SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2557. [2000 : 304] *Proposed by Gord Sinnamon, University of Western Ontario, London, ON and Hans Heinig, McMaster University, Hamilton, ON.*

(a) Show that for all positive sequences $\{x_i\}$ and all integers $n > 0$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \;\le\; 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 x_k^{-1}\,.$$

(b)★ Does the above inequality remain true without the factor of 2?

(c)★ [Proposed by the editors] What is the minimum constant c that can replace the factor 2 in the above inequality?

Solution to part (c) *by Li Chao, student, High School Affiliated to Renmin University of China, Beijing, China.*

The given inequality is equivalent to

$$\sum_{i=1}^{n}\binom{n+2-i}{2}x_i \;<\; c\sum_{i=1}^{n}\frac{1}{x_i}\left(\sum_{j=1}^{i} x_j\right)^2\,.$$

Making the substitution

$$s_k \;=\; \sum_{i=1}^{k} x_i\,,\quad k=0\,,1\,,2\,,\,\ldots\,,n$$

and then simplifying puts the inequality into another equivalent form:

$$\sum_{i=1}^{n}(n+1-i)s_i \;<\; c\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\,.$$

By the Cauchy–Schwartz Inequality we have

$$\begin{aligned}\left(\sum_{i=1}^{n}(n+1-i)s_i\right)^2 \;&\le\; \left(\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)\left(\sum_{i=1}^{n}(n+1-i)^2(s_i-s_{i-1})\right)\\ &=\; \left(\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)\left(\sum_{i=1}^{n}(2n+1-2i)s_i\right)\\ &<\; \left(\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)\left(2\sum_{i=1}^{n}(n+1-i)s_i\right)\,.\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}(n+1-i)s_i \;<\; 2\sum_{i=1}^{n}\frac{s_i^2}{s_i - s_{i-1}}\,,$$

so that taking $c = 2$ ensures that the inequality holds for all positive real numbers $x_1, x_2, \ldots, x_n$.

For each $n > 4$ we will now give a strictly increasing sequence of numbers $s_0, s_1, \ldots, s_n$ such that $s_0 = 0$ and

$$\lim_{n\to\infty}\frac{\left(\displaystyle\sum_{i=1}^{n}(n+1-i)s_i\right)}{\left(\displaystyle\sum_{i=1}^{n}\frac{s_i^2}{s_i - s_{i-1}}\right)} \;=\; 2$$

(this is equivalent to specifying $x_1, x_2, \ldots, x_n$). Take $n > 4$, let $s_0 = 0$ and

$$\begin{aligned} s_i &= \frac{(n-1)(n-2)}{(n-i)(n-i-1)}\,, \quad 1 \le i \le n-2\,; \\ s_n &= 2s_{n-1} \;=\; 4s_{n-2}\,. \end{aligned}$$

Then we have

$$\begin{aligned} &\sum_{i=1}^{n}\frac{s_i^2}{s_i - s_{i-1}} \\ &= 1 + \sum_{i=2}^{n-2}\frac{\dfrac{(n-1)^2(n-2)^2}{(n-i)^2(n-i-1)^2}}{(n-1)(n-2)\left[\dfrac{1}{(n-i)(n-i-1)} - \dfrac{1}{(n-i+1)(n-i)}\right]} \\ &\quad + 6(n-1)(n-2) \\ &= 1 + \frac{(n-1)(n-2)}{2}\sum_{i=2}^{n-2}\left(\frac{2}{n-i-1} - \frac{1}{n-i}\right) \\ &\quad + 6(n-1)(n-2) \\ &= 1 + \frac{(n-1)(n-2)}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-3} + \left(1 - \frac{1}{n-2}\right)\right] \\ &\quad + 6(n-1)(n-2) \\ &= 1 + \frac{(n-1)(n-2)}{2}\bigl[\ln n + O(1)\bigr] \;+\; 6(n-1)(n-2) \\ &= \frac{1}{2}n^2\ln n + O(n^2)\,, \end{aligned}$$

where the notation $f(n) = O\bigl(g(n)\bigr)$ here means that there is a positive constant C such that $|f(n)| \le C|g(n)|$ holds for sufficiently large n, and we have used the fact that $\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\ln n$ tends to Euler's constant, γ, as n tends to infinity.

We also have

$$\begin{aligned}
&\sum_{i=1}^{n}(n+1-i)s_i \\
=\;& \sum_{i=1}^{n-2}(n-i+1)\frac{(n-1)(n-2)}{(n-i)(n-i-1)} \\
&\quad + 4(n-1)(n-2) \\
=\;& (n-1)(n-2)\sum_{i=1}^{n-2}\left(\frac{2}{n-i-1}-\frac{1}{n-i}\right) \\
&\quad + 4(n-1)(n-2) \\
=\;& (n-1)(n-2)\left[1+\frac{1}{2}+\cdots+\frac{1}{n-2}+\left(1-\frac{1}{n-1}\right)\right] \\
&\quad + 4(n-1)(n-2) \\
=\;& (n-1)(n-2)\left[\ln n+O(1)\right] \;+\; 4(n-1)(n-2) \\
=\;& n^2\ln n+O(n^2)\,.
\end{aligned}$$

We now compute

$$\begin{aligned}
\lim_{n\to\infty}\frac{\left(\displaystyle\sum_{i=1}^{n}(n+1-i)s_i\right)}{\left(\displaystyle\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)} &= \lim_{n\to\infty}\frac{n^2\ln n+O(n^2)}{\frac{1}{2}n^2\ln n+O(n^2)} \\
&= \lim_{n\to\infty}\frac{2+2O(n^2)/n^2\ln n}{1+2O(n^2)/n^2\ln n} \;=\; 2\,,
\end{aligned}$$

as desired.

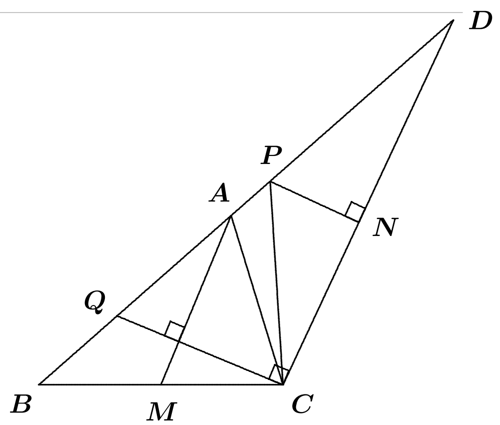
Therefore, the minimum value of the constant c is 2.

No other solutions to part (c) *were submitted.*

3351. [2008 : 298, 300] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $AB > AC$. Let P be a point on the line AB beyond A such that $AP+PC=AB$. Let M be the midpoint of BC, and let Q be the point on the side AB such that $CQ\perp AM$. Prove that $BQ=2AP$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Taichi Maekawa, Takatsuki City, Osaka, Japan.



Let D be the point on the line PB beyond P such that $PD = PC$. Since

$$\begin{aligned} AD &= AP + PD \\ &= AP + PC = AB\,, \end{aligned}$$

A is the midpoint of the segment BD. It follows that $AM||DC$, because M is the midpoint of BC. Hence $\angle DCQ = 90^\circ$. Let $PN \perp DC$ with point N on the line CD. Then $DN = NC$, because triangle CDP is isosceles. Since N is the midpoint of CD and $PN \parallel QC$, it follows that P is the midpoint of QD. Thus $PQ = PD = PC$. Hence

$$AP + AQ = PQ = PC = AB - AP\,,$$

and therefore,

$$2AP = AB - AQ = BQ\,,$$

as claimed.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CHRIS BROYLES and MATTHEW STEIN, Southeast Missouri State University, Cape Girardeau, MO, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany (two solutions); JOHN G. HEUVER, Grande Prairie, AB; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3352. [2008 : 298, 300] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a right-angled triangle with right angle at A. Let I be the incentre of $\triangle ABC$, and let D and E be the intersections of BI and CI with AC and AB, respectively. Prove that

$$\frac{BI \cdot ID}{CI \cdot IE} = \frac{AB}{AC}\,.$$

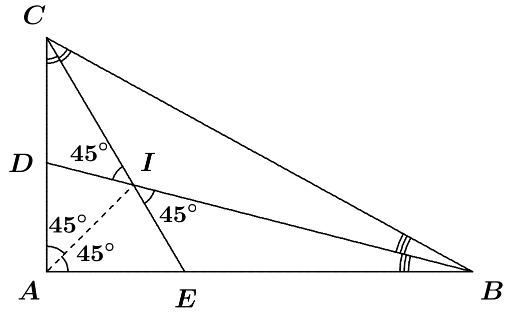
Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India.

Since AI bisects the right angle at A, we have

$$\angle CID = \frac{1}{2}\angle B + \frac{1}{2}\angle C = \frac{1}{2}\left(180^\circ - \angle A\right) = 45^\circ = \angle CAI\,.$$

Hence, triangles CAI and CID are similar, so that

$$\frac{ID}{CI} = \frac{AI}{AC}\,.$$



Likewise, triangles BAI and BIE are similar, and therefore,

$$\frac{BI}{IE} = \frac{AB}{AI}\,.$$

The result now follows by multiplying across the last two equations.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3353. [2008 : 298, 301] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle all of whose side lengths are positive integers.

(a) Determine all such triangles where one angle has twice the measure of a second angle.

(b) Determine all such triangles where two medians are perpendicular.

Composite of solutions by Roy Barbara, Lebanese University, Fanar, Lebanon and Michel Bataille, Rouen, France.

We denote by α, β, and γ the angles opposite the sides $BC = a$, $CA = b$, and $AB = c$ respectively.

(a) Without loss of generality we take $\gamma = 2\alpha$ and show that the solutions are the triangles with

$$a = dn^2\,, \qquad b = d(m^2 - n^2)\,, \qquad c = dmn\,, \tag{1}$$

where d, m, and n are positive integers, m is coprime to n, and $n < m < 2n$.

We use an equivalence proved in the April 2006 issue (see [2006 : 159]) of this journal: A triangle satisfies $\gamma = 2\alpha$ if and only if $c^2 = a(a+b)$.

Now suppose that a, b, and c are the sides of a triangle with $\gamma = 2\alpha$. Let $d = \gcd(a,b)$. Then $a = da'$ and $b = db'$ with a' coprime to b'. Clearly, d also divides c, so $c = dc'$.

Since a' and b' are coprime, then a' and $a' + b'$ are also coprime. Since $(c')^2 = a'(a' + b')$, it follows that there exist coprime positive integers m and n such that $a' = n^2$ and $a' + b' = m^2$.

Thus, (1) holds for positive integers d, m, and n with m coprime to n and $n < m$.

Finally, since ABC is a triangle, $a + c > b$, hence $2n^2 > m(m-n)$, hence $m < 2n$.

Conversely, if (1) holds for positive integers d, m, and n with the aforementioned restrictions, then the inequality $n < m < 2n$ ensures that a, b, and c are the side lengths of a triangle and also the relation $c^2 = a(a+b)$ holds, which ensures that $\gamma = 2\alpha$.

(b) Without loss of generality, we restrict the problem to the case where the medians from B and C are perpendicular.

First we show that the positive real numbers a, b, and c are the sides of a triangle ABC where the medians from B and C are perpendicular if and only if

$$b^2 + c^2 = 5a^2 \quad \text{and} \quad bc > 2a^2\,. \tag{2}$$

If G is the centroid of ABC, then BG is perpendicular to CG if and only if $BC^2 = BG^2 + CG^2$, or

$$\frac{1}{9}\left(2a^2 + 2c^2 - b^2\right) + \frac{1}{9}\left(2a^2 + 2b^2 - c^2\right) = a^2\,,$$

that is, $b^2 + c^2 = 5a^2$.

In addition, $1 > \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{2a^2}{bc}$, hence $bc > 2a^2$.

Conversely, if the conditions in (2) hold, then $(b+c)^2 = 5a^2 + 2bc > a^2$ and $(b-c)^2 = 5a^2 - 2bc < a^2$. Hence, $|b - c| < a < b + c$ and there exists a triangle ABC with sides $BC = a$, $AC = b$, and $AB = c$. The relation $b^2 + c^2 = 5a^2$ holds for this triangle, so the medians from B and C are perpendicular.

Thus the problem amounts to finding the positive integers a, b, and c that satisfy (2). Because of the homogeneity of the conditions, we need only find the solutions such that $\gcd(a,b,c) = 1$; the general solution is then obtained by multiplying these *primitive* solutions by positive integers.

Now let a, b, c be a primitive solution. Then there exist coprime positive integers m and n for which $\dfrac{b-a}{2a-c} = \dfrac{2a+c}{b+a} = \dfrac{m}{n}$, and thus

$$\begin{aligned} a(2m+n) - nb - mc &= 0\,, \\ a(m-2n) + mb - nc &= 0\,. \end{aligned}$$

Solving for a, b, and c yields

$$\begin{aligned} a &= \lambda\left(m^2+n^2\right)\,, \\ b &= \lambda\left(n^2-m^2+4mn\right)\,, \\ c &= 2\lambda\left(m^2-n^2+mn\right)\,, \end{aligned}$$

where λ is a positive rational number. The condition $bc > 2a^2$ reduces to

$$\left(m^2-n^2\right)\left(3mn+2n^2-2m^2\right) > 0\,,$$

which holds if and only if $n < m < 2n$.

Writing $\lambda = \dfrac{u}{v}$, where u and v are positive coprime integers, we obtain

$$\begin{aligned} va &= u\left(m^2+n^2\right)\,, \\ vb &= u\left(n^2-m^2+4mn\right)\,, \\ vc &= 2u\left(m^2-n^2+mn\right)\,. \end{aligned}$$

Now, $\gcd(u,v) = 1$ and $\gcd(a,b,c) = 1$, hence $u = 1$ and

$$(a,b,c) = \left(\frac{m^2+n^2}{v}, \frac{n^2-m^2+4mn}{v}, \frac{2\left(m^2-n^2+mn\right)}{v}\right)\,, \qquad (3)$$

where $v = \gcd\left(m^2+n^2, n^2-m^2+4mn, 2\left(m^2-n^2+mn\right)\right)$.

Conversely, it is easy to show that a triple such as in (3) with m and n coprime and $n < m < 2n$ is a primitive solution.

If p is prime, k is a positive integer, and p^k divides v, then p^k divides both m^2+n^2 and $2\left(n^2-m^2+4mn\right) + 2\left(m^2-n^2+mn\right) = 10mn$.

However, p does not divide n or m, since otherwise it would divide both n and m, and so p^k divides 10. Hence, $p = 2$ or $p = 5$ and $k = 1$, which implies that $v \in \{1, 2, 5, 10\}$.

If v is even then, since v divides m^2+n^2, m, n must be odd.

If 5 divides v then 5 divides $\left(m^2+n^2\right)+\left(n^2-m^2+4mn\right) = 2n(n+2m)$ and $\left(n^2-m^2+4mn\right) - \left(m^2+n^2\right) = 2m(2n-m)$. Since 5 does not divide m or n, then 5 divides both $n+2m$ and $2n-m$. Note that $n+2m = 5k$ and $2n-m = 5l$ is equivalent to $m = 2k-l$ and $n = k+2l$, and that in this case $n < m < 2n$ becomes $0 < 3l < k$.

A complete description of the primitive solutions can now be given:

- $(a,b,c) = \left(\dfrac{m^2+n^2}{10}, \dfrac{n^2-m^2+4mn}{10}, \dfrac{2(m^2-n^2+mn)}{10}\right)$, where $m = 2k-l$, $n = k+2l$; $\gcd(k,l) = 1$; k, l are odd; and $0 < 3l < k$.

- $(a,b,c) = \left(\frac{m^2+n^2}{5}, \frac{n^2-m^2+4mn}{5}, \frac{2(m^2-n^2+mn)}{5}\right)$, where $m = 2k-l$, $n = k+2l$; $\gcd(k,l) = 1$; k, l have opposite parity; and $0 < 3l < k$.

- $(a,b,c) = \left(\frac{m^2+n^2}{2}, \frac{n^2-m^2+4mn}{2}, \frac{2(m^2-n^2+mn)}{2}\right)$, where $\gcd(m,n) = 1$; m, n are odd; $n+2m$ is not divisible by 5; and $0 < n < m < 2n$.

- $(a,b,c) = \left(m^2+n^2, n^2-m^2+4mn, 2(m^2-n^2+mn\right)$, where $\gcd(m,n) = 1$; m, n have opposite parity; $n+2m$ is not divisible by 5; and $0 < n < m < 2n$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (*part* (a) *only*)*; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany* (*part* (a) *only*)*; RICHARD I. HESS, Rancho Palos Verdes, CA, USA* (*part* (a) *only*)*; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria* (*part* (a) *only*)*; RODOLFO LARREA and LUZ RONCAL, Logroño, Spain* (*part* (b) *only*)*; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India* (*part* (a) *only*)*; and TITU ZVONARU, Comănești, Romania* (*part* (a) *only*)*. There were three incomplete solutions submitted.*

Modak remarks that Theorem 44 of [1] *gives all integer solutions to* $b^2 + c^2 = 5a^2$*, while Janous refers to* [2] *where solutions to this equation are obtained by factoring over the Gaussian integers and using the fact that* 5 *is a sum of two squares.*

Geupel notes that Problem 129(a) of [3] *asks for the smallest integral triangle for which one angle is twice another.*

References

[1] L.E. Dickson, *Introduction to the Theory of Numbers*, Dover Publ., 1957.

[2] L.J. Mordell, *Dophantine Equations*, Academic Press, London and New York, 1969.

[3] D.O. Shklarsky, N.N. Khentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book*, Dover Publ., New York, 1962.

3354. [2008 : 298, 301] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{n^2+k^2}{n^2}\right)^{k^3/n^4}.$$

Composite of nearly identical solutions submitted by all the solvers whose names appear below.

The function $f(x) = x^3 \ln\left(1+x^2\right)$ is continuous and hence Riemann integrable on $[0, 1]$. Using integration by parts, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{n^2+k^2}{n^2}\right)^{k^3/n^4} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3\ln\left(1+\left(\frac{k}{n}\right)^2\right)$$

$$= \int_0^1 x^3\ln\left(1+x^2\right)\,dx = \frac{1}{4}x^4\ln\left(1+x^2\right)\Big|_0^1 - \frac{1}{2}\int_0^1\frac{x^5}{1+x^2}\,dx$$

$$= \frac{1}{4}\ln 2 - \frac{1}{2}\int_0^1\left(x^3 - x + \frac{x}{1+x^2}\right)dx$$

$$= \frac{1}{4}\ln 2 - \frac{1}{2}\left(\frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2}\ln\left(1+x^2\right)\right)\Big|_0^1 = \frac{1}{8}.$$

Solvers: GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; OVIDIU FURDUI, Campia Turzii, Cluj, Romania; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3355. [2008 : 299, 301] *Proposed by Todor Yalamov, Sofia University, Sofia, Bulgaria.*

For the triangle ABC let $(x, y)_{ABC}$ denote the line which intersects the union of the segments AB and BC in X and the segment AC in Y such that

$$\frac{\widetilde{AX}}{AB+BC} = \frac{AY}{AC} = \frac{x\cdot AB + y\cdot BC}{(x+y)(AB+BC)},$$

where $\widetilde{AX}$ is either the length of the segment AX if X lies between A and B, or the sum of the lengths of the segments AB and BX if X lies between B and C. Prove that the three lines $(x, y)_{ABC}$, $(x, y)_{BCA}$, and $(x, y)_{CBA}$ intersect in a point dividing the segment NI in the ratio $x : y$, where N is the Nagel point and I the incentre of $\triangle ABC$.

Solution outline by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

As usual we let $a = BC$, $b = CA$, $c = AB$, and $s = \dfrac{a+b+c}{2}$. Set $t = \dfrac{x}{x+y}$; thus, $1 - t = \dfrac{y}{x+y}$, and the ratio of interest becomes a function of t, namely

$$f(t) \;=\; \frac{tc+(1-t)a}{a+c} \;=\; \frac{\widetilde{AX}_t}{a+c} \;=\; \frac{AY_t}{b}\,,$$

where X_t is the point of $AB \cup BC$, and Y_t of AC, that correspond to our parameter t. In this notation, $(x,y)_{ABC} = X_tY_t$. Of course, when $a = c$, $f(t)$ is constant and $(x,y)_{ABC}$ is the line NI for all x, y. This is consistent with what we are to prove unless $a = b = c$; when $\triangle ABC$ is equilateral we have $N = I$ (so there is no line NI), and our three lines intersect at that point. Let us therefore assume that $a \neq c$, so that $f(t)$ is not constant and the lines NI and $(x,y)_{ABC}$ intersect.

We shall see that $(x,y)_{ABC}$ is the line in the family of lines parallel to the bisector of $\angle B$ (where B is the middle vertex in the subscript) that divides the segment NI in the ratio $t:(1-t) = x:y$. Note that as a consequence, $(x,y)_{ABC}$ and $(x,y)_{CBA}$ represent the same line, so the proposer probably intended $(x,y)_{CAB}$ for his third line. For this problem we are concerned with the domain $0 \le t \le 1$; in particular,

(a) $AY_1 = bf(1) = \dfrac{bc}{a+c}$; $CY_1 = b - AY_1 = \dfrac{ab}{a+c}$; Y_1 is the foot of the bisector of $\angle CBA$ (because it divides the side AC in the ratio $c:a$);

(b) $AY_0 = bf(0) = \dfrac{ab}{a+c}$; $CY_0 = \dfrac{bc}{a+c}$;

(c) $X_1 = B$ (because $f(1) = \dfrac{c}{a+c} = \dfrac{AX_1}{a+c}$);

(d) $\widetilde{AX}_0 = (a+c)f(0) = a$: when $c \ge a$, X_0 lies on AB (whence $AX_0 = a$ and $BX_0 = c-a$); otherwise, when $a \ge c$, X_0 lies on BC (whence $CX_0 = c$ and $BX_0 = a-c$).

By items (a) and (c), the line X_1Y_1 bisects $\angle CBA$ and, therefore, it passes through the incentre I. We next will see that the line X_0Y_0 passes through the Nagel point N. To determine N we use the points P, Q, R where the excircles meet the sides BC, CA, AB of $\triangle ABC$, whence

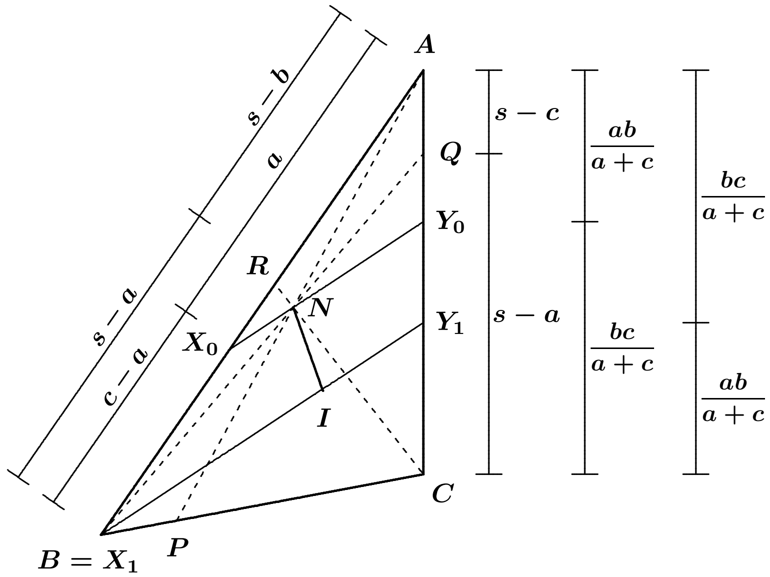
$$BR = CQ = s-a\,, \quad AR = CP = s-b\,, \quad \text{and} \quad AQ = BP = s-c\,.$$

The Nagel point is defined to be the point common to AP, BQ, and CR. Apply Menelaus' theorem to transversal NCR of $\triangle BQA$ to deduce that

$$\frac{BN}{NQ} \;=\; \frac{CA}{QC}\cdot\frac{RB}{AR} \;=\; \frac{b}{s-a}\cdot\frac{s-a}{s-b} \;=\; \frac{b}{s-b}\,. \tag{1}$$

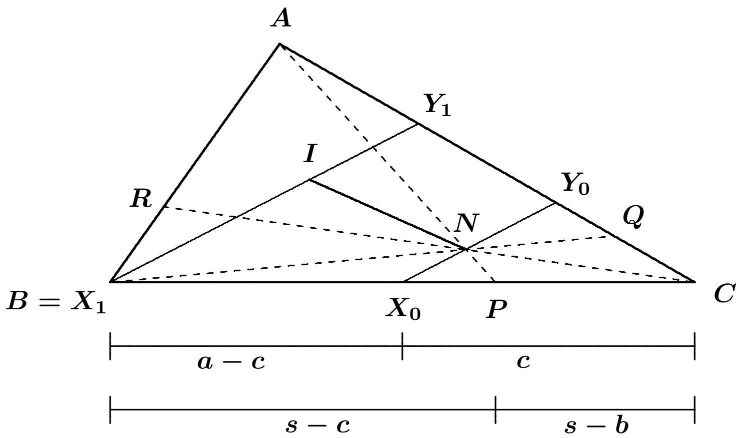
Let $N' = X_0Y_0 \cap BQ$; we want to show that $N' = N$. Here we will need the length

$$QY_0 \;=\; |AY_0 - AQ| \;=\; \left|\frac{ab}{a+c} - (s-c)\right| \;=\; \frac{|a-c|(s-b)}{a+c}\,.$$



When $c > a$ we apply Menelaus' theorem to transversal $N'Y_0X_0$ of $\triangle BQA$ to get

$$\frac{BN'}{N'Q} = \frac{Y_0A}{QY_0} \cdot \frac{X_0B}{AX_0} = \frac{ab}{a+c} \cdot \frac{a+c}{(c-a)(s-b)} \cdot \frac{c-a}{a} = \frac{b}{s-b}.$$



Otherwise, when $a > c$ we apply Menelaus' theorem to transversal $N'Y_0X_0$ of $\triangle BCQ$ to get

$$\frac{BN'}{N'Q} = \frac{Y_0C}{QY_0} \cdot \frac{X_0B}{CX_0} = \frac{bc}{a+c} \cdot \frac{a+c}{(a-c)(s-b)} \cdot \frac{a-c}{c} = \frac{b}{s-b}.$$

In both cases $\frac{BN'}{N'Q}$ equals the value of $\frac{BN}{NQ}$ in equation (1), from which we conclude that $N = N'$, and X_0Y_0 intersects IN at N, as claimed.

It remains to observe that $X_0Y_0 \parallel X_1Y_1$: when $c > a$ (that is, when X_0 lies on AB),

$$AX_0 : AX_1 \;=\; a : c \;=\; \frac{ab}{a+c} : \frac{bc}{a+c} \;=\; AY_0 : AY_1\,;$$

when $a > c$ (and X_0 lies on BC),

$$CX_0 : CX_1 \;=\; c : a \;=\; \frac{bc}{a+c} : \frac{ab}{a+c} \;=\; CY_0 : CY_1\,.$$

Finally, because X_t divides the segment X_0X_1 in the ratio $t : (1-t)$ while Y_t divides Y_0Y_1 in that same ratio, X_tY_t is parallel to both X_0Y_0 and X_1Y_1 for all t. It follows that X_tY_t meets the segment NI in a point that divides it in that same ratio $t : (1-t) = x : y$, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Geupel remarked that the Spieker centre is the midpoint of NI (on the line $X_{1/2}Y_{1/2}$) while the centroid of the triangle is a trisector of NI (on the line $X_{2/3}Y_{2/3}$).

3356. [2008 : 299, 301] *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let $f : [0,\infty) \to \mathbb{R}$ be integrable on $[0,1]$ and have period 1 (that is, $f(x+1) = f(x)$ for all $x \in [0,\infty)$). If $\{x_n\}_{n=0}^{\infty}$ is any strictly increasing, unbounded sequence with $x_0 = 0$ for which $(x_{n+1} - x_n) \to 0$, denote

$$r(n) \;=\; \max\{k \in \mathbb{N} \mid x_k \le n\}\,.$$

(a) Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{r(n)} (x_k - x_{k-1}) f(x_k) \;=\; \int_0^1 f(x)\,dx\,.$$

(b) Prove that

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{f(\ln k)}{k} \;=\; \int_0^1 f(x)\,dx\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $I = \int_0^1 f(x)\,dx$ and

$$S_n \;=\; \sum_{k=r(n)+1}^{r(n+1)} (x_k - x_{k-1}) f(x_k)\,.$$

We first prove that
$$\lim_{n\to\infty} S_n = I\,. \tag{1}$$
Consider the following partition of the interval $[n, n+1]$
$$n \;<\; x_{r(n)+1} \;<\; x_{r(n)+2} \;<\; \cdots \;<\; x_{r(n+1)} \;\le\; n+1\,,$$
where the lengths of the resulting intervals between the points are denoted by Δ_1, Δ_2, $\ldots$, Δ_m and $m = r(n+1) - r(n) + 1$. Consider the Riemann sum $R_n = \sum\limits_{k=1}^{m} f(x_k^*)\Delta_k$, where x_k^* is the right endpoint of the k^{th} interval. Because f is bounded and $(x_{k+1} - x_k) \to 0$, we obtain
$$S_n \;=\; R_n + \big(n - x_{r(n)}\big) f(x_{r(n)+1}) - \big(n + 1 - x_{r(n+1)}\big) f(n+1) \to I$$
as $n \to \infty$.

Next we prove the limit in part (a) in the following equivalent form:
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S_k \;=\; I\,. \tag{2}$$
Let $\epsilon > 0$ be given. By (1) there is an integer N such that $|S_k - I| < \frac{\epsilon}{2}$ whenever $k \ge N$. Let $S = \left|\sum\limits_{k=0}^{N-1} (S_k - I)\right|$. Whenever $n > \max\left\{N,\, \frac{2S}{\epsilon}\right\}$ we have
$$\begin{aligned}\left|\frac{1}{n}\sum_{k=0}^{n-1} S_k - I\right| \;&\le\; \frac{1}{n}\left|\sum_{k=0}^{N-1} (S_k - I)\right| \;+\; \frac{1}{n}\left|\sum_{k=N}^{n-1} (S_k - I)\right| \\ &\le\; \frac{S}{n} \;+\; \frac{n-N}{n}\cdot\frac{\epsilon}{2} \;<\; \epsilon\end{aligned}$$
and (2) is proved.

For $B > 0$, let $r(B) = \max\{k \mid x_k \le B\}$. We next prove that
$$\lim_{B\to\infty} \frac{1}{B} \sum_{k=1}^{r(B)} (x_k - x_{k-1}) f(x_k) \;=\; I\,. \tag{3}$$
Let $n = \lfloor B \rfloor$, $U_n = \sum\limits_{k=0}^{n-1} S_k$, and $V_n = \sum\limits_{k=r(n)+1}^{r(B)} (x_k - x_{k-1}) f(x_k)$. Then $\sum\limits_{k=1}^{r(B)} (x_k - x_{k-1}) f(x_k) = U_n + V_n$ and by (2) we have $\lim\limits_{n\to\infty} \frac{1}{n} U_n = I$; therefore, it suffices to prove that $\frac{1}{B}(U_n + V_n) - \frac{1}{n} U_n \to 0$ as $B \to \infty$. But this follows from the fact that V_n is bounded and the calculation
$$\frac{1}{B}(U_n + V_n) - \frac{1}{n} U_n \;=\; \frac{V_n}{B} - \left(\frac{B - \lfloor B \rfloor}{B}\right)\left(\frac{U_n}{n}\right) \to 0 - 0\cdot I \;=\; 0\,.$$

Finally, we prove the limit in part (b). Let

$$T_n = \frac{1}{\ln n}\sum_{k=1}^{n}\ln\left(1+\frac{1}{k-1}\right)f(\ln k)\,; \qquad W_n = \frac{1}{\ln n}\sum_{k=1}^{n}\frac{f(\ln k)}{k}\,.$$

We must show that $W_n \to I$. It follows from (3) with $x_k = \ln k$, $(k > 0)$ and $n = e^B$ that $T_n \to I$. Therefore, it suffices to prove $T_n - W_n \to 0$. Let $\epsilon > 0$ be given and let $|f| < C$, where C is a constant. Note that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = e$, so there is an integer K such that

$$\left|\ln\left(1+\frac{1}{k-1}\right)^k - 1\right| < \frac{\epsilon}{2C}$$

whenever $k > K$. Moreover, K can be chosen so that $H_K > H_n - \ln n$ for each n, where $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ is the n^{th} Harmonic number. Let

$$T = \left|\sum_{k=1}^{K}\frac{1}{k}\left(\ln\left(1+\frac{1}{k-1}\right)^k - 1\right)f(\ln k)\right|\,.$$

Now if $n > \max\{K,\, e^{2T/\epsilon}\}$, then

$$\begin{aligned}|T_n - W_n| &\le \frac{T}{\ln n} + \frac{1}{\ln n}\left|\sum_{k=K+1}^{n}\frac{1}{k}\left(\ln\left(1+\frac{1}{k-1}\right)^k - 1\right)f(\ln k)\right| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon\,.\end{aligned}$$

This completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France (part (a) only); MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India (part (a) only); PETER Y. WOO, Biola University, La Mirada, CA, USA (part (a) only); and the proposer. There were two incomplete solutions submitted to part (b).

The fact that $S_n \to I$ as $n \to \infty$ implies that $\frac{1}{n}\sum\limits_{k=0}^{n-1} S_k \to I$ as $n \to \infty$ is part of the theory of Cesàro sums, which some solvers quoted directly.

3357. [2008 : 172, 175] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let a be a real number such that $-1 < a \le 1$. Prove that

$$\int_0^1 \frac{x+a}{x^2+2ax+1}\ln(1-x)\,dx = \frac{1}{2}\ln^2\left(2\sin\frac{\theta}{2}\right) + \frac{\theta^2}{8} - \frac{\theta\pi}{4} + \frac{\pi^2}{24}\,,$$

where θ is the unique solution in $(0,\pi]$ of the equation $\cos\theta = -a$.

Solution by Michel Bataille, Rouen, France.

Let

$$f(a) \;=\; \int_0^1 \frac{x+a}{x^2+2ax+1}\ln(1-x)\,dx\,.$$

For $a \in (-1,1]$ and $x \in [0,1)$, we have

$$\left|\frac{x+a}{x^2+2ax+1}\ln(1-x)\right| \;\le\; \frac{|\ln(1-x)|}{1+x} \;\le\; |\ln(1-x)|\,.$$

Since $|\ln(1-x)| = -\ln(1-x)$ is integrable on $[0,1)$, we see that $f(a)$ is well-defined and continuous on $(-1,1]$. [*Ed.: It seems one must also compute* $\left|\frac{x+a}{x^2+2ax+1} - \frac{x+b}{x^2+2bx+1}\right| = \frac{|b-a|\,(1-x^2)}{(x^2+2ax+1)\,(x^2+2bx+1)}$ *and note that* $x^2+2ax+1$ *on* $[0,1)$ *is bounded below by* 1 *if* $0 \le a \le 1$ *or by* $1-a^2$ *if* $-1 < a < 0$*, and then combine this with the salient observation above.*]
For $s \in (0,1)$ we have

$$\begin{aligned}
&\int_0^{1-s} \frac{2(x+a)}{x^2+2ax+1}\ln(1-x)\,dx \\
&\quad= \int_0^{1-s} \ln(1-x)\cdot d\big[\ln\left(x^2+2ax+1\right)\big] \\
&\quad= \ln s\cdot\ln\big((1-s)^2+2a(1-s)+1\big) \;+\; \int_0^{1-s}\frac{\ln\left(x^2+2ax+1\right)}{1-x}\,dx \\
&\quad= \ln s\cdot\ln\left(s^2-(2+2a)s+2+2a\right) \\
&\qquad+ \int_u^1 \frac{\ln\left(u^2-(2+2a)u+2+2a\right)}{u}\,du \\
&\quad= \ln s\cdot\ln(2+2a)+\ln s\cdot\ln\left(1-s+\frac{s^2}{2+2a}\right) \\
&\qquad+ \int_s^1 \frac{\ln(2+2a)}{u}\,du \;+\; \int_s^1 \frac{\ln\left(1-u+\frac{u^2}{2+2a}\right)}{u}\,du \\
&\quad= \ln s\cdot\ln(2+2a)+\varepsilon(s)-\ln s\cdot\ln(2+2a) \\
&\qquad+ \int_s^1 \frac{\ln\left(1-u+\frac{u^2}{2+2a}\right)}{u}\,du\,,
\end{aligned}$$

where $\varepsilon(s) \to 0$ as $s \to 0^+$.

It follows that

$$2f(a) \;=\; \int_0^1 \frac{\ln\left(1-u+\frac{u^2}{2+2a}\right)}{u}\,du$$

or $2f(a) = g(\theta)$ where

$$g(\theta) \;=\; \int_0^1 \frac{\ln\left(1 - u + \dfrac{u^2}{4\sin^2(\theta/2)}\right)}{u}\,du\,. \qquad (1)$$

First, we consider the case $a = 1$ that is, $\theta = \pi$. Making the substitution $u = 2v$, we have

$$f(1) \;=\; \frac{1}{2}\int_0^1 \frac{\ln\left(1 - u + \dfrac{u^2}{4}\right)}{u}\,du \;=\; \int_0^{\frac{1}{2}} \frac{\ln(1-v)}{v}\,dv\,.$$

Next, integrating by parts and then using the well-known identity

$$\int_0^1 \frac{\ln(1-t)}{t}\,dt \;=\; -\int_0^1 \left(\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}\right)dt \;=\; -\sum_{n=1}^{\infty} \frac{1}{n^2}$$

we obtain

$$\begin{aligned} f(1) &= (\ln 2)^2 + \int_0^{\frac{1}{2}} \frac{\ln(v)}{1-v}\,dv \\ &= (\ln 2)^2 + \int_0^1 \frac{\ln(v)}{1-v}\,dv \;-\; \int_{\frac{1}{2}}^1 \frac{\ln(v)}{1-v}\,dv \\ &= (\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt \;-\; \int_0^{\frac{1}{2}} \frac{\ln(1-t)}{t}\,dt \\ &= (\ln 2)^2 - \frac{\pi^2}{6} - f(1)\,, \end{aligned}$$

so that $f(1) = \frac{1}{2}(\ln 2)^2 - \dfrac{\pi^2}{12}$, in agreement with the given formula.

To determine $g(\theta)$ for $\theta \in (0, \pi)$, we differentiate under the integral sign in equation (1):

$$g'(\theta) \;=\; -\frac{\cos(\theta/2)}{\sin(\theta/2)}\int_0^1 \frac{u}{u^2 - 4u\sin^2(\theta/2) + 4\sin^2(\theta/2)}\,du \qquad (2)$$

(To justify this, note that if $\beta \in \left(0, \dfrac{\pi}{2}\right)$ and $\theta \in [\beta, \pi - \beta]$, then for all $u \in [0,1]$,

$$\left|\frac{\cos(\theta/2)}{\sin(\theta/2)} \cdot \frac{u}{u^2 - 4u\sin^2(\theta/2) + 4\sin^2(\theta/2)}\right| \;\le\; \frac{u\cot(\beta/2)}{u^2 + 4\sin^2(\beta/2)(1-u)}\,,$$

and the dominating function is integrable on $[0,1]$.) [*Ed.: One may rewrite the integrand in* (1) *as the integral of its partial derivative with respect to* θ*, then change the order of integration by Fubini's Theorem by the solver's remark, then differentiate each side with respect to* θ *to obtain* (2) .]

Writing the numerator u as $\frac{1}{2}\big(2u - 4\sin^2(\theta/2)\big) + 2\sin^2(\theta/2)$, the calculation of the integral in (2) is straightforward and gives

$$g'(\theta) = 2\cdot\frac{\frac{1}{2}\cos(\theta/2)}{\sin(\theta/2)}\ln\left(2\sin\frac{\theta}{2}\right) + \frac{\theta}{2} - \frac{\pi}{2}.$$

It follows that

$$g(\theta) = \big(\ln(2\sin(\theta/2))\big)^2 + \frac{\theta^2}{4} - \frac{\pi\theta}{2} + C$$

for some constant C. This constant is $\frac{\pi^2}{12}$, easily determined using the continuity of g and the already found value of $f(1) = g(\pi)$. Finally,

$$f(a) = \frac{1}{2}g(\theta) = \frac{1}{2}\ln^2\left(2\sin\frac{\theta}{2}\right) + \frac{\theta^2}{8} - \frac{\theta\pi}{4} + \frac{\pi^2}{24}.$$

Also solved by the proposer. There was one incorrect submission.

3358. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The interior bisector of $\angle BAC$ of triangle ABC meets BC at D. Suppose that

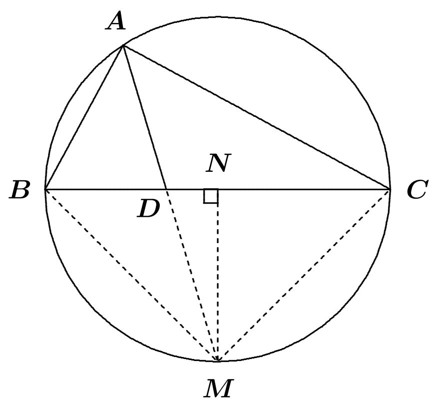
$$\frac{1}{BD^2} + \frac{1}{CD^2} = \frac{2}{AD^2}.$$

Prove that $\angle BAC = 90^\circ$.

I. *Solution by the proposer.*

Let M be the second intersection of AD with the circumcircle of $\triangle ABC$ and let N be the midpoint of BC. Since $\angle BAM = \angle MAC$, we have that $BM = MC$ and $MN \perp BC$. Since $BD \cdot CD = AD \cdot DM$, we obtain

$$\frac{2}{AD^2} = \frac{1}{BD^2} + \frac{1}{CD^2} = \frac{BD^2 + CD^2}{BD^2 \cdot CD^2} = \frac{BD^2 + CD^2}{AD^2 \cdot DM^2},$$



hence,

$$2DM^2 = BD^2 + CD^2. \qquad (1)$$

Since N is the midpoint of BC,

$$BD^2 + CD^2 = 2\left(DN^2 + BN^2\right).$$

Thus, we have from equation (1) that

$$DM^2 \;=\; DN^2 + BN^2 \,. \qquad (2)$$

Since $\angle DNM = 90^\circ$, we have $DM^2 = DN^2 + MN^2$, and comparing this with (2) yields $BN^2 = MN^2$, that is, $BN = MN$. We have now deduced that $\angle MBN = 45^\circ$.

Therefore,

$$\angle BAC \;=\; 2\angle MAC \;=\; 2\angle MBC \;=\; 2\angle MBN \;=\; 90^\circ \,.$$

II. *Solution by Oliver Geupel, Brühl, NRW, Germany.*

We shall prove the following generalization:

$$\frac{2}{AD^2} - \frac{1}{BD^2} - \frac{1}{CD^2} \;\begin{cases} < 0 & \text{if} \quad A < 90^\circ\,, \\ = 0 & \text{if} \quad A = 90^\circ\,, \\ > 0 & \text{if} \quad A > 90^\circ\,. \end{cases}$$

Let $a = BC$, $b = CA$, $c = AB$, $p = BD$, $q = CD$, and $w = AD$. It is a well-known fact that each interior angle bisector divides the opposite side in the ratio of the other two sides. Hence, $p = \dfrac{ac}{b+c}$ and $q = \dfrac{ab}{b+c}$. Another common formula is $w^2 = \dfrac{4b^2c^2\cos^2(A/2)}{(b+c)^2}$. Using these relations as well as the Law of Cosines, we derive

$$\begin{aligned}
&\frac{1}{\dfrac{1}{p^2}+\dfrac{1}{q^2}} \;-\; \frac{1}{\dfrac{2}{w^2}} \\
=\;\; & \frac{1}{\dfrac{(b+c)^2}{a^2c^2}+\dfrac{(b+c)^2}{a^2b^2}} \;-\; \frac{2b^2c^2\cos^2\dfrac{A}{2}}{(b+c)^2} \\
=\;\; & \frac{b^2c^2}{(b+c)^2}\left(\frac{a^2}{b^2+c^2} - 2\cos^2\frac{A}{2}\right) \\
=\;\; & \frac{b^2c^2}{(b+c)^2}\cdot\frac{b^2+c^2-2bc\cos A - \big(b^2+c^2\big)(1+\cos A)}{b^2+c^2} \\
=\;\; & -\frac{b^2c^2\cos A}{b^2+c^2}\,,
\end{aligned}$$

which completes the proof.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, *France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; KEE-WAI LAU, Hong Kong, China; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; NANCY MUELLER and SETH STAHLHEBER, Southeast Missouri State University, Cape Girardeau, MO, USA; K.C. SANDEEP, student, Southeast Missouri State University, Cape Girardeau, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; BIKRAM KUMAR SITOULA, student, Southeast Missouri State University, Cape Girardeau, MO, USA; SOUTHEAST MISSOURI STATE UNIVERSITY MATH CLUB; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania. There were two incorrect solutions submitted.*

3359. [2008 : 300, 302] *Proposed by Ray Killgrove, Vista, CA, USA and David Koster, University of Wisconsin, La Crosse, WI, USA.*

Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = n^2 + n + 1$. Find a subsequence $\{b_n\}_{n=1}^{\infty}$ such that $b_1 = a_1$, $b_2 = a_2$, $b_3 > a_3$, every pair of terms from this subsequence are relatively prime, and there are primes which divide no term of the subsequence.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Define $\{b_n\}$ by $b_1 = a_1 = 3$, $b_2 = a_2 = 7$, and for $n > 1$ let

$$b_{n+1} = \left(\prod_{k=1}^{n} b_k\right)^2 + \left(\prod_{k=1}^{n} b_k\right) + 1\,.$$

Then clearly $\{b_n\}$ is a subsequence of $\{a_n\}$ and $b_3 > a_3$. Since we have $b_n \equiv 1 \pmod{b_m}$ for $m < n$, we see that b_m and b_n are relatively prime. Finally, since all the b_n's are odd, it follows that the prime number 2 divides no term of $\{b_n\}$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposers.

Most solutions were similar to the one featured above.

Geupel remarks that computations of the residues of n^2+n+1 modulo p for $1 \le n < p$ show that no prime $p \in \{2, 5, 11\}$ divides any term of the sequence $\{a_n\}$.

A prime $p \neq 3, 7$ can divide at most one term b_m, $m \ge 3$, of the sequence $\{b_n\}$ in the featured solution, so by deleting at most term from that sequence the prime p can be avoided. Thus, any finite set of primes not containing 3 and 7 can be avoided by a subsequence of $\{a_n\}$ beginning with $b_1 = a_1$ and $b_2 = a_2$.

3360. [2008 : 300, 302] *Proposed by Michel Bataille, Rouen, France.*

For complex numbers a, b, and c, not all zero, let $\mathcal{N}(a,b,c)$ denote the number of solutions $(z_1,z_2,z_3) \in \mathbb{C}^3$ to the system:

$$\begin{aligned} z_1z_3 &= a\,, \\ z_1z_2 + z_2z_3 &= b\,, \\ z_1^2 + z_2^2 + z_3^2 &= c\,. \end{aligned}$$

For which a, b, and c does $\mathcal{N}(a,b,c)$ attain its minimal value?

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

It is straightforward to see that if (z_1,z_2,z_3) is a solution then

$$\begin{aligned} (z_1+z_2+z_3)^2 &= 2a+2b+c\,, \\ (z_1-z_2+z_3)^2 &= 2a-2b+c\,. \end{aligned}$$

Let A be a square root of $2a+2b+c$ and B be a square root of $2a-2b+c$. We then obtain the system of linear equations:

$$\begin{aligned} z_1+z_2+z_3 &= \pm A\,, \\ z_1-z_2+z_3 &= \pm B\,. \end{aligned}$$

Solving yields

$$\begin{aligned} (z_1+z_3,z_2) \;=\; & \left(\frac{A+B}{2},\frac{A-B}{2}\right),\left(\frac{A-B}{2},\frac{A+B}{2}\right), \\ & \left(\frac{-A+B}{2},\frac{-A-B}{2}\right),\left(\frac{-A+B}{2},\frac{-A-B}{2}\right). \end{aligned} \qquad (1)$$

The system of two equations $z_1+z_3 = \dfrac{\pm A \pm B}{2}$ and $z_1z_3 = a$ always has a solution. When such a solution is paired with the corresponding z_2 a solution to the original system is obtained. Also we note that if (z_1,z_2,z_3) is a solution to the system, then $-(z_1,z_2,z_3)$ is a different solution to the system (since $(a,b,c) \neq (0,0,0)$), thus the system has at least two solutions. We claim that this is in fact the minimal value of $\mathcal{N}(a,b,c)$.

If the terms on the right side of (1) are distinct, then $\mathcal{N}(a,b,c) \geq 4$. The only way for two of those terms to be equal is if $A = 0$ or $B = 0$.

If $A = 0$ and $B \neq 0$, then $B^2 = -4b$ and $(z_1+z_3,z_2) = \pm\left(\frac{B}{2},-\frac{B}{2}\right)$. Since $B \neq 0$, in order for the original system to have two solutions, the system $z_1+z_3 = \pm\frac{B}{2}$; $z_1z_3 = a$ must have only one solution; that is, $z_1 = z_3$, and hence $-b = \left(\pm\frac{B}{2}\right)^2 = 4a$. Therefore, $b = -4a$, and since $A = 0$ we have $c = 6a$ and $(z_1,z_2,z_3) = \pm(\alpha,-2\alpha,\alpha)$, where α is a square root of $a \neq 0$.

If $A = B = 0$, then $b = 0$, $c = -2a$, and $(z_1, z_2, z_3) = \pm(\alpha, 0, -\alpha)$, where α is a square root of $-a \neq 0$.

If $A \neq 0$ and $B = 0$, then $A^2 = 4b$ and $(z_1 + z_3, z_2) = \pm\left(\frac{A}{2}, \frac{A}{2}\right)$. Since $A \neq 0$, in order for the original system to have two solutions, the system $z_1 + z_3 = \pm\frac{A}{2}$; $z_1 z_3 = a$ must have only one solution, and hence $b = \left(\pm\frac{A}{2}\right)^2 = 4a$. Therefore, $b = 4a$, and since $B = 0$ we have $c = 6a$ and $(z_1, z_2, z_3) = \pm(\alpha, 2\alpha, \alpha)$, where α is a square root of $a \neq 0$.

In summary, the minimal value of $\mathcal{N}(a, b, c)$ is two, which is attained for triples of the form $(a, 4a, 6a)$, $(a, -4a, 6a)$, or $(a, 0, -2a)$, where a is a nonzero complex number.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ROY BARBARA, Lebanese University, Fanar, Lebanon; and TITU ZVONARU, Comănești, Romania. There was one incomplete solution submitted.

3361. [2008 : 300, 302] *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of triangle ABC meet the sides CA and AB at E and F, respectively. For which points P of the line segment EF do the areas of $\triangle EBC$, $\triangle PBC$, and $\triangle FBC$ form an arithmetic progression?

Composite of solutions by Oliver Geupel, Brühl, NRW, Germany and by Titu Zvonaru, Comănești, Romania.

Let e, f, and p be the distances from the points E, F, and P, respectively, to the line BC. The areas of the triangles EBC, PBC, and FBC form an arithmetic progression if and only if their respective altitudes e, p, and f satisfy

$$2p = e + f\,.$$

If $AB = AC$, then the lines BC and EF are parallel, whence each point P on EF satisfies the desired condition. Otherwise, only the midpoint of EF has the desired property. (This last claim follows immediately from the more familiar theorem that *if EFF' is a triangle with P on side EF, P' on side EF', and $PP' \parallel FF'$, then P is the midpoint of EF if and only if $FF' = 2PP'$*. To prove the claim from the triangle theorem, let the line through E that is parallel to BC meet at F' and P' our altitudes to BC from F and P, respectively. Then P is the midpoint of EF if and only if $FF' = 2PP'$, if and only if $(FF' + e) + e = 2PP' + 2e$, if and only if $f + e = 2p$.)

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Observe that the incircle plays almost no role in the solution—the conclusion holds for any segment EF that lies entirely on one side of the line BC. However, when E and F are the points of tangency of the incircle as in our problem, then $AF = AE$ so that the location of P for which the areas form an arithmetic progression can alternatively be described as lying on the bisector of $\angle BAC$, or as the foot of the perpendicular from A to EF.

3362. [2008 : 172,175] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Prove that

$$\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx \;<\; \frac{\pi^2}{12}\,.$$

Similar solutions by Kee-Wai Lau, Hong Kong, China; Ovidiu Furdui, Campia Turzii, Cluj, Romania; and Missouri State University Problem Solving Group, Springfield, MO, USA.

By Hölders inequality, we have

$$\begin{aligned}
\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx &< \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{1/3} \left(\int_0^1 1^{3/2}dx\right)^{2/3} \\
&= \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{1/3}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx &< \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{2/3} \left(\int_0^1 1^{3}dx\right)^{1/3} \\
&= \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{2/3}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
&\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx \\
&\quad< \int_0^1 \frac{\ln(1+x)}{x}\,dx \\
&\quad= \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n-1}}{n}\,dx \\
&\quad= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 x^{n-1}\,dx \\
&\quad= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \;=\; \frac{\pi^2}{12}\,,
\end{aligned}$$

where the integral and the sum can be interchanged since the sum converges uniformly on $[0, 1]$, and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ follows by straightforward manipulations from the well-known formula $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, and KARL HAVLAK, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Both Geupel and the proposer identified the last integral computed above as essentially a value of the dilogarithm function, one form of which is $\mathrm{Li}_2(x) = \int_x^0 \ln(1-t)/t\,dt$.