MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le 15 décembre 2009. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précéderà l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.


Déterminer toutes les solutions de l'équation

\[
\frac{1}{x-1} + \frac{2}{x-2} + \frac{6}{x-6} + \frac{7}{x-7} = x^2 - 4x - 4. 
\]

M401. Proposé par l'Équipe de Mayhem.

Graham et Vazz sont en train de projeter une nouvelle pelouse aux Quartiers Généraux de CRUX. Graham dit : "Si tu choisis une pelouse 9 mètres plus longue et 8 mètres plus étroite, la surface sera la même." Vazz répond : "Si tu la choisis 12 mètres plus courte et 16 mètres plus large, la surface sera aussi la même." Quelles sont les dimensions de la pelouse projetée ?

M402. Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucene Sava, Berca, Roumanie.

Trouver toutes les paires ordonnées \((a, b)\) d'entiers tels que

\[
a^b b^a + a^b + b^a = 89.
\]
M403. Proposé par Matthew Babbitt, étudiant, Albany Area Math Circle, Fort Edward, NY, É-U.

Jean a écrit un programme sur son ordinateur pour tester si un entier plus grand que 1 est un nombre premier. Sa sœur, Alice, a écrit le code de telle sorte que si l'entrée est impaire, la probabilité que le programme donne une réponse correcte est de 52% et si l'entrée est paire, cette probabilité est 98%. Jean vérifie le programme en testant deux entiers plus grands que 1 choisis au hasard. Quelle est la probabilité que les deux réponses soient correctes ?

M404. Proposé par Bill Sands, Université de Calgary, Calgary, AB.

Un magasin vend des copies d'un certain article à $a$$ pièce, ou $y$$ pour $a$ copies, ou encore $z$$ pour $a$ copies, $a$ et $b$ étant des entiers tels que $1 < a < b$ et $x$, $y$ et $z$ des nombres réels positifs. Pour rendre le rabais "$y$$ pour $a$ copies" intéressant, $y$$ devrait être plus bas que le prix payé pour $a$ achats au prix de $a$$, donc $y < ax$. Pour rendre le deuxième rabais "$z$$ pour $a$ copies" intéressant aussi, on pourrait insister sur une des deux conditions :

(a) $\frac{z}{b} < \frac{y}{a}$ ; c.-à-d. que le prix moyen pour une copie soit moindre que le prix correspondant du premier rabais.

(b) Chaque fois qu'on peut écrire $b = qa + r$ avec $q$ et $r$ des entiers non négatifs, alors on a $a < y + rx$; c.-à-d. qu'il en coûterait plus cher d'acheter $b$ copies en combinant le premier rabais avec l'achat individuel de copies supplémentaires plutôt que d'opter pour le deuxième rabais.

Montrer que si la condition (a) est satisfaite, alors la condition (b) l'est aussi. Donner aussi un exemple pour montrer que la condition (b) pourrait être satisfaite sans que la condition (a) le soit.

M405. Proposé par George Apostolopoulos, Messolonghi, Grèce.

Trouver une formule donnant la valeur de la somme

$$17 + 187 + 1887 + 18887 + \cdots + 188\ldots87,$$

où le dernier terme contient exactement $n$ chiffres 8.


Le carré $ABCD$ est inscrit dans le huitième d'un cercle de rayon 1 et de centre $O$ de sorte qu'il ait un sommet sur chaque rayon et les sommets $B$ et $C$ sur l'arc. Le carré $EFGH$ est inscrit dans le triangle $DOA$ de sorte que $E$ et $H$ soient sur les rayons et $F$ et $G$ soient sur $AD$. Dans le problème M295 [2007 : 200, 202 ; solution 2008 : 203-204], on a vu que l'aire du carré $ABCD$ est $\frac{2 - \sqrt{3}}{3}$. Trouver l'aire du carré $EFGH$. 

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\textbf{M381.} Correction. \textit{Proposed by Mihály Benze, Brasov, Romania.}

Determine all of the solutions to the equation

\[ \frac{1}{x - 1} + \frac{2}{x - 2} + \frac{6}{x - 6} + \frac{7}{x - 7} = x^2 - 4x - 4. \]

\textbf{M401.} \textit{Proposed by the Mayhem Staff.}

Graham and Vazz were marking out a new lawn at \textit{CRUX} Headquarters. Graham said: "If you make the lawn 9 metres longer and 8 metres narrower, the area will be the same". Vazz said: "If you make it 12 metres shorter and 16 metres wider, the area will still be the same". What are the dimensions of the lawn?

\textbf{M402.} \textit{Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.}

Determine all ordered pairs \((a, b)\) of positive integers such that

\[ a^b b^a + a^b + b^a = 89. \]

\textbf{M403.} \textit{Proposed by Matthew Babbitt, student, Albany Area Math Circle, Fort Edward, NY, USA.}

Jason wrote a computer program that tests if an integer greater than 1 is prime. His devious sister Alice has edited the code so that if the input is odd, the probability that the program gives the correct output is 52\% and if the input is even, the probability that the program gives the correct output is 98\%. Jason tests the program by inputting two random integers each greater than 1. What is the probability that both outputs are correct?

\textbf{M404.} \textit{Proposed by Bill Sands, University of Calgary, Calgary, AB.}

A store sells copies of a certain item at \$x\ each, or \at \textit{a} items for \$y\, or \at \textit{b} items for \$z\, where \textit{a} and \textit{b} are positive integers satisfying \(1 < a < b\) and \(x, y,\) and \(z\) are positive real numbers. To make \"\textit{a} items for \$y\" a sensible bargain, \$\textit{y}\ should be less than buying \textit{a} separate items; in other words we should have \(y < ax\). To make \"\textit{b} items for \$z\" also a sensible bargain, we could insist on one of two conditions:

\begin{enumerate}
\item[(a)] \(\frac{x}{b} < \frac{y}{a}\); that is, the average price of an item under the \"\textit{b} items for \$z\" deal is less than under the \"\textit{a} items for \$y\" deal.
\item[(b)] Whenever we can write \(b = qa + r\) for nonnegative integers \(q\) and \(r\), then \(z < qy + rx\) holds; that is, it should always cost more to buy \textit{b} items by buying a combination of \textit{a} items plus individual items, than by choosing the \"\textit{b} items for \$z\" deal.
\end{enumerate}

Show that if condition (a) is true, then condition (b) is also true. Give an example to show that condition (b) could be true while condition (a) is false.
M405. Proposed by George Apostolopoulos, Messolonghi, Greece.

Determine a closed form expression for the sum

\[ 17 + 187 + 1887 + 18887 + \cdots + 18 \ldots 87, \]

where the last term contains exactly \( n \) 8's.

M406. Proposed by Constantino Ligouras, student, E. Majorana Scientific High School, Putignano, Italy.

Square \( ABCD \) is inscribed in one-eighth of a circle of radius 1 and centre \( O \) so that there is one vertex on each radius and two vertices \( B \) and \( C \) on the arc. Square \( EFGH \) is inscribed in \( \triangle DOA \) so that \( E \) and \( H \) lie on the radii, and \( F \) and \( G \) lie on \( AD \). In problem M295 [2007 : 200, 202; solution 2008 : 203-204], we saw that the area of square \( ABCD \) is \( \frac{2 - \sqrt{2}}{3} \). Determine the area of square \( EFGH \).

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**Mayhem Solutions**

M369. Proposed by the Mayhem Staff.

A rectangle has vertices \( A(0, 0) \), \( B(6, 0) \), \( C(6, 4) \), and \( D(0, 4) \). A horizontal line is drawn through \( P(4, 3) \), meeting \( BC \) at \( M \) and \( AD \) at \( N \). A vertical line is drawn through \( P \), meeting \( AB \) at \( Q \) and \( CD \) at \( R \). Prove that \( AP \), \( DM \), and \( BR \) all pass through the same point.

*Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.*

Since the sides of rectangle \( ABCD \) are parallel to the axes, any point on \( BC \) has \( x \)-coordinate 6 and any point on \( CD \) has \( y \)-coordinate 4. Thus, \( M \) has coordinates \((6, 3)\) and \( R \) has coordinates \((4, 4)\).

The line through \( A(0, 0) \) and \( P(4, 3) \) has slope \( \frac{3 - 0}{4 - 0} = \frac{3}{4} \) and passes through the origin, so has equation \( y = \frac{3}{4}x \).

The line through \( D(0, 4) \) and \( M(6, 3) \) has equation \( \frac{y - 4}{x - 0} = \frac{4 - 3}{0 - 6} \), or equivalently \( y = -\frac{1}{6}x + 4 \).

The line through \( B(6, 0) \) and \( R(4, 4) \) has equation \( \frac{y - 0}{x - 6} = \frac{4 - 4}{6 - 4} \), or equivalently \( y = -2x + 12 \).

Next, we find the point of intersection of the lines \( AP \) and \( DM \) by equating to obtain \( \frac{3}{4}x = -\frac{1}{6}x + 4 \), or \( \frac{11}{12}x = 4 \), or \( x = \frac{48}{11} \). Since \( y = \frac{3}{4}x \)
on line $AP$, then $y = \frac{3}{4} \left( \frac{48}{11} \right) = \frac{36}{11}$. Therefore, the point of intersection of lines $AP$ and $DM$ is $\left( \frac{48}{11}, \frac{36}{11} \right)$.

Next, we find the point of intersection of lines $AP$ and $BR$ by equating to obtain $\frac{3}{4}x = -2x + 12$, or $\frac{11}{4}x = 12$, or $x = \frac{48}{11}$. As before, we see that the point of intersection of lines $AP$ and $BR$ is $\left( \frac{48}{11}, \frac{36}{11} \right)$.

Since line $AP$ intersects lines $DM$ and $BR$ at the same point, it follows that all three lines are concurrent, meeting at $\left( \frac{48}{11}, \frac{36}{11} \right)$.

Also solved by EDIN AJANOVIC, student. First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messaughi, Greece; JACLYN CHANG, student, Western Canada High School, Calgary, AB; JULIA CLINE, student, Walt Whitman High School, Bethesda, MD, USA; KATHERINE JANELL EYRE, student, Angelo State University, San Angelo, TX, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SANCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; RICARD PEIRO, IES "Abastos", Valencia, Spain; BRUCE SHAWVER, Memorial University of Newfoundland, St. John’s, N.L; and NECULAI STANCIU, Saint Mureci Sava Technical High School, Berca, România.

After finding the first point of intersection, it would suffice to show that this point lies on line $BR$. Strictly speaking, we have proven that the lines through $A$ and $P$, $D$ and $M$, and $B$ and $R$ all pass through the same point; line segments $DM$ and $BR$ pass through this point, but line segment $AP$ does not contain the point $\left( \frac{48}{11}, \frac{36}{11} \right)$ (though its extension does).

**M370. Proposed by the Mayhem Staff.**

(a) Prove that $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$ for all angles $A$ and $B$.

(b) Prove that $\cos C + \cos D = 2 \cos \left( \frac{C + D}{2} \right) \cos \left( \frac{C - D}{2} \right)$ for all angles $C$ and $D$.

(c) Determine the exact value of $\cos 20° + \cos 60° + \cos 100° + \cos 140°$, without using a calculator.

**Solution by Courtis G. Chrysostomos, Larissa, Greece.**

(a) Applying the sum and difference formulae for the cosine, we obtain

$$
\cos(A + B) + \cos(A - B)
= \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B
= 2 \cos A \cos B.
$$

(b) Applying the formula from (a) with $A = \frac{C + D}{2}$ and $B = \frac{C - D}{2}$, we obtain $A + B = C$ and $A - B = D$, and so

$$
\cos C + \cos D = 2 \cos \left( \frac{C + D}{2} \right) \cos \left( \frac{C - D}{2} \right).
$$
(c) We rearrange the given expression and apply the formula from (b):
\[
\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ \\
= \cos 140^\circ + \cos 20^\circ + \cos 100^\circ + \cos 60^\circ \\
= 2 \cos \left( \frac{140^\circ + 20^\circ}{2} \right) \cos \left( \frac{140^\circ - 20^\circ}{2} \right) + \cos 100^\circ + \cos 60^\circ \\
= 2 \cos 80^\circ \cos 60^\circ + \cos 100^\circ + \cos 60^\circ \\
= 2 \cos 80^\circ \left( \frac{1}{2} \right) + \cos 100^\circ + \frac{1}{2} = \cos 80^\circ + \cos 100^\circ + \frac{1}{2} \\
= \cos 80^\circ + (-\cos 80^\circ) + \frac{1}{2} = \frac{1}{2}.
\]

Also solved by EDIN AJANOVIC, student, First Bosnian High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JULIA CLINE, student, Walt Whitman High School, Bethesda, MD, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; RALPH LOZANO, student, Missouri State University, Missouri, USA; HUGO LUYO SANCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; RICARD PEIRO, IES "Abastos", Valencia, Spain; NECULAI STANDIU, Saint Murenic Sava Technological High School, Berez, Romania; VASILE TEODOROVICI, Toronto, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were two incomplete solutions submitted.


Suppose that the line segment \(AB\) has length 3 and \(C\) is on \(AB\) with \(AC = 2\). Equilateral triangles \(ACF\) and \(CBE\) are constructed on the same side of \(AB\). If \(K\) is the midpoint of \(FC\), determine the area of \(\triangle AKE\).

Solution by Antonio Godoy Toharia, Madrid, Spain.

Since \(K\) is the midpoint of segment \(FC\), we have \(KC = \frac{1}{2}(FC) = 1 = CE\). Because \(\triangle ACF\) and \(\triangle CBE\) are equilateral, \(\angle FCA = \angle ECB = 60^\circ\), so we have \(\angle KCE = 60^\circ\).

Therefore \(\triangle KCE\) is also equilateral, and so \(KE = 1\).

Since \(\angle EKC = \angle KCA\), then \(KE\) is parallel to \(AB\), and so we can think of \(\triangle AKE\) as having base \(KE = 1\) and height equal to the vertical distance between the lines through \(KE\) and \(AB\).

The height of \(\triangle CBE\) is \(h = 1 \sin(60^\circ) = \frac{\sqrt{3}}{2}\). Thus, the distance between \(KE\) and \(AB\) is \(\frac{\sqrt{3}}{2}\).

Therefore, the area of \(\triangle AKE\) is equal to \(\frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{4}\).

Also solved by EDIN AJANOVIC, student, First Bosnian High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG.
M372. Proposed by the Mayhem Staff.

A real number \( x \) satisfies \( x^3 = x + 1 \). Determine integers \( a, b, \) and \( c \) so that \( x^7 = ax^2 + bx + c \).

Solution by Paul Bracken, University of Texas, Edinburg, TX, USA.

Since \( x^3 = x + 1 \), we have

\[
x^3 \cdot x^3 = (x + 1)(x + 1) = x^2 + 2x + 1,
\]

which yields

\[
x^7 = x \cdot x^6 = x(x^2 + 2x + 1) = x^3 + 2x^2 + x
\]

\[
= (x + 1) + 2x^2 + x = 2x^2 + 2x + 1.
\]

Thus, if \( a = 2, b = 2, \) and \( c = 1 \), we have \( x^7 = ax^2 + bx + c \).

Also solved by EDIN AJANOVIĆ, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APPOSTOLOPOULOS, Messologi, Greece; CAO MINH QUANG, Nguyễn Bình Khiêm High School, Vinh Long, Vietnam; ANTONIO GODOY TOHARIA, Madrid, Spain; JOSE HERNANDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SANCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; RICARD PEIRO, IES “Abastos”, Valencia, Spain; and JIHXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. It can be proven using algebra that \( a = 2, b = 2, \) and \( c = 1 \) is the only solution.

M373. Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

The side lengths of a triangle are three consecutive positive integers and the largest angle in the triangle is twice the smallest one. Determine the side lengths of the triangle.

1. Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Suppose that the side lengths of the triangle are \( n - 1, n, \) and \( n + 1 \) for some positive integer \( n \geq 3 \). (Smaller values of \( n \) do not give a triangle.) Suppose that the smallest angle is \( \theta \), which is opposite the shortest side (of length \( n - 1 \)). Thus, the largest angle is \( 2\theta \), which is opposite the longest side (of length \( n + 1 \)).

By the Law of Sines, \( \frac{n - 1}{\sin \theta} = \frac{n + 1}{\sin 2\theta} \) and so \( \frac{n - 1}{\sin \theta} = \frac{n + 1}{2 \sin \theta \cos \theta} \). Since \( \sin \theta \neq 0 \), then \( \cos \theta = \frac{n + 1}{2(n - 1)} \).
Now, by the Law of Cosines,

\[
(n - 1)^2 = n^2 + (n + 1)^2 - 2n(n + 1) \cos \theta ; \\
(n - 1)^2 = n^2 + (n + 1)^2 - 2n(n + 1) \frac{n + 1}{2(n - 1)} ; \\
n^2 - 2n + 1 = n^2 + n^2 + 2n + 1 - \frac{n(n + 1)^2}{n - 1} ; \\
\frac{n(n + 1)^2}{n - 1} = n^2 + 4n ; \\
n(n^2 + 2n + 1) = (n^2 + 4n)(n - 1) ; \\
n^2 + 2n + 1 = (n + 4)(n - 1) \quad \text{(since } n \neq 0) ; \\
n^2 + 2n + 1 = n^2 + 3n - 4 ; \\
n = 5 .
\]

Therefore, \( n = 5 \), and the side lengths are 4, 5, and 6.

II. Solution by Vasile Teodorovici, Toronto, ON.

Let the triangle be \( ABC \), with side lengths \( AB = m, AC = m + 1, \) and \( BC = m + 2 \) for some positive integer \( m \geq 2 \). Then the smallest angle is \( \angle ACB \), which we label \( \theta \), and the largest is \( \angle BAC \), which we label \( 2\theta \).

Let \( AD \) be the angle bisector of \( \angle BAC \), with \( D \) on \( BC \). Let \( BD = x \) and \( DC = y \). Note that \( BC = AB + 2 \), so \( x + y = m + 2 \).

By the Angle Bisector Theorem,
\[
\frac{AC}{AB} = \frac{DC}{DB} \quad \text{so} \quad \frac{m + 1}{m} = \frac{y}{x} .
\]

Thus, \( \frac{m + 1}{m} + 1 = \frac{y}{x} + 1 \) and so
\[
\frac{2m + 1}{m} = \frac{x + y}{x} = \frac{m + 2}{x} .
\]

This gives \( x = \frac{m(m + 2)}{2m + 1} \) and \( y = \frac{m + 1}{m} x = \frac{(m + 1)(m + 2)}{2m + 1} \).

Next, we note that \( \angle BAD = \angle BCA = \theta \) so \( \triangle BAD \) is similar to \( \triangle BCA \), whence \( \frac{BD}{BA} = \frac{BA}{BC} \) or \( \frac{m(m + 2)}{2m + 1} / m = \frac{m}{m + 2} \), which gives the equivalent equations \( (m + 2)^2 = m(2m + 1) \), and \( m^2 + 4m + 4 = 2m^2 + m \), and \( 0 = m^2 - 3m - 4 \).

Since \( m \) is a positive integer, then \( m = 4 \), and so the side lengths are 4, 5, and 6.

Also solved by EDIN AJANOVIĆ, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; COURTIS G. CHRYSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes.
\textbf{M374. Proposed by Mihály Benze, Brasov, Romania.} \\
Suppose that \( p \) is a fixed prime number with \( p \geq 3 \). Determine the number of solutions to \( x^3 + y^3 = x^2y + xy^2 + p^{2009} \), where \( x \) and \( y \) are integers.

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

The equation can be rewritten as
\[
p^{2009} = x^3 - x^2y - xy^2 + y^3 = x^2(x - y) - y^2(x - y) = (x - y)(x^2 - y^2) = (x - y)^2(x + y).
\]

Since \( x \) and \( y \) are integers and the only divisors of \( p^{2009} \) are of the form \( \pm p^n \), then \( x - y = \pm p^k \) for some integer \( k \) with \( 0 \leq k \leq 1004 \). Thus, \( (x - y)^2 = p^{2k} \) and so \( x + y = p^{2009-2k} \). (The upper bound on \( k \) comes from the fact that \( (x - y)^2 \) is also a divisor of \( p^{2009} \).)

This yields
\[
x = \frac{1}{2}(x + y) + (x - y) = \frac{1}{2}(p^{2009-2k} \pm p^k),
\]
\[
y = \frac{1}{2}(x + y) - (x - y) = \frac{1}{2}(p^{2009-2k} \mp p^k).
\]

These are all integers since \( p \) is odd (in fact this is still true if \( p = 2 \) as long as \( k > 0 \)). Each pair \( (x, y) \) is distinct since for each pair the values of \( x + y \) and \( x - y \) are different.

Since there are 1005 possible values for \( k \) in the range \( 0 \leq k \leq 1004 \), there are then \( 2(1005) = 2010 \) solutions.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. There were four incorrect solutions and one incomplete solution submitted.

\textbf{M375. Proposed by Neculai Staniciu, Saint Mucenic Sava Technological High School, Berca, Romania.} \\
Determine all real solutions to the system of equations
\[
\frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} = 4; \quad x^2 + y^2 + z^2 = 9; \quad xyz = \frac{9}{2}.
\]

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

There are exactly four solutions for \( (x, y, z) \), namely \( \left( \frac{\sqrt{6}}{2}, \sqrt{3}, \frac{3\sqrt{2}}{2} \right) \), \( \left( -\frac{\sqrt{6}}{2}, -\sqrt{3}, -\frac{3\sqrt{2}}{2} \right) \), \( \left( -\frac{\sqrt{6}}{2}, \sqrt{3}, -\frac{3\sqrt{2}}{2} \right) \), and \( \left( \frac{\sqrt{6}}{2}, -\sqrt{3}, \frac{3\sqrt{2}}{2} \right) \).
Note that if \((x, y, z)\) is a solution, then so are \((x, -y, -z), (-x, y, -z),\) and \((-x, -y, z)\). (No other combination of minus signs will work, since an even number of minus signs is necessary to make the sign of \(xyz\) correct.) Hence, we may assume that \(x, y, z > 0\) (since none of \(x, y, z\) can equal 0).

The Cauchy–Schwarz Inequality says that if \(a, b, c, A, B,\) and \(C\) are real numbers, then

\[
(a^2 + b^2 + c^2)(A^2 + B^2 + C^2) \geq (aA + bB + cC)^2,
\]

with equality if and only if \((a, b, c)\) is a scalar multiple of \((A, B, C)\).

By the Cauchy–Schwarz Inequality, we have

\[
36 = (x^2 + y^2 + z^2) \left( \frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} \right) \geq \left( x \left( \frac{1}{x} \right) + y \left( \frac{2}{y} \right) + z \left( \frac{3}{z} \right) \right)^2 = 36.
\]

Since equality holds, then \((x, y, z)\) must be a scalar multiple of \(\left( \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right)\).

Thus, \(x = \frac{k}{x}, y = \frac{2k}{y},\) and \(z = \frac{3k}{z}\), for some \(k > 0\), and so \(x^2 = k, y^2 = 2k,\) and \(z^2 = 3k\).

Substituting back into the original second equation, we obtain \(6k = 9,\) and so \(k = \frac{3}{2}.\) whence \(x = \sqrt{\frac{3}{2}} = \sqrt{\frac{6}{2}},\) \(y = \sqrt{3},\) and \(z = \sqrt{\frac{9}{2}} = \frac{3\sqrt{2}}{2}.\) This gives the four solutions above, as claimed.

Also solved by GEORGE APOTOLEPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; COURTISG. CHRYSSOSTOMO S. Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and RICARD PEIRO, IES “Abastos”, Valencia, Spain. There were two incomplete solutions submitted.

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**Problem of the Month**

**Ian VanderBurgh**

Sometimes we mathematicians like to use a sledgehammer when perhaps a smaller instrument would be in order.

**Problem 1** (2009 Gauss Contest, Grade 7) If \(x, y,\) and \(z\) are positive integers with \(xy = 18, xz = 3,\) and \(yz = 6,\) what is the value of \(x + y + z?\)

\[(A) 6 \quad (B) 10 \quad (C) 25 \quad (D) 11 \quad (E) 8\]

If we weren’t in Grade 7, then here are two ways that we might tackle this problem:
Solution 1 Since $xy = 18$, $xz = 3$, and $yz = 6$, by multiplying these equations we have $(xy)(xz)(yz) = 18(3)(6)$, or $x^2y^2z^2 = 18^2$, or $(xyz)^2 = 18^2$.

Since $x$, $y$, and $z$ are positive, then $xyz > 0$, so $xyz = 18$. We can now combine this with the original three equations as follows.

Since $xy = 18$, we have $\frac{xyz}{xy} = \frac{18}{18}$, or $z = 1$. Since $xz = 3$, we have $\frac{xyz}{xz} = \frac{18}{3}$, or $y = 6$. Since $yz = 6$, we have $\frac{xyz}{yz} = \frac{18}{6}$, or $x = 3$.

Therefore, $x + y + z = 3 + 6 + 1 = 10$.

Solution 2 Since $xy = 18$ and $xz = 3$, then $\frac{xy}{xz} = \frac{18}{3}$, or $\frac{y}{z} = 6$, or $y = 6z$. Since $yz = 6$, then $(6z)z = 6$, or $z^2 = 1$. Since $z > 0$, then $z = 1$, and so $y = 6z = 6$. Since $xz = 3$ and $z = 1$, then $x = 3$.

Therefore, $x + y + z = 3 + 6 + 1 = 10$.

These are two great solutions using standard techniques for solving systems of equations. But they’re hardly suitable for Grade 7 students. This problem is a perfectly good Grade 7 problem, though.

Solution 3 We are told that $x$, $y$, and $z$ are positive integers. Since $xz = 3$ and 3 is a prime, then $x$ and $z$ must be 1 and 3, or 3 and 1, respectively.

Let’s look at the case $x = 1$ and $z = 3$. Since $x = 1$ and $xy = 18$ and the number that we multiply 1 by to get 18 is 18 itself, then $y = 18$. Therefore, $yz = 18(3) = 54$, which disagrees with the equation $yz = 6$. Thus, $x = 1$ and $z = 3$ is not the correct combination. (For the record, I never choose the correct line in the grocery store either.)

So we look at the case $x = 3$ and $z = 1$. Since $x = 3$ and $xy = 18$ and the number that we multiply 3 by to get 18 is 6, then $y = 6$. (The product $yz$ does in fact equal 6, as required). Thus, $x = 3$, $y = 6$, and $z = 1$.

Therefore, $x + y + z = 3 + 6 + 1 = 10$.

It’s easy to get “trapped” into using the high powered techniques that we know, but sometimes there is a nicer solution that uses less machinery.

Here is a problem in a similar vein that has developed a following in the math contest world.

Problem 2 (1988 U.K. Schools Mathematical Challenge) Weighing the baby at the clinic was a problem. The baby would not keep still and caused the scales to wobble. So I held the baby and stood on the scales while the nurse read off 78 kg. Then the nurse held the baby while I read off 69 kg. Finally, I held the nurse while the baby read off 137 kg. What is the combined weight of all three (in kg)?

(A) 142  (B) 147  (C) 206  (D) 215  (E) 284

Again, there is a standard system-of-equations type solution which is worth seeing. To save confusion, we’ll give the narrator a name chosen at random, say, Tony.
Solution 1 Let Tony's weight be $x$ kg, let the baby's weight be $y$ kg, and let the nurse's weight be $z$ kg.

The combined weight of Tony and the baby is 78 kg, so $x + y = 78$. The combined weight of the baby and the nurse is 69 kg, so $y + z = 69$. The combined weight of Tony and the nurse is 137 kg, so $x + z = 137$.

We can show this system of equations nicely in a visual way:

$$
\begin{align*}
  x + y &= 78, \\
  y + z &= 69, \\
  x + z &= 137.
\end{align*}
$$

By the way, laying out the equations in this fashion is a really useful thing to do and gives you a much better idea of what to do than by writing

$$
\begin{align*}
  x + y &= 78, \\
  y + z &= 69, \\
  x + z &= 137.
\end{align*}
$$

In any event, the "nice" way of writing the equations allows us to see that adding the three equations is a really good idea. When we do this, we obtain $2x + 2y + 2z = 78 + 69 + 137$, or $2(x + y + z) = 284$, or $x + y + z = 142$.

Notice that we don't actually need to determine $x$, $y$, and $z$ at all!

But, again, we don't need to do anything nearly that fancy. In fact, we can get away without doing any algebra at all.

Solution 2 First, we look at the fact that the combined weight of Tony and the baby is 78 kg and the combined weight of the nurse and the baby is 69 kg. Since the baby's weight is included in both of these totals, then Tony must be $78 - 69 = 9$ kg heavier than the nurse (that is, the difference between Tony's weight and the nurse's weight is 9 kg).

But the combined weight of Tony and the nurse is 137 kg. We want to find two numbers that add to 137, one of which is 9 greater than the other. To find these numbers, we can subtract 9 from 137 to get 128 and then divide by 2 to get 64. The numbers 64 and $64 + 9 = 73$ differ by 9 and add to 137, so must be the weights, in kg, of the nurse and Tony, respectively.

Since the combined weight of Tony and the baby is 78 kg and Tony weighs 73 kg, then the weight of the baby is 5 kg.

Therefore, the combined weight of all three is $64 + 73 + 5 = 142$ kg.

So put your sledgehammer back in the garage, and think before you leap.