A Study of Knight's Tours on the Surface of a Cube

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1 Introduction

The classical puzzle of a knight's tour was, is, and will always be fascinating. The reason is its simplicity as well as its complexity, which have endless charm. For centuries, the traditional study of the knight's tour was mostly confined to square boards. Later, Schubert [1], Gibbins [2], and Stewart [3] delved into the knight's tour puzzle in 3-dimensional space. More recently, Kumar [4] looked into the possibilities of knight's tours in cubes and cuboids having magic properties. Awani Kumar, Francis Gaspalou, and Guenter Stertenbrink have discovered a magic knight's tour inside a $4 \times 4 \times 4$ cube. However, a lot remains to be explored and discovered even on its surface. A tour of a knight on a square board is called a *magic tour* if the sum of the numbers in each row and column is the same (the *magic constant*). Perusal of the literature reveals that knight's tours have been constructed on the surfaces of cubes smaller or larger than $4 \times 4 \times 4$ but, astonishingly, not on a $4 \times 4 \times 4$ cube itself. We know that a knight's tour is not possible on a 4×4 board, but is it possible on the surface of a $4 \times 4 \times 4$ cube? Will such tours be open or closed? Can they have magic properties? What about magic tours on the surfaces of larger cubes? We shall look at these questions.

2 Knight's tours on surfaces of small cubes

The reader can visualize the knight's move on the face of a cube in a much better way when it is unfolded, as shown in Figure 1. On a conventional board a knight can cover only up to eight squares. However, this need not be the case when it is moving on the surface of a cube or cuboid. In fact, it can cover up to 10 squares, as shown in Figure 2a. Some readers may find this strange because the consecutive knight's moves are not equidistant, as on a conventional board. Well, don't worry. It is because a cube can be unfolded along its edges in many different ways. Look at Figure 2b and the knight's move to the square labelled "6" is clear. If it is ever unclear, then appropriately unfold a cube along its edges using a pair of scissors!

A knight's tour is even possible on the smallest possible cube measuring $1 \times 1 \times 1$, as shown in Figure 3. The discerning reader will observe that here the numbers are arranged as on a conventional die. That is, the numbers on opposite faces add up to 7.

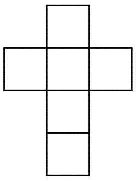

Figure 1

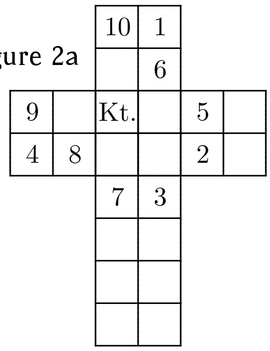

Figure 2a

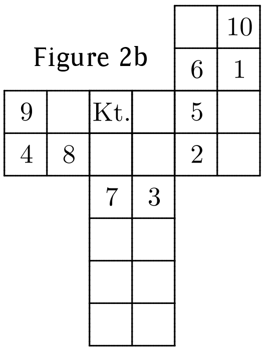

Figure 2b

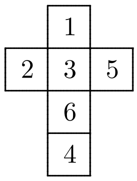

Figure 3

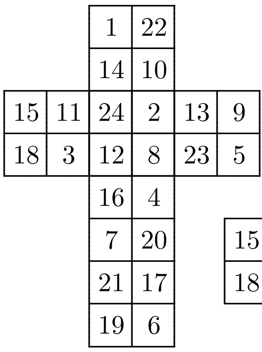

Figure 4

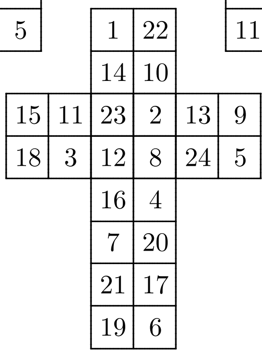

Figure 5

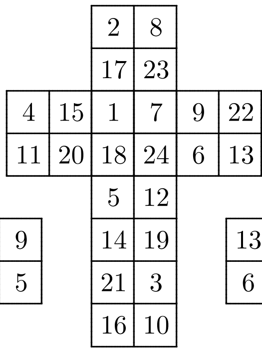

Figure 6

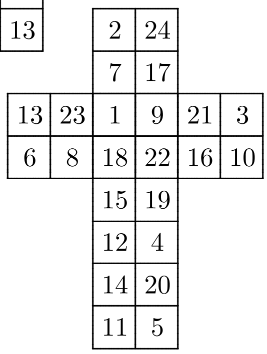

Figure 7

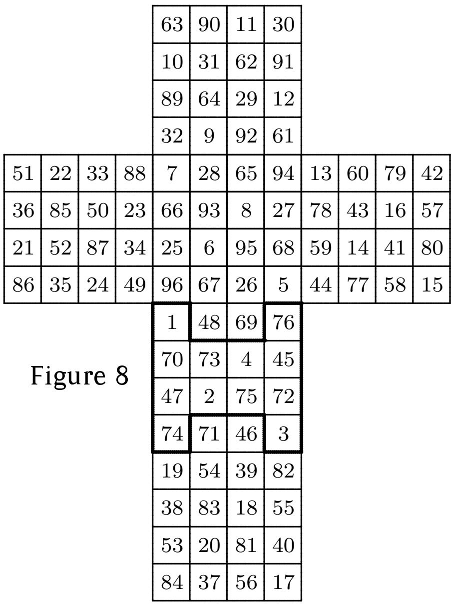

Figure 8

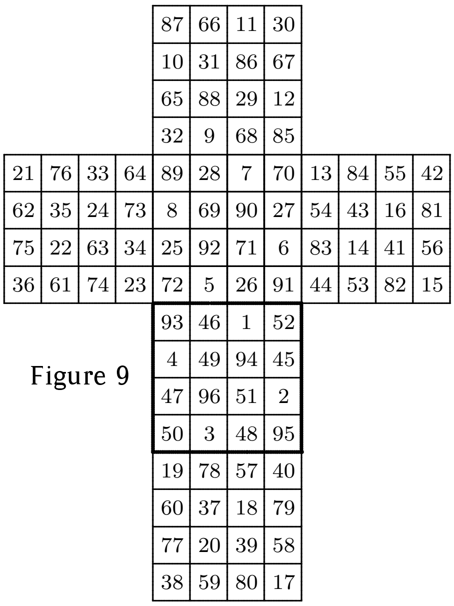

Figure 9

Watkins [5] and Pickover [6] have given a knight's tour on the surface of a $2 \times 2 \times 2$ cube. Figure 4 and Figure 5 are two examples of closed and open tours, respectively, on the surface of a $2 \times 2 \times 2$ cube. There are millions of such tours and, therefore, these are of little interest. However, tours having magic properties are a different story. Figure 6 and Figure 7 are open and closed tours, respectively, having magic properties. In these tours, the sum of the numbers on each face is 50, the *magic sum*. There are thousands of such tours. In fact, $2 \times 2 \times 2$ is the smallest cube on which such interesting tours with magic faces are possible. But how can such tours be constructed? Well, the secret lies in the systematic movement of the knight over the faces of the cube. Altogether, there are 24 squares. Put them into four groups: 1 to 6, 7 to 12, 13 to 18, and 19 to 24. Start from any face and complete the knight's tour in such a way that each face has numbers from all the groups. With a little patience and perseverance, tours with magic properties can be constructed.

Before proceeding any further, let us prove the following theorem.

Theorem For odd n, a knight's tour on the surface of an $n \times n \times n$ cube cannot have a magic sum on its faces.

Proof: Assume there is a magic sum. Altogether, there are $6n^2$ numbers, which sum up to $3n^2(6n^2+1)$. The magic sum (the sum on each face) is thus $\frac{n^2(6n^2+1)}{2}$. Since n is odd, the numerator is odd. Thus, the sum of the numbers on each face is not an integer, a contradiction. ∎

3 Knight's tours on the surface of a $4 \times 4 \times 4$ cube

Professional and amateur mathematicians have been studying the knight's tour inside a $4 \times 4 \times 4$ cube for over a century but, astonishingly, no one has looked for them on its surface! There are billions of tours on the surface of a $4 \times 4 \times 4$ cube, and consequently these are of little interest. However, tours having magic properties are, again, a different story. We have seen that the faces of a $2 \times 2 \times 2$ cube can have magic properties and this is true for tours on a $4 \times 4 \times 4$ cube too. In fact, this is true for all even-sided cubes. Later, we will see that a doubly-even cube (say, $8 \times 8 \times 8$) can have both rows and columns magic but a singly-even cube (say, $6 \times 6 \times 6$) has only the rows or the columns (but not both) magic. More interesting and challenging is the construction of tours in which all rows and columns have magic properties. Figure 8 is one such tour in which five faces are magic in rows and columns and only 4 lines (2 rows and 2 columns) are just a hair's breadth away from the magic constant. The discerning reader must have observed that these tours are made up of regular quads. Here also the numbers are arranged in four groups: 1 to 24, 25 to 48, 49 to 72, and 73 to 96. All these groups are represented in each row and each column of the faces. The same holds true for the four 2×2 squares on each of the faces. Figure 9 shows a reentrant tour. Curiously, it can be converted to a magic tour of two knights by interchanging the squares 46 and 48. One knight covers the squares 1 to 48 and the other covers 49 to 96. It is a four-fold cyclic almost magic tour. That is, it retains its almost magic properties (44 magic lines) when starting from the square 25, 49, or 73. Readers are encouraged to improve on this.

4 Knight's tours on the surface of a $6 \times 6 \times 6$ cube

The knight's tour on the surface of $6 \times 6 \times 6$ cube has gotten scantily little attention. Perusal of the literature reveals that, in spite of billions of possible tours, only Watkins [5] has given a solution found by Arden Rzewnicki and Jesse Howard. The author has enumerated 88 semimagic tours (only rows or columns are magic) on a 6×6 board and by carefully selecting and linking these tours, one can obtain hundreds of reentrant tours on the surface of a $6 \times 6 \times 6$ cube. One such example is shown in Figure 10. Since it is a closed tour, by selecting a suitable starting point (square 19 as 1, square 20 as 2, and so on), we can get a tour having up to 12 magic lines with magic constant 651. In fact, by choosing any of the six starting point squares 19, 55, 91, 127, 163, or 199, we can get tours having 12 magic lines. Jelliss [7] has proved that there cannot be magic tours on singly-even boards (that is, 6×6, 10×10, and so forth). But what about these on the surfaces of singly-even cubes? The author conjectures that they do not exist. The best the author could achieve is shown in Figure 11. All six faces there are semimagic and altogether there are 40 magic lines. So, the magic sum has been achieved up to 55%. Readers are challenged to improve on this.

5 Knight's tours on the surface of an $8 \times 8 \times 8$ cube

This is the oldest problem of the lot, over 200 years old. Dudeney [8] writes that the problem was raised by Vandermonde, an 18^{th} century musician and mathematician. Dudeney gave a solution by completing a tour on a face and then proceeding to the next face. Subsequent writers, namely Petkovic [9], Pickover [6], and Watkins [5] have merely reproduced Dudeney's solution, giving an erroneous impression it is the only possible tour. In fact, there are trillions of such tours. Using powerful computers and intelligent programs, the international team of Mackay, Meyrignac, and Stertenbrink [10] enumerated all of the 280 magic tours on the 8×8 board. By judiciously selecting and linking these tours, one can get hundreds of tours and by rearranging the knight's tour, we can get magic tours on all six faces (magic constant $= 1540$). Figure 12 is one such tour; all its rows and columns are magic. The diagonal sums are twice the magic constant on two of the faces but, in spite of intense effort, the author could not obtain any diagonally magic face or a reentrant magic tour. Readers are encouraged to look for them.

						213	190	187	204	201	196						
						186	205	212	197	188	203						
						191	214	189	202	195	200						
						206	185	198	211	182	209						
						215	192	183	208	199	194						
						184	207	216	193	210	181						
69	46	59	44	57	52	1	10	35	28	25	12	172	147	180	157	174	145
60	43	68	53	66	37	34	29	2	11	8	27	179	156	173	146	163	158
47	70	45	58	51	56	3	36	9	26	13	24	148	171	162	175	152	167
42	61	54	67	38	65	30	33	20	5	16	7	155	178	153	166	159	164
71	48	63	40	55	50	21	4	31	18	23	14	170	149	176	161	168	151
62	41	72	49	64	39	32	19	22	15	6	17	177	154	169	150	165	160
						88	93	80	95	82	105						
						73	102	89	104	79	96						
						92	87	94	81	106	83						
						101	74	103	90	97	78						
						86	91	76	99	84	107						
						75	100	85	108	77	98						
						124	109	128	137	122	111						
						129	138	123	110	127	136						
						116	125	130	139	112	121						
						131	140	117	126	135	144						
						118	115	142	133	120	113						
						141	132	119	114	143	134						

Figure 10

						123	116	101	94	125	118						
						100	95	124	117	102	93						
						115	122	99	108	119	126						
						96	107	112	121	92	103						
						111	114	105	98	109	120						
						106	97	110	113	104	91						
195	30	25	32	197	184	1	212	201	10	13	210	136	81	142	75	144	73
24	189	196	185	26	33	18	11	2	211	202	9	141	90	135	82	127	76
29	194	31	190	183	198	3	200	213	12	209	14	134	137	80	143	74	83
188	23	186	27	34	19	214	17	4	205	8	203	89	140	87	130	77	128
193	28	21	36	191	182	199	206	215	6	15	208	86	133	138	79	84	131
22	187	192	181	20	35	216	5	16	207	204	7	139	88	85	132	129	78
						180	165	38	47	178	49						
						37	52	179	50	39	46						
						164	173	166	45	48	177						
						53	44	51	172	169	40						
						174	163	42	167	176	171						
						43	54	175	170	41	168						
						160	157	162	55	60	57						
						155	68	159	58	149	62						
						158	161	156	61	56	59						
						67	154	69	148	63	150						
						70	147	152	65	72	145						
						153	66	71	146	151	64						

Figure 11

								167	170	183	220	185	222	195	198								
								182	219	168	171	194	197	186	223								
								169	166	217	184	221	188	199	196								
								218	181	172	165	200	193	224	187								
								179	216	201	208	173	164	189	210								
								204	207	180	213	192	209	174	161								
								215	178	205	202	163	176	211	190								
								206	203	214	177	212	191	162	175								
39	54	331	348	333	350	35	50	15	354	17	384	11	358	23	378	242	159	144	225	240	255	146	129
330	347	38	53	36	51	334	351	18	383	14	355	22	379	10	359	227	142	241	160	145	130	239	256
55	40	345	332	349	34	49	336	353	16	381	20	357	12	377	24	158	243	226	143	254	237	132	147
346	329	56	37	52	335	352	33	382	19	356	13	380	21	360	9	141	228	157	244	131	148	253	238
41	58	327	344	323	62	337	48	3	366	29	372	5	364	25	376	156	245	140	229	152	249	236	133
328	343	42	57	340	45	322	63	32	369	4	365	28	373	8	361	139	230	153	248	135	234	149	252
59	326	341	44	61	324	47	338	367	2	371	30	363	6	375	26	246	155	232	137	250	151	134	235
342	43	60	325	46	339	64	321	370	31	368	1	374	27	362	7	231	138	247	154	233	136	251	150
								66	83	302	317	300	315	70	87								
								303	318	67	84	69	86	299	314								
								320	65	82	301	316	297	88	71								
								81	304	319	68	85	72	313	298								
								94	291	80	305	296	311	74	89								
								79	306	93	292	73	90	295	312								
								290	95	308	77	92	293	310	75								
								307	78	289	96	309	76	91	294								
								102	283	268	117	288	97	270	115								
								267	118	101	284	269	116	287	98								
								120	103	282	265	100	285	114	271								
								281	266	119	104	113	272	99	286								
								106	121	264	277	260	127	112	273								
								263	280	105	124	109	274	257	128								
								122	107	278	261	276	259	126	111								
								279	262	123	108	125	110	275	258								

Figure 12

6 Conclusion and an Acknowledgement

We have seen that a knight's tour is possible on the surfaces of cubes of various sizes. Cubes of size $4 \times 4 \times 4$ and larger can have magic rows and magic columns on their faces.

The author has felt that in India, getting the references is more difficult than discovering a magic tour on a cube. The author is grateful to Takaya Iwamoto for providing photocopies of [2] and [3].

References

[1] H. Schubert (1904); Mathematische Mussestunden Eine Sammlung von Geduldspielen, Kunststücken und Unterhaltungs-aufgaben mathematischer Natur. (Leipzig), p. 235-7, $4\times4\times4$ closed tours, p. 238, $3\times4\times6$ closed tour. [Mathematical Association Library, Leicester University]

[2] N.M. Gibbins (1944); Chess in 3 and 4 dimensions, *Mathematical Gazette*, May 1944, pp. 46-50.

[3] J. Stewart (1971); Solid Knight's Tours, *Journal of Recreational Mathematics* Vol. 4, No. 1, January 1971, p. 1.

[4] A. Kumar (2006); Studies in Tours of the Knight in Three Dimensions, *The Games and Puzzles Journal* #43 (electronic), January-April 2006.

[5] J.J. Watkins (2005); *Across the Board, The Mathematics of Chessboard Problems*, Princeton University Press, 2005, pp. 10-11, pp. 88-89, pp. 93-94.

[6] C.A. Pickover (2004); *The Zen of Magic Squares, Circles, and Stars*, Princeton University Press, 2004, pp. 214-216.

[7] G.P. Jelliss (2003); Existence Theorems on Magic Tours, *The Games and Puzzles Journal* #25 (electronic), January-March 2003.

[8] H.E. Dudeney (1970); *Amusements in Mathematics*, Dover Publications, New York, 1970, p. 103, p. 229.

[9] M. Petkovic (1997); *Mathematics and Chess*, Dover Publications, New York, 1997, p. 53, p. 64.

[10] G. Stertenbrink (2003); Computing Magic Knight Tours, `http://magictour.free.fr/`

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