \(x, y, z > 0\) (since none of \(x, y, z\) can equal 0).

The Cauchy–Schwarz Inequality says that if \(a, b, c, A, B,\) and \(C\) are real numbers, then

\[
(a^2 + b^2 + c^2)(A^2 + B^2 + C^2) \geq (aA + bB + cC)^2,
\]

with equality if and only if \((a, b, c)\) is a scalar multiple of \((A, B, C)\).

By the Cauchy–Schwarz Inequality, we have

\[
36 = \left( x^2 + y^2 + z^2 \right) \left( \frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} \right)
\]

\[
\geq \left( x \left( \frac{1}{x} \right) + y \left( \frac{2}{y} \right) + z \left( \frac{3}{z} \right) \right)^2 = 36.
\]

Since equality holds, then \((x, y, z)\) must be a scalar multiple of \(\left( \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right)\).

Thus, \(x = \frac{k}{x}, y = \frac{2k}{y},\) and \(z = \frac{3k}{z}\), for some \(k > 0\), and so \(x^2 = k, y^2 = 2k,\) and \(z^2 = 3k.\)

Substituting back into the original second equation, we obtain \(6k = 9,\)
and so \(k = \frac{3}{2},\) whence \(x = \sqrt{\frac{3}{2}}, y = \sqrt{3},\) and \(z = \sqrt{\frac{9}{2}} = \frac{3\sqrt{2}}{2}.\) This gives the four solutions above, as claimed.

Also solved by GEORGE APOTOLEPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; C U R T I S G. CHRYSSOSTOMOS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and RICARD PEIRO, IES “Abastos”, Valencia, Spain. There were two incomplete solutions submitted.

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**Problem of the Month**

Ian VanderBurgh

Sometimes we mathematicians like to use a sledgehammer when perhaps a smaller instrument would be in order.

**Problem 1** (2009 Gauss Contest, Grade 7) If \(x, y,\) and \(z\) are positive integers with \(xy = 18, xz = 3,\) and \(yz = 6,\) what is the value of \(x + y + z?\)

(A) 6 \hspace{10mm} (B) 10 \hspace{10mm} (C) 25 \hspace{10mm} (D) 11 \hspace{10mm} (E) 8

If we weren’t in Grade 7, then here are two ways that we might tackle this problem:
Solution 1 Since \( xy = 18 \), \( xz = 3 \), and \( yz = 6 \), by multiplying these equations we have \((xy)(xz)(yz) = 18(3)(6)\), or \( x^2y^2z^2 = 18^2 \), or \((xyz)^2 = 18^2\).

Since \( x \), \( y \), and \( z \) are positive, then \( xyz > 0 \), so \( xyz = 18 \). We can now combine this with the original three equations as follows.

Since \( xy = 18 \), we have \( \frac{xyz}{xy} = \frac{18}{18} \) or \( z = 1 \). Since \( xz = 3 \), we have \( \frac{xyz}{xz} = \frac{18}{3} \), or \( y = 6 \). Since \( yz = 6 \), we have \( \frac{xyz}{yz} = \frac{18}{6} \), or \( x = 3 \).

Therefore, \( x + y + z = 3 + 6 + 1 = 10 \). ■

Solution 2 Since \( xy = 18 \) and \( xz = 3 \), then \( \frac{xy}{xz} = \frac{18}{3} \), or \( \frac{y}{z} = 6 \), or \( y = 6z \).

Since \( yz = 6 \), then \((6z)z = 6 \), or \( z^2 = 1 \). Since \( z > 0 \), then \( z = 1 \), and so \( y = 6z = 6 \). Since \( xz = 3 \) and \( z = 1 \), then \( x = 3 \).

Therefore, \( x + y + z = 3 + 6 + 1 = 10 \). ■

These are two great solutions using standard techniques for solving systems of equations. But they're hardly suitable for Grade 7 students. This problem is a perfectly good Grade 7 problem, though.

Solution 3 We are told that \( x \), \( y \), and \( z \) are positive integers. Since \( xz = 3 \) and 3 is a prime, then \( x \) and \( z \) must be 1 and 3, or 3 and 1, respectively.

Let's look at the case \( x = 1 \) and \( z = 3 \). Since \( x = 1 \) and \( xy = 18 \) and the number that we multiply 1 by to get 18 is 18 itself, then \( y = 18 \).

Therefore, \( yz = 18(3) = 54 \), which disagrees with the equation \( yz = 6 \). Thus, \( x = 1 \) and \( z = 3 \) is not the correct combination. (For the record, I never choose the correct line in the grocery store either.)

So we look at the case \( x = 3 \) and \( z = 1 \). Since \( x = 3 \) and \( xy = 18 \) and the number that we multiply 3 by to get 18 is 6, then \( y = 6 \). (The product of \( yz \) does in fact equal 6, as required.) Thus, \( x = 3 \), \( y = 6 \), and \( z = 1 \).

Therefore, \( x + y + z = 3 + 6 + 1 = 10 \). ■

It's easy to get "trapped" into using the high powered techniques that we know, but sometimes there is a nicer solution that uses less machinery.

Here is a problem in a similar vein that has developed a following in the math contest world.

Problem 2 (1988 UK Schools Mathematical Challenge) Weighing the baby at the clinic was a problem. The baby would not keep still and caused the scales to wobble. So I held the baby and stood on the scales while the nurse read off 78 kg. Then the nurse held the baby while I read off 69 kg. Finally, I held the nurse while the baby read off 137 kg. What is the combined weight of all three (in kg)?

(A) 142  (B) 147  (C) 206  (D) 215  (E) 284

Again, there is a standard system-of-equations type solution which is worth seeing. To save confusion, we'll give the narrator a name chosen at random, say, Tony.
Solution 1 Let Tony's weight be \( x \) kg, let the baby's weight be \( y \) kg, and let the nurse's weight be \( z \) kg.

The combined weight of Tony and the baby is 78 kg, so \( x + y = 78 \).
The combined weight of the baby and the nurse is 69 kg, so \( y + z = 69 \). The combined weight of Tony and the nurse is 137 kg, so \( x + z = 137 \).

We can show this system of equations nicely in a visual way:

\[
\begin{align*}
x + y &= 78, \\
y + z &= 69, \\
x + z &= 137.
\end{align*}
\]

By the way, laying out the equations in this fashion is a really useful thing to do and gives you a much better idea of what to do than by writing

\[
\begin{align*}
x + y &= 78, \\
y + z &= 69, \\
x + z &= 137.
\end{align*}
\]

In any event, the "nice" way of writing the equations allows us to see that adding the three equations is a really good idea. When we do this, we obtain \( 2x + 2y + 2z = 78 + 69 + 137 \), or \( 2(x + y + z) = 284 \), or \( x + y + z = 142 \).

Notice that we don't actually need to determine \( x, y, \) and \( z \) at all!

But, again, we don't need to do anything nearly that fancy. In fact, we can get away without doing any algebra at all.

Solution 2 First, we look at the fact that the combined weight of Tony and the baby is 78 kg and the combined weight of the nurse and the baby is 69 kg.

Since the baby's weight is included in both of these totals, then Tony must be \( 78 - 69 = 9 \) kg heavier than the nurse (that is, the difference between Tony's weight and the nurse's weight is 9 kg).

But the combined weight of Tony and the nurse is 137 kg. We want to find two numbers that add to 137, one of which is 9 greater than the other. To find these numbers, we can subtract 9 from 137 to get 128 and then divide by 2 to get 64. The numbers 64 and 64 + 9 = 73 differ by 9 and add to 137, so must be the weights, in kg, of the nurse and Tony, respectively.

Since the combined weight of Tony and the baby is 78 kg and Tony weighs 73 kg, then the weight of the baby is 5 kg.

Therefore, the combined weight of all three is 64 + 73 + 5 = 142 kg. ■

So put your sledgehammer back in the garage, and think before you leap.