MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Please send your solutions to the problems in this edition by **15 November 2009**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

Each of the following Mayhem problems is specially dedicated to the memory of Jim Totten, hence the special numbering used below. The numbering of the regular Mayhem problems will resume in subsequent issues.

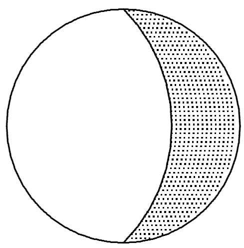
The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

Totten–M1. *Proposed by Shawn Godin, Cairine Wilson Secondary School, Orleans, ON.*

Ancient Egyptians wrote all fractions in terms of distinct unit fractions (that is, in terms of distinct fractions with numerators of 1). For example, instead of writing $\frac{11}{12}$, they would write $\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$. The unit fraction $\frac{1}{2}$ can be written in terms of other unit fractions as $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$. Find an infinite family of unit fractions each of which can be written as the sum of two unit fractions.

Totten–M2. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The boundary of the shadow on the moon is always a circular arc. On a certain day, the moon is seen with the shadow passing through diametrically opposite points. If the centre of the circular arc forming the shadow is on the circumference of the moon, determine the exact proportion of the moon that is not in shadow.



Totten–M3. *Proposed by John Ciriani, Kamloops, BC.*

Prove that the quadratic equation $ax^2 + bx + c = 0$ does not have a rational root if a, b, and c are odd integers.

Totten–M4. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

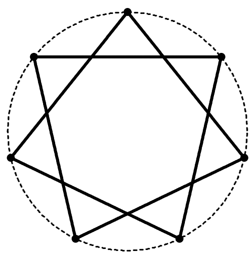
In a survey, some students were asked whether they liked the colour orange. Exactly 2% of the boys in the survey liked orange, while exactly 59% of the girls in the survey liked orange. Altogether, exactly 17% of the students in the survey liked orange. Find the smallest possible number of students in the survey.

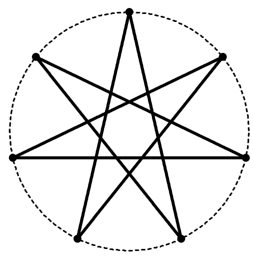
Totten–M5. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $a \neq 1$ be a positive real number. Determine all pairs of positive integers (x, y) such that $\log_a x - \log_a y = \log_a(x - y)$.

Totten–M6. *Proposed by Suzanne Feldberg, Thompson Rivers University, Kamloops, BC.*

It is widely known how to draw a 5-pointed star quickly. To make it symmetric, one places 5 vertices at $72°$ intervals about a circle and connects the vertices with line segments of equal length without lifting one's pen. By starting from a fixed point and using the same method, one can draw two different (and symmetric) 7-pointed stars without lifting one's pen.

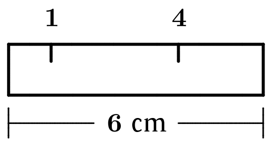




How many different 6-pointed, 8-pointed, or 9-pointed stars can one draw this way? How many different n-pointed stars can one draw this way?

Totten–M7. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

An unmarked ruler is known to be exactly 6 cm in length. It is possible to exactly measure all integer lengths from 1 cm to 6 cm using only 2 marks, as shown in the diagram, at 1 cm and 4 cm, since $2 = 6 - 4$, $3 = 4 - 1$, and $5 = 6 - 1$.



Suppose that an unmarked ruler is known to be exactly 30 cm in length.

(a) Find a way of placing 9 or fewer marks on the ruler to be able to exactly measure all integer lengths from 1 cm to 30 cm.

(b) Prove that at least 7 marks are needed to be able to exactly measure all integer lengths from 1 cm to 30 cm.

(c)★ Determine the smallest number of marks required on the ruler to be able to exactly measure all integer lengths from 1 cm to 30 cm.

Totten–M8. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Let T be the set of all ordered triples (a, b, c) of positive integers such that $a < b < c$. We say that two triples (a, b, c) and (u, v, w) are equivalent if $a : b : c = u : v : w$. We use this relation to partition T into equivalence classes. The triple (a, b, c) is *geometric* if $ac = b^2$ (that is, its terms form a geometric sequence) and *harmonic* if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ (that is, the reciprocals of its terms form an arithmetic sequence).

(a) Verify that if (a, b, c) is geometric, then all triples equivalent to it are also geometric.

(b) Verify that if (a, b, c) is harmonic, then all triples equivalent to it are also harmonic.

(c) Let G be the set of equivalence classes of geometric triples and H be the set of equivalence classes of harmonic triples. Determine a one-to-one correspondence between G and H.

Totten–M9. *Proposed by Kirk Evenrude, Kamloops, BC.*

A train 900 m long, travelling at 90 km/h, approaches a 100 m long bridge.

(a) How many seconds does it take the train to clear the bridge?

(b) Suppose that, just as the train reaches the bridge, it begins to slow down at the rate of 0.2 m/s^2. Now how long does it take to clear the bridge?

Totten–M10. *Proposed by Nicholas Buck, College of New Caledonia, Prince George, BC.*

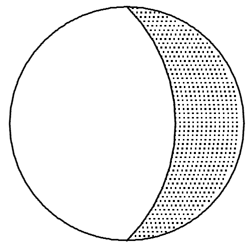
Show that if p is a prime number, and A and B are positive integers such that p divides A, p^2 does not divide A, and p does not divide B, then the Diophantine equation $Ax^2 + By^2 = p^{2008}$ does not have any solutions in positive integers x and y.

Totten–M1. *Proposé par Shawn Godin, École secondaire Cairine Wilson, Orléans, ON.*

Les anciens égyptiens écrivaient toutes leurs fractions en termes de fractions unitaires distinctes, c'est-à-dire de fractions distinctes avec 1 comme numérateur. Par exemple, au lieu d'écrire $\frac{11}{12}$, ils auraient écrit $\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$. La fraction unitaire $\frac{1}{2}$ peut être écrite en termes d'autres fractions unitaires comme $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$. Trouver une famille infinie de fractions unitaires, chacune pouvant être écrite comme somme de deux fractions unitaires.

Totten–M2. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

La frontière de l'ombre sur la lune est toujours un arc de cercle. Un certain jour, on peut constater que l'ombre passe par des points diamétralement opposés. Si le centre de l'arc circulaire formant cette ombre se trouve sur la circonférence de la lune, déterminer la portion exacte de la lune qui n'est pas dans l'ombre.



Totten–M3. *Proposé par John Ciriani, Kamloops, BC.*

Montrer que l'équation quadratique $ax^2+bx+c=0$ n'a pas de racine rationnelle si a, b et c sont des entiers impairs.

Totten–M4. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

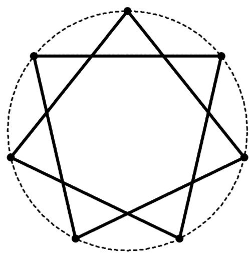
Lors d'un sondage, on a demandé à certains étudiants s'ils aimaient la couleur orange. Chez les garçons exactement 2% ont répondu par oui, tandis que chez les filles, la proportion exacte de oui a été de 59%. Comme au total, exactement 17% des répondants ont dit aimer l'orange, on demande de trouver quel est le plus petit nombre possible de participants au sondage.

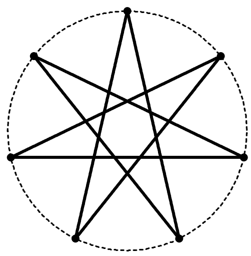
Totten–M5. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $a \neq 1$ un nombre réel positif. Trouver toutes les paires d'entiers positifs (x, y) telles que $\log_a x - \log_a y = \log_a(x-y)$.

Totten–M6. *Proposé par Suzanne Feldberg, Université Thompson Rivers, Kamloops, BC.*

On sait bien comment dessiner rapidement une étoile à 5 sommets. Pour en faire une symétrique, on place les 5 sommets à intervalles de $72°$ sur un cercle et on relie les sommets par des segments rectilignes de longueur égale sans lever son crayon. En partant d'un point fixé et en utilisant la même méthode, on peut dessiner sans lever son crayon deux étoiles différentes (et symétriques) à 7 sommets.

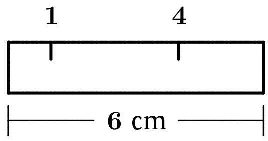




Combien d'étoiles différentes à 6 sommets, 8 sommets, ou 9 sommets peut-on ainsi dessiner? Combien d'étoiles différentes à n sommets peut-on ainsi dessiner?

Totten–M7. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

On utilise une règle sans graduation dont on sait qu'elle mesure exactement 6 cm. Il est possible de mesurer exactement toutes les longueurs entières de 1 cm à 6 cm avec seulement 2 marques, comme indiqué dans la figure ci-contre, à 1 cm et 4 cm, car $2 = 6 - 4$, $3 = 4 - 1$ et $5 = 6 - 1$.



Supposons maintenant qu'on utilise une règle d'exactement 30 cm de long.

(a) Trouver un moyen de placer 9 marques sur la règle afin d'être capable de mesurer exactement toutes les longueurs entières, de 1 cm à 30 cm.

(b) Montrer qu'au moins 7 marques sont nécessaires pour être capable de mesurer exactement toutes les longueurs entières, de 1 cm à 30 cm.

(c)★ Déterminer le plus petit nombre de marques à placer sur la règle afin d'être capable de mesurer exactement toutes les longueurs entières, de 1 cm à 30 cm.

Totten–M8. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Soit T l'ensemble de toutes les triplets ordonnés d'entiers positifs (a, b, c) tels que $a < b < c$. On dit que deux triplets (a, b, c) et (u, v, w) sont équivalents si $a : b : c = u : v : w$. On utilise cette équivalence pour partitionner T en classes d'équivalence. Le triplet (a, b, c) est dit *géométrique* si $ac = b^2$ (c.-à-d. ses éléments forment une suite géométrique) et *harmonique* si $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ (c.-à-d. les inverses de ses éléments forment une suite arithmétique).

(a) Vérifier que si (a, b, c) est géométrique, alors tous les triplets qui lui sont équivalents sont aussi géométriques.

(b) Vérifier que si (a, b, c) est harmonique, alors tous les triplets qui lui sont équivalents sont aussi harmoniques.

(c) Soit G l'ensemble des classes d'équivalence de triplets géométriques et H l'ensemble des classes d'équivalence de triplets harmoniques. Trouver une correspondance biunivoque entre G et H.

Totten–M9. *Proposé par Kirk Evenrude, Kamloops, BC.*

Un train de 900 m de long s'approche d'un pont d'une longueur de 100 m à la vitesse de 90 km/h.

(a) En combien de secondes le train traversera-t-il le pont?

(b) Supposons qu'au moment d'atteindre le pont, le train décélère de 0.2 m/s^2. Combien de temps mettra-t-il cette fois pour traverser le pont?

Totten–M10. *Proposé par Nicholas Buck, Collège de New Caledonia, Prince George, BC.*

Montrer que si p est un nombre premier, et si A et B sont des entiers positifs tels que p divise A, p^2 ne divise pas A, et p ne divise pas B, alors l'équation diophantienne $Ax^2 + By^2 = p^{2008}$ n'a aucune solution en entiers positifs x et y.

Mayhem Solutions

M363. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that A is a six-digit positive integer and B is the positive integer formed by writing the digits of A in reverse order. Prove that $A - B$ is a multiple of 9.

Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.

Since A is a six-digit positive integer, it can be expressed in the form $10^5a + 10^4b + 10^3c + 10^2d + 10^1e + f$, where a is a positive integer and b, c, d, e, and f are nonnegative integers. Since B is formed by writing the digits of A in reverse order, B is of the form $10^5f + 10^4e + 10^3d + 10^2c + 10^1b + a$.

(a) Vérifier que si (a, b, c) est géométrique, alors tous les triplets qui lui sont équivalents sont aussi géométriques.

(b) Vérifier que si (a, b, c) est harmonique, alors tous les triplets qui lui sont équivalents sont aussi harmoniques.

(c) Soit G l'ensemble des classes d'équivalence de triplets géométriques et H l'ensemble des classes d'équivalence de triplets harmoniques. Trouver une correspondance biunivoque entre G et H.

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Their difference, $A - B$, can then be simplified as follows:

$$\begin{aligned} A - B &= \left(10^5 a + 10^4 b + 10^3 c + 10^2 d + 10^1 e + f\right) \\ &\quad - \left(10^5 f + 10^4 e + 10^3 d + 10^2 c + 10^1 b + a\right) \\ &= 10^5 a - a + 10^4 b - 10^1 b + 10^3 c - 10^2 c \\ &\quad + 10^2 d - 10^3 d + 10^1 e - 10^4 e + f - 10^5 f \\ &= 99999a + 9990b + 900c - 900d - 9990e - 99999f \\ &= 9(11111a + 1110b + 100c - 100d - 1110e - 11111f)\,. \end{aligned}$$

Since the digits of A are integers, sums and differences of multiples of these digits are integers too. Thus, the difference of A and B is a multiple of nine.

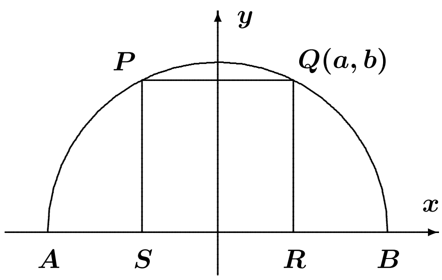
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M364. *Proposed by the Mayhem Staff.*

A semicircle of radius 2 is drawn with diameter AB. The square $PQRS$ is drawn with P and Q on the semicircle and R and S on AB. Is the area of the square less than or greater than one-half of the area of the semicircle?

Solution by Peter Chien, student, Central Elgin Collegiate, St. Thomas, ON, modified by the editor.

Place AB on the x-axis with the midpoint of AB (that is, the centre of the semicircle) at the origin. The full circle has radius 2 and centre $(0,0)$, and so has equation $x^2 + y^2 = 4$.



Let Q have coordinates (a, b), with a and b positive. Since Q is on the semicircle, then $a^2 + b^2 = 4$. Since $PQRS$ is a square, which must sit symmetrically inside the semicircle, then we also have $b = 2a$.

Thus, $a^2 + (2a)^2 = 4$, hence $a^2 = \frac{4}{5}$ and so $a = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$. Since the length of one side of $PQRS$ is $b = 2a = \frac{4\sqrt{5}}{5}$, then the area of the square is $\left(\frac{4\sqrt{5}}{5}\right)^2 = \frac{80}{25} = \frac{16}{5} = 3.2$.

One-half the area of the semicircle is $\frac{1}{2}\times\frac{1}{2}\times\pi\times 2^2 = \pi < 3.2$. The area of the square is therefore greater than one-half the area of the semicircle.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M365. *Proposed by Alexander Gurevich, student, University of Waterloo, Waterloo, ON.*

Let D be the family of lines of the form $y = nx + n^2$, with $n \geq 2$ a positive integer. Let H be the family of lines of the form $y = m$, where $m \geq 2$ is a positive integer. Prove that a line from H has a prime y-intercept if and only if this line does not intersect any line from D at a point with an x-coordinate that is a nonnegative integer.

Solution by Alex Song, student, Elizabeth Ziegler Public School, Waterloo, ON, modified by the editor.

Let $m \geq 2$ be a positive integer. Consider a line $y = m$ from H. Its y-intercept is m. It suffices to prove two things:

(a) if m is prime, then for any positive integer n with $n \geq 2$, the simultaneous equations $y = m$ and $y = nx + n^2$ do not have a solution for x that is a nonnegative integer, and

(b) if m is composite, then there is a positive integer $n \geq 2$ such that the simultaneous equations $y = m$ and $y = nx + n^2$ do have a solution for x that is a nonnegative integer.

Putting this another way, we must prove that if m is prime, then the equation $m = nx + n^2$ does not have a nonnegative integer solution for x for any $n \geq 2$, and if m is composite, then the equation $m = nx + n^2$ does have a nonnegative integer solution for x for some $n \geq 2$.

Assume that m is prime and that n is a positive integer with $n \geq 2$. Suppose also that $m = nx+n^2 = n(x+n)$ has an integer solution for x with $x \geq 0$. Note that m has only two positive divisors, namely m and 1. Since $n \geq 2$, then n must equal m, and so $x+n = 1$, which gives $x = 1-n \leq -1$. Thus, x is not a positive integer or zero. This is a contradiction, so there is no n for which $m = nx + n^2$ has nonnegative integer solutions for x.

Assume next that m is composite. Then $m = pq$ for some integers p and q with $2 \le p \le q$. Thus, we want to find integers n and x with $n \ge 2$ and $x \ge 0$ such that $m = n(x+n) = pq$. Setting $n = p$ and $x + p = q$ satisfies the equation. Here, $x = q - p \ge 0$, so the restrictions on x and n are satisfied. Thus, for some n there is a nonnegative solution for x.

Therefore, a line from H has a prime y-intercept if and only if this line does not intersect any line from D at a point with an x-coordinate that is a nonnegative integer.

Also solved by SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M366. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The roots of the equation $x^3 + bx^2 + cx + d = 0$ are p, q, and r. Find a quadratic equation with roots $(p^2 + q^2 + r^2)$ and $(p + q + r)$.

Solution by Shamil Asgarli, student, Burnaby South Secondary School, Burnaby, BC.

Since p, q, and r are the roots of the cubic equation, we can factor the left side of the equation to get $(x-p)(x-q)(x-r) = 0$. Expanding yields $x^3 - (p+q+r)x^2 + (pq+qr+rp)x - pqr = 0$. Comparing coefficients with the original equation, we obtain $p + q + r = -b$ while $pq + qr + rp = c$.

Since $(p+q+r)^2 = p^2 + q^2 + r^2 + 2(pq + qr + rp)$, then we have $(-b)^2 = p^2 + q^2 + r^2 + 2c$, so $p^2 + q^2 + r^2 = b^2 - 2c$.

A possible quadratic equation with the desired roots is therefore

$$\Big(x - (b^2 - 2c)\Big)\Big(x - (-b)\Big) \;=\; 0\,,$$

or $x^2 + \big(b - b^2 + 2c\big)x + \big(2bc - b^3\big) = 0$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M367. *Proposed by George Tsapakidis, Agrinio, Greece.*

For the positive real numbers a, b, and c we have $a + b + c = 6$. Determine the maximum possible value of $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

Solution by José Hernández Santiago, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico.

Applying the Arithmetic Mean–Geometric Mean Inequality to positive real numbers a, b, and c we obtain $abc \leq \left(\frac{a+b+c}{3}\right)^3 = 8$ and consequently $\sqrt{abc} \leq 2\sqrt{2}$. (Equality holds here if and only if $a = b = c$ and so if and only if $a = b = c = 2$.)

We can then further conclude that

$$\begin{aligned} a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} &= \sqrt{a}\sqrt{abc} + \sqrt{b}\sqrt{abc} + \sqrt{c}\sqrt{abc} \\ &= \sqrt{abc}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \\ &\leq 2\sqrt{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right). \qquad (1) \end{aligned}$$

Now,

$$\begin{aligned} \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 &= a + b + c + 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}\right) \\ &= 6 + 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}\right), \end{aligned}$$

since $a + b + c = 6$.

Applying the AM–GM Inequality once more yields $2\sqrt{xy} \leq x + y$, thus we obtain $2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}\right) \leq a + b + a + c + b + c = 2(a + b + c)$. (Again, equality holds if and only if $a = b = c = 2$.)

Hence, $\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \leq 6 + 2(a + b + c) = 18$ and consequently $\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{18} = 3\sqrt{2}$.

From (1), it follows that $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq \left(2\sqrt{2}\right)\left(3\sqrt{2}\right) = 12$, so the maximum possible value is 12, which we have seen is achieved when $a = b = c = 2$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M368. *Proposed by J. Walter Lynch, Athens, GA, USA.*

An infinite series of positive rational numbers $a_1 + a_2 + a_3 + \cdots$ is the fastest converging infinite series with a sum of 1, $a_1 = \frac{1}{2}$, and each a_i having numerator 1. (By "fastest converging", we mean that each term a_n is successively chosen to make the sum $a_1 + a_2 + \cdots + a_n$ as close to 1 as possible.) Determine a_5 and describe a recursive procedure for finding a_n.

Solution by Robert Sheets, Southeast Missouri State University, Cape Girardeau, MO, USA.

Let s_n be the n^{th} partial sum of the infinite series. Then $a_1 = \frac{1}{2}$ and $s_1 = \frac{1}{2}$. Since all terms of the infinite series are positive, none of the terms can be negative or zero, thus no partial sum can be greater than or equal to 1. We therefore need $a_2 < 1 - s_1 = \frac{1}{2}$. Seeking the largest such a_2, we find that $a_2 = \frac{1}{3}$ since a_2 is a fraction of positive integers with numerator equal to 1.

Then $s_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ which gives $a_3 < \frac{1}{6}$, whence $a_3 = \frac{1}{7}$. Similarly, we find that $s_3 = \frac{41}{42}$, giving $a_4 = \frac{1}{43}$, and $s_4 = \frac{1805}{1806}$, giving $a_5 = \frac{1}{1807}$.

Next, we describe a recursive procedure to find a_n. We conjecture that if $a_n = \frac{1}{k}$, then $s_n = \frac{k(k-1)-1}{k(k-1)}$ and $a_{n+1} = \frac{1}{k(k-1)+1}$. (Note that $2(1)+1 = 3$, $3(2)+1 = 7$, $7(6)+1 = 43$, and $43(42)+1 = 1807$, so these equations hold for $n = 1$, 2, 3, and 4.)

We prove these recursive relations by induction. We have already verified the base cases above. Suppose that for some positive integers n and k we have $a_n = \frac{1}{k}$, $s_n = \frac{k(k-1)-1}{k(k-1)}$, and $a_{n+1} = \frac{1}{k(k-1)+1}$. We prove that the relations will also hold for s_{n+1} and a_{n+2}. Let $t = k(k-1)+1$; this means that $a_{n+1} = \frac{1}{t}$ and $s_n = \frac{t-2}{t-1}$. Then

$$\begin{aligned} s_{n+1} &= s_n + a_{n+1} \\ &= \frac{t-2}{t-1} + \frac{1}{t} = \frac{t^2-2t+t-1}{t(t-1)} \\ &= \frac{t^2-t-1}{t(t-1)} = \frac{t(t-1)-1}{t(t-1)}. \end{aligned}$$

Finally, since $a_{n+2} < 1 - s_{n+1} = \frac{1}{t(t-1)}$ and a_{n+2} is the largest fraction with numerator 1 satisfying this property, we find that $a_{n+2} = \frac{1}{t(t-1)+1}$, as required.

Therefore, the result holds by induction and for all positive integers n, if $a_n = \frac{1}{k}$ for some positive integer k, then $a_{n+1} = \frac{1}{k(k-1)+1}$.

Also solved by ARKADY ALT, San Jose, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON. There was one incomplete solution submitted.

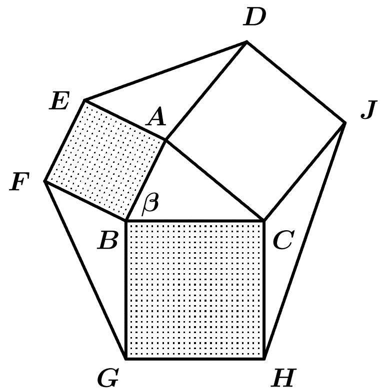
Hess noted that the sequence of denominators appearing in the series is A000058 *in the* On-Line Encyclopedia of Integer Sequences (`http://www.research.att.com/~njas/sequences/`) *and is known as Sylvester's sequence. It is a curious fact that two consecutive terms of Sylvester's sequence differ by a square, since the number k in the sequence is followed by k^2-k+1, yielding a difference of $(k^2-k+1)-k = (k-1)^2$.*

Problem of the Month

Ian VanderBurgh

Among Jim Totten's many interests was his involvement in mathematics outreach. In particular, Jim was a driving force behind the Cariboo College High School Mathematics Contest. He edited a volume of problems taken from those contests written between 1973 and 1992. This month, we look at one of the problems from this volume.

Problem (1989 Cariboo College High School Mathematics Contest, Senior Final Round, Part B) In the figure, $AEFB$, $BGHC$, and $ACJD$ are squares constructed on the sides of $\triangle ABC$. If the (combined) area of the two shaded squares equals the area of the rest of the figure, show that the area of $\triangle ABC$ equals the area of $\triangle FBG$ and then find the number of degrees in $\angle ABC$.



This problem actually appears to be the "poster child" for this contest, as it appears on the cover of the compilation book and appears as a kind of logo elsewhere on the web.

Before we look at the solution, there is one really useful formula upon which we should agree. If in $\triangle XYZ$ we have $XY = z$ and $YZ = x$, then the area of $\triangle XYZ$ is equal to $\frac{1}{2}xz\sin(\angle XYZ)$. This formula is a great alternative to the standard area formula "$\frac{1}{2}bh$" if all you have is one angle of a triangle and the lengths of the two sides enclosing it. We'll derive this formula after the solution to the problem.

Solution Let $BC = a$, $AC = b$, and $AB = c$. Since $AEFB$ is a square, then $BF = AE = c$. Since $BGHC$ is a square, then $BG = CH = a$. Since $ACJD$ is a square, then $AD = CJ = b$.

Since $\angle ABC = \beta$ and $\angle ABF = \angle CBG = 90°$, it then follows that $\angle FBG = 360° - \beta - 90° - 90° = 180° - \beta$.

We first need to prove that $\triangle ABC$ and $\triangle FBG$ have equal areas. From the formula in the preamble, the area of $\triangle ABC$ is $\frac{1}{2}(BC)(AB)\sin(\angle ABC)$ or $\frac{1}{2}ac\sin\beta$. Similarly, the area of $\triangle FBG$ is $\frac{1}{2}(BG)(FB)\sin(\angle FBG)$ or $\frac{1}{2}ac\sin(180° - \beta)$.

But $\sin\beta = \sin(180° - \beta)$ for any angle β, so the two areas are equal.

While it may not be immediately obvious why this helps, we can stall a bit by noting that we can use the same argument to conclude that the area of

$\triangle EAD$ and the area of $\triangle HCJ$ are each equal to the area of $\triangle ABC$. Can you see why?

At this point, we should probably use the piece of information that we were given, namely, that the combined area of square $AEFB$ and square $BGHC$ equals the area of the rest of the figure. We use the short-hand $|AEFB|$ to denote the area of figure $AEFB$. Thus, we are told that

$$|AEFB| + |BGHC| = |FBG| + |ABC| + |EAD| + |HCJ| + |ACJD| .$$

Using some of what we know so far, this becomes

$$c^2 + a^2 = 4|ABC| + b^2 ,$$

or

$$c^2 + a^2 = 4\left(\frac{1}{2}ac\sin\beta\right) + b^2 .$$

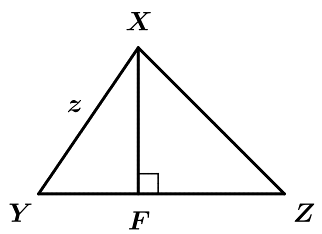
Remember, we're trying to find β. We seem to have too many other pieces of information floating around to have any hope of doing this. But at this point, the amazing pattern recognition abilities of the brain might kick in. This equation looks somewhat similar to a law that we often use. This might prompt us to try applying that law. Do you see what I'm getting at?

Applying the Law of Cosines in $\triangle ABC$ gives $b^2 = a^2 + c^2 - 2ac\cos\beta$. Substituting this into the last equation, we obtain

$$\begin{aligned} c^2 + a^2 &= 2ac\sin\beta + (a^2 + c^2 - 2ac\cos\beta) ; \\ 2ac\cos\beta &= 2ac\sin\beta ; \\ \cos\beta &= \sin\beta , \end{aligned}$$

since $ac > 0$. Since $\cos\beta = \sin\beta$, and β is an angle in a triangle, then $\beta = 45^\circ$, and we are done. ∎

To me, this was quite surprising. Well, the answer itself wasn't so surprising since these problems almost always have 30°, 45°, or 60° as an answer. But, it was surprising to me that the angle β was completely determined from the given information while no other information (side lengths or angles) can be determined.



Before wrapping up the column this month, we should go back and look at the formula from the preamble. Suppose that $\angle XYZ$ is acute. Drop a perpendicular from X to F on YZ.

Then the area of $\triangle XYZ$ is equal to $\frac{1}{2}(YZ)(XF)$. But $YZ = x$ and we also have $XF = XY\sin(\angle XYZ) = z\sin(\angle XYZ)$, so the area equals $\frac{1}{2}xz\sin(\angle XYZ)$, as required. Can you prove this in the case that angle Y is obtuse or a right angle?

The Tanker Problem

Ross Honsberger

When I walk my puppy with his retractable leash, he is forever getting behind, catching up and running past, cutting across in front, and getting behind again. When I was having my after-lunch nap the other day, this reminded me of those old "patrol boat circling an ocean liner" problems and I mused over the following situation.

A security patrol boat repeatedly circles a supertanker that is a gigantic rectangular box 450 metres long and 50 metres across. The ocean is calm and the tanker travels at a constant speed along a straight path. The patrol boat goes up the left side, across the front, down the right side, and across the back, and keeps doing it over and over.

The patrol boat travels in only two directions of the compass – when going parallel to the path of the tanker, it travels in straight lanes parallel to the tanker, one on each side at a distance of 25 metres from it, and when crossing in front or behind, it goes straight across perpendicular to the path of the tanker.

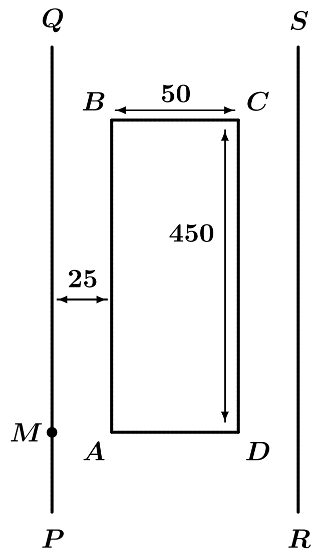


Figure 1

Neglecting the dimensions of the patrol boat (that is, considering it to be represented geometrically by a point) and given that it goes constantly at twice the speed of the tanker and that its turns are instantaneous, *what is the shortest distance that the patrol boat must travel in completing one cycle around the tanker*?

I felt the problem would be an easy recreation with a pleasing analysis and I expected a solution along the following lines.

Let the tanker and the lanes of the patrol boat be labeled as in Figure 1 and let the boat travel towards Q when it is in lane PQ.

Consider a cycle that begins at M on PQ; that is, to begin the boat is abreast of the stern AD of the tanker. While the boat travels 900 m up PQ, the tanker will have advanced 450 m and the boat will be abreast of the front BC of the tanker at the point E on PQ (Figure 2).

If the boat were to start crossing between its lanes at this point, it would crash into the tanker at a point 12.5 m down the side. In order to cross in front of the tanker, the boat must get $25 + 50 = 75$ m across the gap to the right hand edge of the tanker before the tanker advances to its level. Therefore, it needs to have a lead of 37.5 m when it starts its turn, for the tanker will advance 37.5 m while the boat is going across this 75 m.

In order to get a 37.5 m lead, the boat must continue up PQ another 75 m beyond E (while the boat is doing this the tanker advances 37.5 m to yield a net lead of 37.5 m).

Summarizing, the boat and the tanker are abreast at E and BC; as the boat advances the 75 m to F, the tanker advances 37.5 m to UV; the boat turns at F and while going the 75 m along FK, the tanker advances 37.5 m to GK. Thus, the boat and the tanker just miss each other at K. And while the boat completes the remaining 25 m along KL to its other lane, the tanker takes a 12.5 m lead at YN.

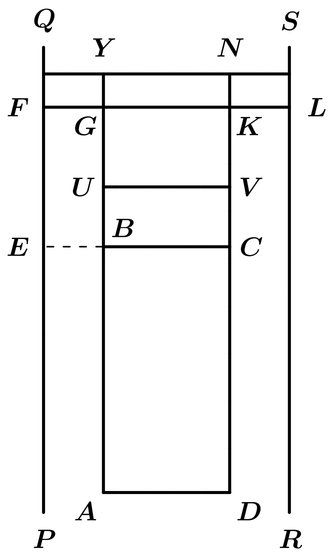


Figure 2

Thus, when the boat turns down SR it is already 12.5 m down that side of the tanker. Now, the boat need go only to a point H that is 12.5 m from the far end before making its turn across the back (while it crosses the 25 m gap toward the tanker, the tanker moves ahead this 12.5 m and the boat just misses the bottom right corner of the tanker). Hence the boat needs to cover a distance of only $450 - 12.5 - 12.5 = 425$ m along this side before turning. Since it goes twice as fast as the tanker, while the boat covers two-thirds of this distance, the tanker moves ahead the other third, and so the boat need travel down that side only two-thirds of 425 m, that is, a distance of $\frac{850}{3}$ m.

Now, from just missing the tanker at its bottom right corner, the boat still has to go 75 m to get back to lane PQ. While the boat is doing this, the tanker gets ahead 37.5 m, and so, in order to catch up and complete its cycle, the patrol boat must go another 75 m up PQ to get abreast of the tanker.

Altogether, then, the patrol boat must travel a grand total of

$$900 + 75 + 100 + \frac{850}{3} + 100 + 75 \;=\; 1533\frac{1}{3}\ \text{m}\,.$$

This solution blithely assumes that the patrol boat and the tanker harmlessly slide by each other when they arrive simultaneously at the tanker's right hand corners. However, it makes a great deal more sense to interpret such an event as a collision: both vessels can hardly move into the same position at the same time without fatal consequences for the patrol boat. Fortunately, this nagging uncertainty is easily removed by adding the condition that *the two vessels are never to get closer to each other than* 25 m.

This is readily accommodated by having the boat get a lead of 62.5 m at F, before turning along the dangerous 75 m stretch FK between the lanes. This gives the boat a net lead of $62.5 - 37.5 = 25$ m when it reaches the right hand side of the tanker, thus avoiding any possibility of a collision and providing the 25 m separation at this critical juncture.

In order to get 62.5 m ahead, the boat must go another 125 m up PQ beyond E to a point F (Figure 3). Thus, when the patrol boat reaches F, the tanker is at UV, and the patrol boat is 62.5 m ahead.

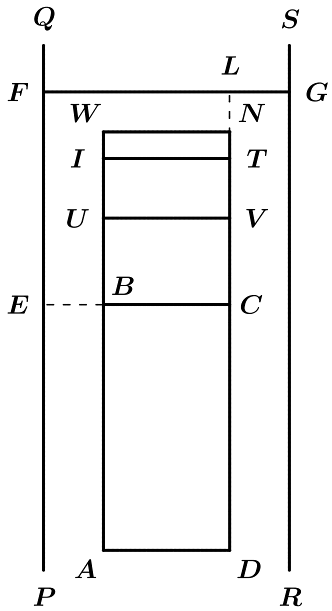


Figure 3

Now the boat turns along FL, and when it reaches L, the tanker has advanced 37.5 m to IT, putting the tanker 25 m behind. While the boat proceeds 25 m along LG to the other lane, the tanker cuts its lead to 12.5 m at WN.

Thus, when the boat turns down SR, it is already 12.5 m ahead. In order to remain 25 m from the other end of the tanker when it goes back across the first lane, it will need to go farther down the side to a point H that is 12.5 m beyond the other end. Then, when it turns and closes the 25 m stretch to the nearer side of the tanker, the tanker will have advanced another 12.5 m to yield a separation of 25 m between them. Hence the patrol boat must cover $450 + 12.5 + 12.5 = 475$ m along this side before making its turn. As before, it needs to travel down that side only two-thirds of this distance, that is, $\frac{950}{3}$ m.

Thus, when passing the right rear corner of the tanker, the boat is already 25 m behind, and still has 75 m of the gap to negotiate to get back to lane PQ. While the boat is doing this, the tanker gets ahead another 37.5 m, for a total of 62.5 m, and so, in order to catch up and complete its cycle, the patrol boat must go another 125 m up PQ to get abreast of the tanker.

Altogether, then, the patrol boat must travel a grand total of

$$900 + 125 + 100 + \frac{950}{3} + 100 + 125 \;=\; 1666\tfrac{2}{3}\text{ m}.$$

This seemed most satisfactory until the brilliant solution of my colleague Larry Rice (retired from the University of Toronto Schools and the University of Waterloo) revealed another unwarranted assumption in my solution. Since the patrol boat had a 25 m lead when it reached the right hand edge of the tanker, it went unchallenged that the minimum 25 m distance between the boat and the front right corner of the tanker would be maintained as the boat completed the final 25 m of the crossing to its other lane.

Unfortunately this is not true; it is not hard to show that the vessels do get closer together than 25 m when the boat is completing the crossing.

So, how much of a lead would the boat have to have when it passes the right hand edge of the tanker in order to maintain a 25 m separation? Since 25 m isn't enough, perhaps a lead of 27 m would do? Again, it is not difficult to see that 27 m is not enough. So, how about 28 m? Bingo: 28 m is sufficient, with a little to spare. Consequently, any lead greater than 28 m would also suffice. But we seek the shortest cycle of the boat around the tanker. Thus we need to determine the smallest acceptable lead. We know it's some value, $25+y$, between 27 and 28 m. Here y is not a variable, but a well-defined fixed value.

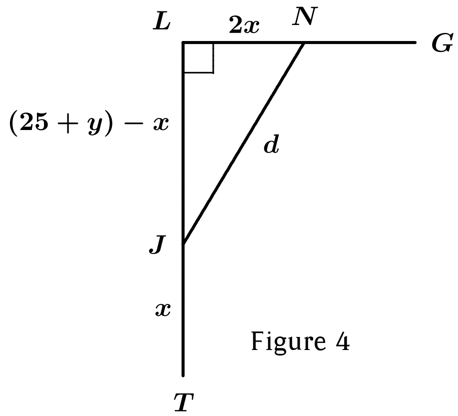


Figure 4

In Figure 4, let the boat pass the right hand edge of the tanker with a lead TL of $(25+y)$ metres. As the tanker advances x metres to J on TL, the boat advances $2x$ metres along LG to the point N. We want to determine the smallest positive value y such that the segment $d = JN$ is at least 25 m for all x between 0 and 12.5.

By the Pythagorean Theorem, we have

$$\begin{aligned} d^2 &= (2x)^2 + (25+y-x)^2 \\ &= 4x^2 + (25+y)^2 - 2x(25+y) + x^2 \\ &= 5x^2 + y^2 + 50y + 625 - 50x - 2xy\,, \end{aligned}$$

and for $d^2 \geq 625$, we need

$$5x^2 + y^2 + 50y - 50x - 2xy \;\geq\; 0\,.$$

Since the only variable here is x, we may write

$$f(x) \;=\; 5x^2 + y^2 + 50y - 50x - 2xy\,,$$

where we require $f(x) \geq 0$ for $0 \leq x \leq 12.5$.

Differentiating, we get

$$f'(x) \;=\; 10x - 50 - 2y \;=\; 2(5x - 25 - y)\,,$$

which vanishes for $x = 5 + \frac{1}{5}y$. Since the graph of $f(x)$ is a parabola opening upwards, its minimum occurs when $f'(x) = 0$.

Hence

$$\begin{aligned}\min f(x) &= 5\left(5+\frac{1}{5}y\right)^2 + y^2 + 50y - 50\left(5+\frac{1}{5}y\right) - 2y\left(5+\frac{1}{5}y\right)\\ &= \frac{4}{5}y^2 + 40y - 125\,.\end{aligned}$$

Now, we don't want this minimum to be greater than 0, for in that case d would always *exceed* 25 and a lead of $25+y$ would be greater than it need be. So for minimum y, we require the minimum to equal 0, and

$$\frac{4}{5}y^2 + 40y - 125 = 0\,,$$

or

$$4y^2 + 200y - 625 = 0\,.$$

The number y being positive, we obtain

$$\begin{aligned}y &= \frac{-200+\sqrt{200^2-4(4)(-625)}}{2(4)} = \frac{-200+\sqrt{50000}}{8}\\ &= \frac{25}{2}\sqrt{5} - 25 \approx 2.95085\,,\end{aligned}$$

or 2.951 to three decimal places of accuracy.

Thus a lead of 27.951 would assure a universal 25 m separation and provide a shortest cycle to a reasonable degree of accuracy.

We have seen that an additional 125 m beyond E gives the boat a 25 m lead when it reaches the right hand edge of the tanker. Hence a lead of 27.951 m would require the boat to go a further $2(2.951) = 5.902$ m for a total of 130.902 m up its left lane before starting its turn across the front. Moreover, its lead is now $12.5 + 2.951 = 15.451$ m when it starts down the right lane.

To produce the same separation at the other end of the tanker, the boat needs to go this same distance, 15.451 m, farther down this lane than the end of the tanker, for a total passage of $450+2(15.451) = 480.902$ m. As before, the boat needs to cover only two-thirds of this distance, namely 320.601 m. After returning the 100 m to its left lane, the boat would be another 50 m behind, for a total of 65.451 m, and therefore needs to go another 130.902 m up PQ in order to complete the cycle. Altogether, then, the boat travels approximately a total of

$$900 + 130.902 + 100 + 320.601 + 100 + 130.902 = 1682.405 \text{ m}\,.$$

Finally, let's consider Larry's wonderful solution – a completely different approach.

The unknown distances in the cycle are those travelled by the patrol boat in the lanes PQ and RS; the distances travelled in crossing between the lanes is always $100 + 100 = 200$ m. Larry obtains these unknown distances from the basic formula

$$\text{distance} = \text{speed} \times \text{time}\,.$$

If we denote the speed of the tanker by v, then the speed of the boat is $2v$, and it remains only to determine the times t_1 and t_2 that the boat spends in lanes PQ and RS respectively, since the distance travelled in lane PQ will be $(2v)t_1$ and the distance travelled in lane RS will be $(2v)t_2$.

Knowing only the *ratio* of the speeds of the boat and the tanker, you might well wonder how he is going to obtain anything of value about these times. Larry very ingeniously enlists the aid of the captain of the tanker in this matter.

To this end, consider the observations of the captain as he watches the boat circle his tanker (Figure 5). First, he sees the boat run up lane PQ from E to F. At F, the boat turns across the lanes in front of him and, as it proceeds, it keeps getting closer to him because the tanker is moving ahead. Upon reaching the other lane, the boat proceeds along it and eventually turns back to lane PQ. As it crosses back, the tanker, still going forward, moves into the lead. Finally, the boat gets abreast of the tanker up lane PQ. Thus the captain sees the boat cycle the tanker along the sides of a trapezoid $EFGH$.

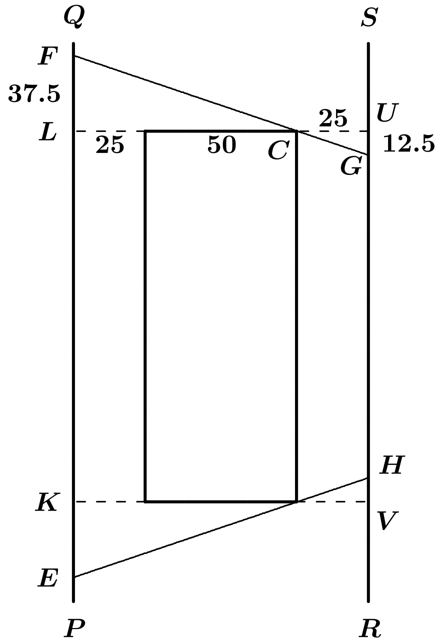


Figure 5

Between two points on a slanted side of the trapezoid, the boat moves twice as far toward the other lane as it does toward, or away from, the tanker, implying that the slopes of these sides are $\pm\frac{1}{2}$. Thus the trapezoid is isosceles and enjoys the symmetry of such a figure.

Since the slope of FG is $-\frac{1}{2}$ and $LC = 75$ m, it follows that $LF = 37.5$ m. The symmetrical part EK is therefore also 37.5 m, making the total length of side $EF = 450 + 2(37.5) = 525$ m. Similarly, since $CU = 25$ m, then $UG = HV = 12.5$ m, making the length of side $GH = 450 - 2(12.5) = 425$ m.

Now, time equals distance divided by speed, so the time t_1 that it takes the boat to traverse EF is 525 divided by the boat's speed along EF. While the boat goes at speed of $2v$ relative to the ocean, the captain sees it proceed up EF at the speed at which it overtakes the tanker; that is, at the relative speed of $2v - v = v$. Hence the captain calculates $t_1 = \dfrac{525}{v}$.

Along side GH, the captain observes the boat is going at a relative speed of $2v + v = 3v$, and so he calculates $t_2 = \frac{425}{3v}$.

It follows that the distance the boat travels up and down the lanes is

$$2v\left(\frac{525}{v}\right) + 2v\left(\frac{425}{3v}\right) = 1050 + \frac{850}{3} = 1333\frac{1}{3}\,\text{m}.$$

Adding in the 200 m travelled in crossing between lanes, we obtain the same earlier grand total of $1533\frac{1}{3}$ m for the first case, in which we generously allowed the vessels to slide by each other without incident.

Allowing an additional 25 m buffer at each crossing between the lanes would increase each of the parallel sides of the trapezoid by 50 m and yield the corresponding times $t_1 = \frac{575}{v}$ and $t_2 = \frac{475}{3v}$.

The resulting cycle, then, would have a total length of

$$2v\left(\frac{575}{v}\right) + 2v\left(\frac{475}{3v}\right) + 200 = 1150 + 316\frac{2}{3} + 200 = 1666\frac{2}{3}\,\text{m},$$

as found earlier.

Finally, (Figure 6), a universal separation of 25 m, would add an extra $2(12.5)(\sqrt{5}) \approx 25(2.236) = 55.9$ m to each of the parallel sides of the original trapezoid $EFGH$.

This is because the right-angled $\triangle XYC$ has legs in the ratio $1 : 2$. Can you see why?

Thus, $t_1 = \frac{580.9}{v}$ and $t_2 = \frac{480.9}{3v}$ which yields a cycle length of approximately

$$\begin{aligned} & 2(580.9) + 2\left(\frac{480.9}{3}\right) + 200 \\ = \;& 1161.8 + 320.6 + 200 \\ = \;& 1682.4\,\text{m}, \end{aligned}$$

in close agreement with our previous result.

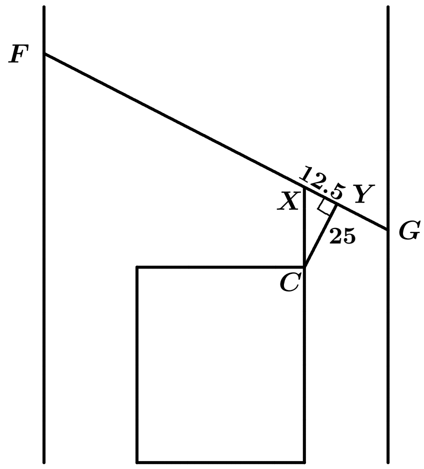


Figure 6

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