The Tanker Problem

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When I walk my puppy with his retractable leash, he is forever getting behind, catching up and running past, cutting across in front, and getting behind again. When I was having my after-lunch nap the other day, this reminded me of those old "patrol boat circling an ocean liner" problems and I mused over the following situation.

A security patrol boat repeatedly circles a supertanker that is a gigantic rectangular box 450 metres long and 50 metres across. The ocean is calm and the tanker travels at a constant speed along a straight path. The patrol boat goes up the left side, across the front, down the right side, and across the back, and keeps doing it over and over.

The patrol boat travels in only two directions of the compass – when going parallel to the path of the tanker, it travels in straight lanes parallel to the tanker, one on each side at a distance of 25 metres from it, and when crossing in front or behind, it goes straight across perpendicular to the path of the tanker.

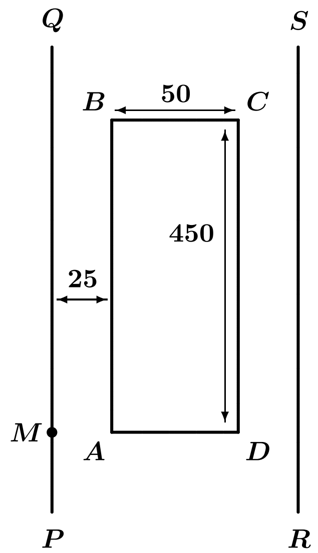

Figure 1

Neglecting the dimensions of the patrol boat (that is, considering it to be represented geometrically by a point) and given that it goes constantly at twice the speed of the tanker and that its turns are instantaneous, *what is the shortest distance that the patrol boat must travel in completing one cycle around the tanker*?

I felt the problem would be an easy recreation with a pleasing analysis and I expected a solution along the following lines.

Let the tanker and the lanes of the patrol boat be labeled as in Figure 1 and let the boat travel towards Q when it is in lane PQ.

Consider a cycle that begins at M on PQ; that is, to begin the boat is abreast of the stern AD of the tanker. While the boat travels 900 m up PQ, the tanker will have advanced 450 m and the boat will be abreast of the front BC of the tanker at the point E on PQ (Figure 2).

If the boat were to start crossing between its lanes at this point, it would crash into the tanker at a point 12.5 m down the side. In order to cross in front of the tanker, the boat must get $25 + 50 = 75$ m across the gap to the right hand edge of the tanker before the tanker advances to its level. Therefore, it needs to have a lead of 37.5 m when it starts its turn, for the tanker will advance 37.5 m while the boat is going across this 75 m.

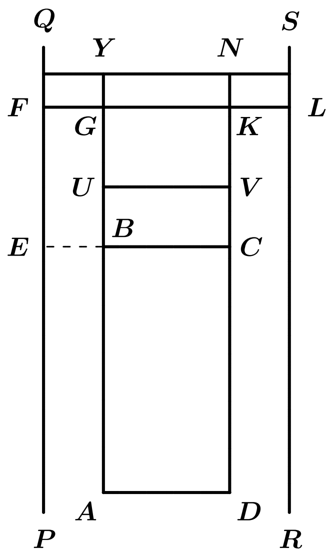

Figure 2

In order to get a 37.5 m lead, the boat must continue up PQ another 75 m beyond E (while the boat is doing this the tanker advances 37.5 m to yield a net lead of 37.5 m).

Summarizing, the boat and the tanker are abreast at E and BC; as the boat advances the 75 m to F, the tanker advances 37.5 m to UV; the boat turns at F and while going the 75 m along FK, the tanker advances 37.5 m to GK. Thus, the boat and the tanker just miss each other at K. And while the boat completes the remaining 25 m along KL to its other lane, the tanker takes a 12.5 m lead at YN.

Thus, when the boat turns down SR it is already 12.5 m down that side of the tanker. Now, the boat need go only to a point H that is 12.5 m from the far end before making its turn across the back (while it crosses the 25 m gap toward the tanker, the tanker moves ahead this 12.5 m and the boat just misses the bottom right corner of the tanker). Hence the boat needs to cover a distance of only $450 - 12.5 - 12.5 = 425$ m along this side before turning. Since it goes twice as fast as the tanker, while the boat covers two-thirds of this distance, the tanker moves ahead the other third, and so the boat need travel down that side only two-thirds of 425 m, that is, a distance of $\frac{850}{3}$ m.

Now, from just missing the tanker at its bottom right corner, the boat still has to go 75 m to get back to lane PQ. While the boat is doing this, the tanker gets ahead 37.5 m, and so, in order to catch up and complete its cycle, the patrol boat must go another 75 m up PQ to get abreast of the tanker.

Altogether, then, the patrol boat must travel a grand total of

$$900 + 75 + 100 + \frac{850}{3} + 100 + 75 \;=\; 1533\frac{1}{3}\,\text{m}\,.$$

This solution blithely assumes that the patrol boat and the tanker harmlessly slide by each other when they arrive simultaneously at the tanker's right hand corners. However, it makes a great deal more sense to interpret such an event as a collision: both vessels can hardly move into the same position at the same time without fatal consequences for the patrol boat. Fortunately, this nagging uncertainty is easily removed by adding the condition that *the two vessels are never to get closer to each other than* 25 m.

This is readily accommodated by having the boat get a lead of 62.5 m at F, before turning along the dangerous 75 m stretch FK between the lanes. This gives the boat a net lead of $62.5 - 37.5 = 25$ m when it reaches the right hand side of the tanker, thus avoiding any possibility of a collision and providing the 25 m separation at this critical juncture.

In order to get 62.5 m ahead, the boat must go another 125 m up PQ beyond E to a point F (Figure 3). Thus, when the patrol boat reaches F, the tanker is at UV, and the patrol boat is 62.5 m ahead.

Now the boat turns along FL, and when it reaches L, the tanker has advanced 37.5 m to IT, putting the tanker 25 m behind. While the boat proceeds 25 m along LG to the other lane, the tanker cuts its lead to 12.5 m at WN.

Thus, when the boat turns down SR, it is already 12.5 m ahead. In order to remain 25 m from the other end of the tanker when it goes back across the first lane, it will need to go farther down the side to a point H that is 12.5 m beyond the other end. Then, when it turns and closes the 25 m stretch to the nearer side of the tanker, the tanker will have advanced another 12.5 m to yield a separation of 25 m between them. Hence the patrol boat must cover $450 + 12.5 + 12.5 = 475$ m along this side before making its turn. As before, it needs to travel down that side only two-thirds of this distance, that is, $\frac{950}{3}$ m.

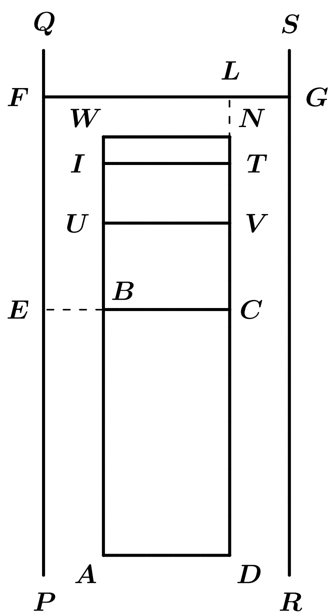

Figure 3

Thus, when passing the right rear corner of the tanker, the boat is already 25 m behind, and still has 75 m of the gap to negotiate to get back to lane PQ. While the boat is doing this, the tanker gets ahead another 37.5 m, for a total of 62.5 m, and so, in order to catch up and complete its cycle, the patrol boat must go another 125 m up PQ to get abreast of the tanker.

Altogether, then, the patrol boat must travel a grand total of

$$900 + 125 + 100 + \frac{950}{3} + 100 + 125 \;=\; 1666\tfrac{2}{3}\text{ m}.$$

This seemed most satisfactory until the brilliant solution of my colleague Larry Rice (retired from the University of Toronto Schools and the University of Waterloo) revealed another unwarranted assumption in my solution. Since the patrol boat had a 25 m lead when it reached the right hand edge of the tanker, it went unchallenged that the minimum 25 m distance between the boat and the front right corner of the tanker would be maintained as the boat completed the final 25 m of the crossing to its other lane.

Unfortunately this is not true; it is not hard to show that the vessels do get closer together than 25 m when the boat is completing the crossing.

So, how much of a lead would the boat have to have when it passes the right hand edge of the tanker in order to maintain a 25 m separation? Since 25 m isn't enough, perhaps a lead of 27 m would do? Again, it is not difficult to see that 27 m is not enough. So, how about 28 m? Bingo: 28 m is sufficient, with a little to spare. Consequently, any lead greater than 28 m would also suffice. But we seek the shortest cycle of the boat around the tanker. Thus we need to determine the smallest acceptable lead. We know it's some value, $25 + y$, between 27 and 28 m. Here y is not a variable, but a well-defined fixed value.

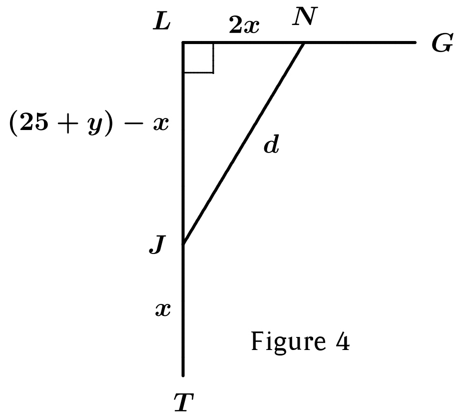

Figure 4

In Figure 4, let the boat pass the right hand edge of the tanker with a lead TL of $(25+y)$ metres. As the tanker advances x metres to J on TL, the boat advances $2x$ metres along LG to the point N. We want to determine the smallest positive value y such that the segment $d = JN$ is at least 25 m for all x between 0 and 12.5.

By the Pythagorean Theorem, we have

$$\begin{aligned} d^2 &= (2x)^2 + (25 + y - x)^2 \\ &= 4x^2 + (25 + y)^2 - 2x(25 + y) + x^2 \\ &= 5x^2 + y^2 + 50y + 625 - 50x - 2xy\,, \end{aligned}$$

and for $d^2 \geq 625$, we need

$$5x^2 + y^2 + 50y - 50x - 2xy \;\geq\; 0\,.$$

Since the only variable here is x, we may write

$$f(x) \;=\; 5x^2 + y^2 + 50y - 50x - 2xy\,,$$

where we require $f(x) \geq 0$ for $0 \leq x \leq 12.5$.

Differentiating, we get

$$f'(x) \;=\; 10x - 50 - 2y \;=\; 2(5x - 25 - y)\,,$$

which vanishes for $x = 5 + \frac{1}{5}y$. Since the graph of $f(x)$ is a parabola opening upwards, its minimum occurs when $f'(x) = 0$.

Hence

$$\begin{aligned}\min f(x) &= 5\left(5+\frac{1}{5}y\right)^2 + y^2 + 50y - 50\left(5+\frac{1}{5}y\right) - 2y\left(5+\frac{1}{5}y\right)\\ &= \frac{4}{5}y^2 + 40y - 125\,.\end{aligned}$$

Now, we don't want this minimum to be greater than 0, for in that case d would always *exceed* 25 and a lead of $25 + y$ would be greater than it need be. So for minimum y, we require the minimum to equal 0, and

$$\frac{4}{5}y^2 + 40y - 125 = 0\,,$$

or

$$4y^2 + 200y - 625 = 0\,.$$

The number y being positive, we obtain

$$\begin{aligned}y &= \frac{-200+\sqrt{200^2-4(4)(-625)}}{2(4)} = \frac{-200+\sqrt{50000}}{8}\\ &= \frac{25}{2}\sqrt{5} - 25 \approx 2.95085\,,\end{aligned}$$

or 2.951 to three decimal places of accuracy.

Thus a lead of 27.951 would assure a universal 25 m separation and provide a shortest cycle to a reasonable degree of accuracy.

We have seen that an additional 125 m beyond E gives the boat a 25 m lead when it reaches the right hand edge of the tanker. Hence a lead of 27.951 m would require the boat to go a further $2(2.951) = 5.902$ m for a total of 130.902 m up its left lane before starting its turn across the front. Moreover, its lead is now $12.5 + 2.951 = 15.451$ m when it starts down the right lane.

To produce the same separation at the other end of the tanker, the boat needs to go this same distance, 15.451 m, farther down this lane than the end of the tanker, for a total passage of $450 + 2(15.451) = 480.902$ m. As before, the boat needs to cover only two-thirds of this distance, namely 320.601 m. After returning the 100 m to its left lane, the boat would be another 50 m behind, for a total of 65.451 m, and therefore needs to go another 130.902 m up PQ in order to complete the cycle. Altogether, then, the boat travels approximately a total of

$$900 + 130.902 + 100 + 320.601 + 100 + 130.902 = 1682.405 \text{ m}\,.$$

Finally, let's consider Larry's wonderful solution – a completely different approach.

The unknown distances in the cycle are those travelled by the patrol boat in the lanes PQ and RS; the distances travelled in crossing between the lanes is always $100 + 100 = 200$ m. Larry obtains these unknown distances from the basic formula

$$\text{distance} = \text{speed} \times \text{time}\,.$$

If we denote the speed of the tanker by v, then the speed of the boat is $2v$, and it remains only to determine the times t_1 and t_2 that the boat spends in lanes PQ and RS respectively, since the distance travelled in lane PQ will be $(2v)t_1$ and the distance travelled in lane RS will be $(2v)t_2$.

Knowing only the *ratio* of the speeds of the boat and the tanker, you might well wonder how he is going to obtain anything of value about these times. Larry very ingeniously enlists the aid of the captain of the tanker in this matter.

To this end, consider the observations of the captain as he watches the boat circle his tanker (Figure 5). First, he sees the boat run up lane PQ from E to F. At F, the boat turns across the lanes in front of him and, as it proceeds, it keeps getting closer to him because the tanker is moving ahead. Upon reaching the other lane, the boat proceeds along it and eventually turns back to lane PQ. As it crosses back, the tanker, still going forward, moves into the lead. Finally, the boat gets abreast of the tanker up lane PQ. Thus the captain sees the boat cycle the tanker along the sides of a trapezoid $EFGH$.

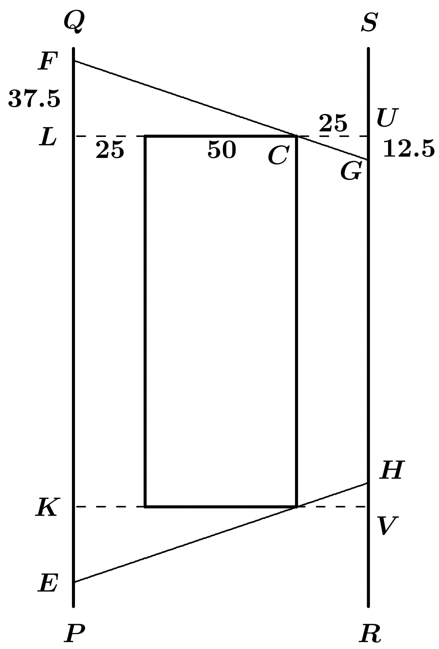

Figure 5

Between two points on a slanted side of the trapezoid, the boat moves twice as far toward the other lane as it does toward, or away from, the tanker, implying that the slopes of these sides are $\pm\frac{1}{2}$. Thus the trapezoid is isosceles and enjoys the symmetry of such a figure.

Since the slope of FG is $-\frac{1}{2}$ and $LC = 75$ m, it follows that $LF = 37.5$ m. The symmetrical part EK is therefore also 37.5 m, making the total length of side $EF = 450 + 2(37.5) = 525$ m. Similarly, since $CU = 25$ m, then $UG = HV = 12.5$ m, making the length of side $GH = 450 - 2(12.5) = 425$ m.

Now, time equals distance divided by speed, so the time t_1 that it takes the boat to traverse EF is 525 divided by the boat's speed along EF. While the boat goes at speed of $2v$ relative to the ocean, the captain sees it proceed up EF at the speed at which it overtakes the tanker; that is, at the relative speed of $2v - v = v$. Hence the captain calculates $t_1 = \dfrac{525}{v}$.

Along side GH, the captain observes the boat is going at a relative speed of $2v + v = 3v$, and so he calculates $t_2 = \frac{425}{3v}$.

It follows that the distance the boat travels up and down the lanes is

$$2v\left(\frac{525}{v}\right) + 2v\left(\frac{425}{3v}\right) = 1050 + \frac{850}{3} = 1333\frac{1}{3}\,\text{m}.$$

Adding in the 200 m travelled in crossing between lanes, we obtain the same earlier grand total of $1533\frac{1}{3}$ m for the first case, in which we generously allowed the vessels to slide by each other without incident.

Allowing an additional 25 m buffer at each crossing between the lanes would increase each of the parallel sides of the trapezoid by 50 m and yield the corresponding times $t_1 = \frac{575}{v}$ and $t_2 = \frac{475}{3v}$.

The resulting cycle, then, would have a total length of

$$2v\left(\frac{575}{v}\right) + 2v\left(\frac{475}{3v}\right) + 200 = 1150 + 316\frac{2}{3} + 200 = 1666\frac{2}{3}\,\text{m},$$

as found earlier.

Finally, (Figure 6), a universal separation of 25 m, would add an extra $2(12.5)(\sqrt{5}) \approx 25(2.236) = 55.9$ m to each of the parallel sides of the original trapezoid $EFGH$.

This is because the right-angled $\triangle XYC$ has legs in the ratio $1 : 2$. Can you see why?

Thus, $t_1 = \frac{580.9}{v}$ and $t_2 = \frac{480.9}{3v}$ which yields a cycle length of approximately

$$\begin{aligned} & 2(580.9) + 2\left(\frac{480.9}{3}\right) + 200 \\ = \; & 1161.8 + 320.6 + 200 \\ = \; & 1682.4\,\text{m}, \end{aligned}$$

in close agreement with our previous result.

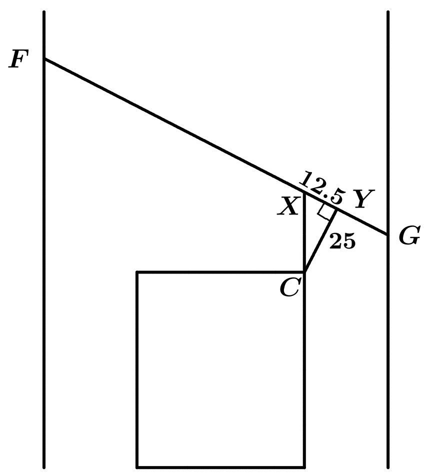

Figure 6

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