(a) Vérifier que si (a, b, c) est géométrique, alors tous les triplets qui lui sont équivalents sont aussi géométriques.

(b) Vérifier que si (a, b, c) est harmonique, alors tous les triplets qui lui sont équivalents sont aussi harmoniques.

(c) Soit G l'ensemble des classes d'équivalence de triplets géométriques et H l'ensemble des classes d'équivalence de triplets harmoniques. Trouver une correspondance biunivoque entre G et H.

Totten–M9. *Proposé par Kirk Evenrude, Kamloops, BC.*

Un train de 900 m de long s'approche d'un pont d'une longueur de 100 m à la vitesse de 90 km/h.

(a) En combien de secondes le train traversera-t-il le pont?

(b) Supposons qu'au moment d'atteindre le pont, le train décélère de 0.2 m/s^2. Combien de temps mettra-t-il cette fois pour traverser le pont?

Totten–M10. *Proposé par Nicholas Buck, Collège de New Caledonia, Prince George, BC.*

Montrer que si p est un nombre premier, et si A et B sont des entiers positifs tels que p divise A, p^2 ne divise pas A, et p ne divise pas B, alors l'équation diophantienne $Ax^2 + By^2 = p^{2008}$ n'a aucune solution en entiers positifs x et y.

Mayhem Solutions

M363. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that A is a six-digit positive integer and B is the positive integer formed by writing the digits of A in reverse order. Prove that $A - B$ is a multiple of 9.

Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.

Since A is a six-digit positive integer, it can be expressed in the form $10^5a + 10^4b + 10^3c + 10^2d + 10^1e + f$, where a is a positive integer and b, c, d, e, and f are nonnegative integers. Since B is formed by writing the digits of A in reverse order, B is of the form $10^5f + 10^4e + 10^3d + 10^2c + 10^1b + a$.

Their difference, $A - B$, can then be simplified as follows:

$$\begin{aligned} A - B &= \left(10^5 a + 10^4 b + 10^3 c + 10^2 d + 10^1 e + f\right) \\ &\quad - \left(10^5 f + 10^4 e + 10^3 d + 10^2 c + 10^1 b + a\right) \\ &= 10^5 a - a + 10^4 b - 10^1 b + 10^3 c - 10^2 c \\ &\quad + 10^2 d - 10^3 d + 10^1 e - 10^4 e + f - 10^5 f \\ &= 99999a + 9990b + 900c - 900d - 9990e - 99999f \\ &= 9(11111a + 1110b + 100c - 100d - 1110e - 11111f)\,. \end{aligned}$$

Since the digits of A are integers, sums and differences of multiples of these digits are integers too. Thus, the difference of A and B is a multiple of nine.

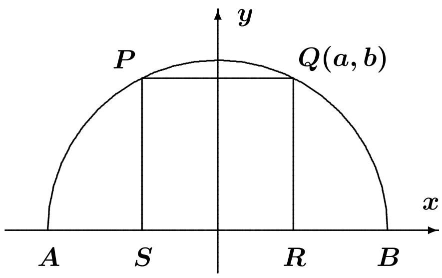
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M364. *Proposed by the Mayhem Staff.*

A semicircle of radius 2 is drawn with diameter AB. The square $PQRS$ is drawn with P and Q on the semicircle and R and S on AB. Is the area of the square less than or greater than one-half of the area of the semicircle?

Solution by Peter Chien, student, Central Elgin Collegiate, St. Thomas, ON, modified by the editor.

Place AB on the x-axis with the midpoint of AB (that is, the centre of the semicircle) at the origin. The full circle has radius 2 and centre $(0,0)$, and so has equation $x^2 + y^2 = 4$.



Let Q have coordinates (a, b), with a and b positive. Since Q is on the semicircle, then $a^2 + b^2 = 4$. Since $PQRS$ is a square, which must sit symmetrically inside the semicircle, then we also have $b = 2a$.

Thus, $a^2 + (2a)^2 = 4$, hence $a^2 = \frac{4}{5}$ and so $a = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$. Since the length of one side of $PQRS$ is $b = 2a = \frac{4\sqrt{5}}{5}$, then the area of the square is $\left(\frac{4\sqrt{5}}{5}\right)^2 = \frac{80}{25} = \frac{16}{5} = 3.2$.

One-half the area of the semicircle is $\frac{1}{2} \times \frac{1}{2} \times \pi \times 2^2 = \pi < 3.2$. The area of the square is therefore greater than one-half the area of the semicircle.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M365. *Proposed by Alexander Gurevich, student, University of Waterloo, Waterloo, ON.*

Let D be the family of lines of the form $y = nx + n^2$, with $n \geq 2$ a positive integer. Let H be the family of lines of the form $y = m$, where $m \geq 2$ is a positive integer. Prove that a line from H has a prime y-intercept if and only if this line does not intersect any line from D at a point with an x-coordinate that is a nonnegative integer.

Solution by Alex Song, student, Elizabeth Ziegler Public School, Waterloo, ON, modified by the editor.

Let $m \geq 2$ be a positive integer. Consider a line $y = m$ from H. Its y-intercept is m. It suffices to prove two things:

(a) if m is prime, then for any positive integer n with $n \geq 2$, the simultaneous equations $y = m$ and $y = nx + n^2$ do not have a solution for x that is a nonnegative integer, and

(b) if m is composite, then there is a positive integer $n \geq 2$ such that the simultaneous equations $y = m$ and $y = nx + n^2$ do have a solution for x that is a nonnegative integer.

Putting this another way, we must prove that if m is prime, then the equation $m = nx + n^2$ does not have a nonnegative integer solution for x for any $n \geq 2$, and if m is composite, then the equation $m = nx + n^2$ does have a nonnegative integer solution for x for some $n \geq 2$.

Assume that m is prime and that n is a positive integer with $n \geq 2$. Suppose also that $m = nx+n^2 = n(x+n)$ has an integer solution for x with $x \geq 0$. Note that m has only two positive divisors, namely m and 1. Since $n \geq 2$, then n must equal m, and so $x+n = 1$, which gives $x = 1-n \leq -1$. Thus, x is not a positive integer or zero. This is a contradiction, so there is no n for which $m = nx + n^2$ has nonnegative integer solutions for x.

Assume next that m is composite. Then $m = pq$ for some integers p and q with $2 \le p \le q$. Thus, we want to find integers n and x with $n \ge 2$ and $x \ge 0$ such that $m = n(x+n) = pq$. Setting $n = p$ and $x + p = q$ satisfies the equation. Here, $x = q - p \ge 0$, so the restrictions on x and n are satisfied. Thus, for some n there is a nonnegative solution for x.

Therefore, a line from H has a prime y-intercept if and only if this line does not intersect any line from D at a point with an x-coordinate that is a nonnegative integer.

Also solved by SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M366. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The roots of the equation $x^3 + bx^2 + cx + d = 0$ are p, q, and r. Find a quadratic equation with roots $(p^2 + q^2 + r^2)$ and $(p + q + r)$.

Solution by Shamil Asgarli, student, Burnaby South Secondary School, Burnaby, BC.

Since p, q, and r are the roots of the cubic equation, we can factor the left side of the equation to get $(x-p)(x-q)(x-r) = 0$. Expanding yields $x^3 - (p+q+r)x^2 + (pq+qr+rp)x - pqr = 0$. Comparing coefficients with the original equation, we obtain $p + q + r = -b$ while $pq + qr + rp = c$.

Since $(p+q+r)^2 = p^2 + q^2 + r^2 + 2(pq + qr + rp)$, then we have $(-b)^2 = p^2 + q^2 + r^2 + 2c$, so $p^2 + q^2 + r^2 = b^2 - 2c$.

A possible quadratic equation with the desired roots is therefore

$$\Big(x - (b^2 - 2c)\Big)\Big(x - (-b)\Big) \;=\; 0\,,$$

or $x^2 + \big(b - b^2 + 2c\big)x + \big(2bc - b^3\big) = 0$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M367. *Proposed by George Tsapakidis, Agrinio, Greece.*

For the positive real numbers a, b, and c we have $a + b + c = 6$. Determine the maximum possible value of $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

Solution by José Hernández Santiago, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico.

Applying the Arithmetic Mean–Geometric Mean Inequality to positive real numbers a, b, and c we obtain $abc \leq \left(\frac{a+b+c}{3}\right)^3 = 8$ and consequently $\sqrt{abc} \leq 2\sqrt{2}$. (Equality holds here if and only if $a = b = c$ and so if and only if $a = b = c = 2$.)

We can then further conclude that

$$\begin{aligned} a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} &= \sqrt{a}\sqrt{abc} + \sqrt{b}\sqrt{abc} + \sqrt{c}\sqrt{abc} \\ &= \sqrt{abc}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \\ &\leq 2\sqrt{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) . \qquad (1) \end{aligned}$$

Now,

$$\begin{aligned} \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 &= a + b + c + 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}\right) \\ &= 6 + 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}\right) , \end{aligned}$$

since $a + b + c = 6$.

Applying the AM–GM Inequality once more yields $2\sqrt{xy} \leq x + y$, thus we obtain $2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}) \leq a + b + a + c + b + c = 2(a + b + c)$. (Again, equality holds if and only if $a = b = c = 2$.)

Hence, $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \leq 6 + 2(a + b + c) = 18$ and consequently $\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{18} = 3\sqrt{2}$.

From (1), it follows that $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq (2\sqrt{2})(3\sqrt{2}) = 12$, so the maximum possible value is 12, which we have seen is achieved when $a = b = c = 2$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M368. *Proposed by J. Walter Lynch, Athens, GA, USA.*

An infinite series of positive rational numbers $a_1 + a_2 + a_3 + \cdots$ is the fastest converging infinite series with a sum of 1, $a_1 = \frac{1}{2}$, and each a_i having numerator 1. (By "fastest converging", we mean that each term a_n is successively chosen to make the sum $a_1 + a_2 + \cdots + a_n$ as close to 1 as possible.) Determine a_5 and describe a recursive procedure for finding a_n.

Solution by Robert Sheets, Southeast Missouri State University, Cape Girardeau, MO, USA.

Let s_n be the n^{th} partial sum of the infinite series. Then $a_1 = \frac{1}{2}$ and $s_1 = \frac{1}{2}$. Since all terms of the infinite series are positive, none of the terms can be negative or zero, thus no partial sum can be greater than or equal to 1. We therefore need $a_2 < 1 - s_1 = \frac{1}{2}$. Seeking the largest such a_2, we find that $a_2 = \frac{1}{3}$ since a_2 is a fraction of positive integers with numerator equal to 1.

Then $s_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ which gives $a_3 < \frac{1}{6}$, whence $a_3 = \frac{1}{7}$. Similarly, we find that $s_3 = \frac{41}{42}$, giving $a_4 = \frac{1}{43}$, and $s_4 = \frac{1805}{1806}$, giving $a_5 = \frac{1}{1807}$.

Next, we describe a recursive procedure to find a_n. We conjecture that if $a_n = \frac{1}{k}$, then $s_n = \frac{k(k-1)-1}{k(k-1)}$ and $a_{n+1} = \frac{1}{k(k-1)+1}$. (Note that $2(1)+1 = 3$, $3(2)+1 = 7$, $7(6)+1 = 43$, and $43(42)+1 = 1807$, so these equations hold for $n = 1$, 2, 3, and 4.)

We prove these recursive relations by induction. We have already verified the base cases above. Suppose that for some positive integers n and k we have $a_n = \frac{1}{k}$, $s_n = \frac{k(k-1)-1}{k(k-1)}$, and $a_{n+1} = \frac{1}{k(k-1)+1}$. We prove that the relations will also hold for s_{n+1} and a_{n+2}. Let $t = k(k-1)+1$; this means that $a_{n+1} = \frac{1}{t}$ and $s_n = \frac{t-2}{t-1}$. Then

$$\begin{aligned} s_{n+1} &= s_n + a_{n+1} \\ &= \frac{t-2}{t-1} + \frac{1}{t} = \frac{t^2 - 2t + t - 1}{t(t-1)} \\ &= \frac{t^2 - t - 1}{t(t-1)} = \frac{t(t-1)-1}{t(t-1)}\,. \end{aligned}$$

Finally, since $a_{n+2} < 1 - s_{n+1} = \frac{1}{t(t-1)}$ and a_{n+2} is the largest fraction with numerator 1 satisfying this property, we find that $a_{n+2} = \frac{1}{t(t-1)+1}$, as required.

Therefore, the result holds by induction and for all positive integers n, if $a_n = \frac{1}{k}$ for some positive integer k, then $a_{n+1} = \frac{1}{k(k-1)+1}$.

Also solved by ARKADY ALT, San Jose, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON. There was one incomplete solution submitted.

Hess noted that the sequence of denominators appearing in the series is A000058 *in the* On-Line Encyclopedia of Integer Sequences (`http://www.research.att.com/~njas/sequences/`) *and is known as Sylvester's sequence. It is a curious fact that two consecutive terms of Sylvester's sequence differ by a square, since the number k in the sequence is followed by $k^2 - k + 1$, yielding a difference of $(k^2 - k + 1) - k = (k-1)^2$.*