A Duality for Bicentric Quadrilaterals

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In memory of Jim Totten

Bicentric quadrilaterals are convex quadrilaterals that have both an incircle and a circumcircle. As every problem solver probably knows, such a quadrilateral is easily constructed from its future incircle. Below, we review this property that we formulate as an internal theorem about the determination of a bicentric quadrilateral. The purpose of this note is to state and prove an external theorem which is, in a sense, the dual of the internal one.

**Internal Theorem** A bicentric quadrilateral is uniquely determined by three of its sides tangent to a given circle.

This results immediately from the following well-known characterization:

Let $PQRS$ be a convex quadrilateral inscribed in a circle $\gamma$. The tangents to $\gamma$ at $P$, $Q$, $R$, and $S$ are the sidelines of a bicentric quadrilateral if and only if $QS$ is perpendicular to $PR$.

For completeness, here is a short proof, in which $\angle(\ell, \ell')$ denotes the directed angle between lines $\ell$ and $\ell'$, measured modulo $\pi$.

**Proof:** Let $I$ be the centre of $\gamma$ and $A$, $B$, $C$, and $D$ be the vertices of the quadrilateral formed by the tangents, as in the figure. Observing that $A$, $P$, $I$, and $S$ are concyclic, we have

$$\angle(AB, AD) = \angle(AP, AS) = \angle(IP, IS) = 2\angle(QP, QS).$$

Similarly,

$$\angle(CB, CD) = 2\angle(PQ, PR),$$

so that $\angle(AB, AD) = \angle(CB, CD)$ if and only if $\angle(QP, QS) = \angle(PQ, PR)$ mod $\frac{\pi}{2}$, that is, if and only if $QS \perp PR$ (since $QS$ and $PR$ are not parallel). \[\square\]

We are guided to the next theorem by considering the circumcircle to be the dual of the incircle, and recalling the duality between a tangent to a circle and its point of tangency.
**External Theorem** A bicentric quadrilateral is uniquely determined by three of its vertices lying on a given circle.

As above, the proof rests upon a characterization of the bicentric quadrilateral and will provide a construction of this quadrilateral.

We begin by showing that the conjunction of two simple properties distinguishes bicentric quadrilaterals among all quadrilaterals inscribed in a given circle. The following lemma is inspired by (and is an extension of) a problem posed in this journal (see [2], [3]).

**Lemma** Let \(ABCD\) be a convex quadrilateral inscribed in a circle \(\Gamma\) with diagonals \(AC\) and \(BD\) meeting at \(X\). Then \(ABCD\) is bicentric if and only if

(a) \(B\) and \(D\) are on the same side of the perpendicular bisector of \(AC\), and

(b) \(BX = \left(\cos^2 \frac{B}{2}\right) BD\).

**Proof:** First, we assume that \(ABCD\) is bicentric. Then,

\[
BA - BC = DA - DC \quad (1)
\]

so that \(BA - BC\) and \(DA - DC\) certainly have the same sign and (a) holds.

Let us denote area by 

\(\left[\cdot\right]\) and for simplicity let \(AB = a, BC = b, CD = c,\)

\(DA = d,\) and \(AC = e.\) Observing that

\(D = \angle ADC = \pi - B,\)

we have

\[
\frac{BX}{BD} = \frac{\left[ABX\right]}{\left[ABD\right]} = \frac{\left[CBX\right]}{\left[CBD\right]} = \frac{\left[ABC\right]}{\left[ABCD\right]} = \frac{ab \sin B}{ab \sin B + cd \sin D} = \frac{ab}{ab + cd}.
\]

Furthermore, from the Law of Cosines we deduce that

\[
\cos B = \frac{a^2 + b^2 - c^2}{2ab} = \frac{e^2 - c^2 - d^2}{2cd} = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.
\]

Squaring each side of (1), we obtain \(a^2 + b^2 - c^2 - d^2 = 2(ab - cd)\) and so

\[
\cos B = \frac{ab - cd}{ab + cd} = 2 \cdot \frac{BX}{BD} - 1 \quad \text{and (b) follows}.
\]

Conversely, if (b) holds, then \((a - b)^2 = (c - d)^2\) follows readily from the above calculations. Since \(a - b\) and \(d - c\) have the same sign by part (a), we have \(a - b = d - c\), that is, \(AB + CD = BC + DA\), which implies that \(ABCD\) has an incircle (see [1]).

For a proof of the external theorem, consider a triangle \(ABC\) with circumcircle \(\Gamma\). We are looking for a point \(D\) on \(\Gamma\) such that \(ABCD\) is bicentric. Let \(Y\) be the foot of the perpendicular from \(B\) to \(AC\), and let \(Y'\) be the point such that \(BY = \left(\cos^2 \frac{B}{2}\right) BY'\); equivalently, \(YY' = \left(\tan \frac{B}{2}\right) BY\).
The lemma implies that the desired point \( D \) must lie both on \( \Gamma \) and on the line \( m \) through \( Y' \) perpendicular to \( BY \) and on the same side as \( B \) of the perpendicular bisector \( n \) of \( AC \). It remains to show that \( m \) necessarily contains a point of \( \Gamma \). To this end, let \( n \) intersect \( AC \) at \( M \) and \( \Gamma \) at \( U \) and \( V \), where \( U \) and \( B \) are on the same arc \( AC \).

Now \( \angle AUC = B \) and \( AVCU \) is cydic, hence \( \angle AVC = \pi - B \) and we have the relations \( U M = \frac{1}{2} \left( \cot \frac{B}{2} \right) AC \) and \( VM = \frac{1}{2} \left( \tan \frac{B}{2} \right) AC \). Observing that \( BY \leq U M \), we obtain

\[
YY' \leq \left( \tan^2 \frac{B}{2} \right) \cdot \frac{1}{2} \left( \cot \frac{B}{2} \right) AC = \frac{1}{2} \left( \tan \frac{B}{2} \right) AC = VM.
\]

It follows that \( m \) lies between \( AC \) and the tangent to \( \Gamma \) at \( V \). In other words, except if \( B = U \), \( m \) intersects \( \Gamma \) in two points, one of which is the desired point \( D \). The lemma then ensures that \( ABCD \) is bicentric.

To conclude, the figure at right shows how to quickly construct \( D \) using \( K \) on the bisector \( BV \) of \( \angle ABC \) with \( KC \) perpendicular to \( BC \), since

\[
\frac{BC}{BK} = \cos \frac{B}{2} = \frac{BK}{BC} \quad \text{implies that} \quad \frac{BC}{BC'} = \cos^2 \frac{B}{2}.
\]

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References


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