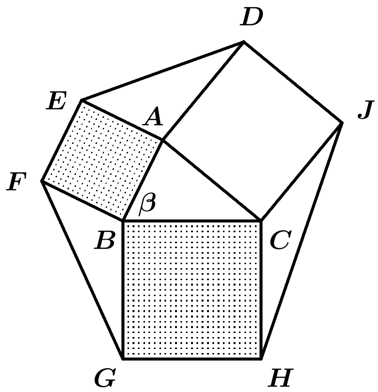
Problem of the Month

Ian VanderBurgh

Among Jim Totten's many interests was his involvement in mathematics outreach. In particular, Jim was a driving force behind the Cariboo College High School Mathematics Contest. He edited a volume of problems taken from those contests written between 1973 and 1992. This month, we look at one of the problems from this volume.

Problem (1989 Cariboo College High School Mathematics Contest, Senior Final Round, Part B) In the figure, $AEFB$, $BGHC$, and $ACJD$ are squares constructed on the sides of $\triangle ABC$. If the (combined) area of the two shaded squares equals the area of the rest of the figure, show that the area of $\triangle ABC$ equals the area of $\triangle FBG$ and then find the number of degrees in $\angle ABC$.



This problem actually appears to be the "poster child" for this contest, as it appears on the cover of the compilation book and appears as a kind of logo elsewhere on the web.

Before we look at the solution, there is one really useful formula upon which we should agree. If in $\triangle XYZ$ we have $XY = z$ and $YZ = x$, then the area of $\triangle XYZ$ is equal to $\frac{1}{2}xz\sin(\angle XYZ)$. This formula is a great alternative to the standard area formula "$\frac{1}{2}bh$" if all you have is one angle of a triangle and the lengths of the two sides enclosing it. We'll derive this formula after the solution to the problem.

Solution Let $BC = a$, $AC = b$, and $AB = c$. Since $AEFB$ is a square, then $BF = AE = c$. Since $BGHC$ is a square, then $BG = CH = a$. Since $ACJD$ is a square, then $AD = CJ = b$.

Since $\angle ABC = \beta$ and $\angle ABF = \angle CBG = 90°$, it then follows that $\angle FBG = 360° - \beta - 90° - 90° = 180° - \beta$.

We first need to prove that $\triangle ABC$ and $\triangle FBG$ have equal areas. From the formula in the preamble, the area of $\triangle ABC$ is $\frac{1}{2}(BC)(AB)\sin(\angle ABC)$ or $\frac{1}{2}ac\sin\beta$. Similarly, the area of $\triangle FBG$ is $\frac{1}{2}(BG)(FB)\sin(\angle FBG)$ or $\frac{1}{2}ac\sin(180° - \beta)$.

But $\sin\beta = \sin(180° - \beta)$ for any angle β, so the two areas are equal.

While it may not be immediately obvious why this helps, we can stall a bit by noting that we can use the same argument to conclude that the area of

$\triangle EAD$ and the area of $\triangle HCJ$ are each equal to the area of $\triangle ABC$. Can you see why?

At this point, we should probably use the piece of information that we were given, namely, that the combined area of square $AEFB$ and square $BGHC$ equals the area of the rest of the figure. We use the short-hand $|AEFB|$ to denote the area of figure $AEFB$. Thus, we are told that

$$|AEFB| + |BGHC| = |FBG| + |ABC| + |EAD| + |HCJ| + |ACJD| .$$

Using some of what we know so far, this becomes

$$c^2 + a^2 = 4|ABC| + b^2 ,$$

or

$$c^2 + a^2 = 4\left(\frac{1}{2}ac\sin\beta\right) + b^2 .$$

Remember, we're trying to find β. We seem to have too many other pieces of information floating around to have any hope of doing this. But at this point, the amazing pattern recognition abilities of the brain might kick in. This equation looks somewhat similar to a law that we often use. This might prompt us to try applying that law. Do you see what I'm getting at?

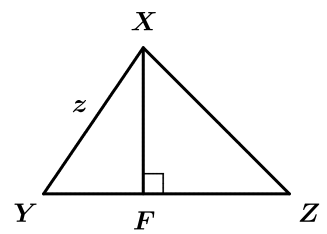
Applying the Law of Cosines in $\triangle ABC$ gives $b^2 = a^2 + c^2 - 2ac\cos\beta$. Substituting this into the last equation, we obtain

$$\begin{aligned} c^2 + a^2 &= 2ac\sin\beta + (a^2 + c^2 - 2ac\cos\beta) ; \\ 2ac\cos\beta &= 2ac\sin\beta ; \\ \cos\beta &= \sin\beta , \end{aligned}$$

since $ac > 0$. Since $\cos\beta = \sin\beta$, and β is an angle in a triangle, then $\beta = 45°$, and we are done. ∎

To me, this was quite surprising. Well, the answer itself wasn't so surprising since these problems almost always have $30°$, $45°$, or $60°$ as an answer. But, it was surprising to me that the angle β was completely determined from the given information while no other information (side lengths or angles) can be determined.

Before wrapping up the column this month, we should go back and look at the formula from the preamble. Suppose that $\angle XYZ$ is acute. Drop a perpendicular from X to F on YZ.



Then the area of $\triangle XYZ$ is equal to $\frac{1}{2}(YZ)(XF)$. But $YZ = x$ and we also have $XF = XY\sin(\angle XYZ) = z\sin(\angle XYZ)$, so the area equals $\frac{1}{2}xz\sin(\angle XYZ)$, as required. Can you prove this in the case that angle Y is obtuse or a right angle?