A Duality for Bicentric Quadrilaterals

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In memory of Jim Totten

Bicentric quadrilaterals are convex quadrilaterals that have both an incircle and a circumcircle. As every problem solver probably knows, such a quadrilateral is easily constructed from its future incircle. Below, we review this property that we formulate as an *internal* theorem about the determination of a bicentric quadrilateral. The purpose of this note is to state and prove an *external* theorem which is, in a sense, the dual of the internal one.

Internal Theorem A bicentric quadrilateral is uniquely determined by three of its sides tangent to a given circle.

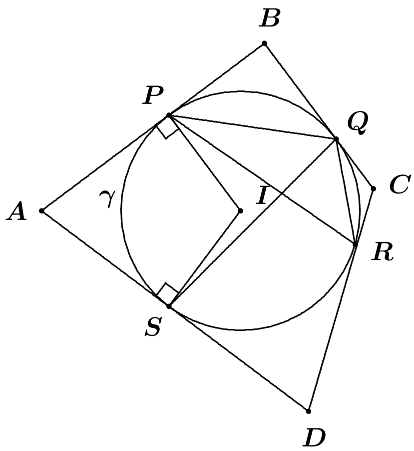
This results immediately from the following well-known characterization:

> Let $PQRS$ be a convex quadrilateral inscribed in a circle γ. The tangents to γ at P, Q, R, and S are the sidelines of a bicentric quadrilateral if and only if QS is perpendicular to PR.

For completeness, here is a short proof, in which $\angle(\ell, \ell')$ denotes the directed angle between lines ℓ and ℓ', measured modulo π.

Proof: Let I be the centre of γ and A, B, C, and D be the vertices of the quadrilateral formed by the tangents, as in the figure. Observing that A, P, I, and S are concyclic, we have

$$\angle(AB, AD) = \angle(AP, AS) = \angle(IP, IS) = 2\angle(QP, QS).$$



Similarly,

$$\angle(CB, CD) = 2\angle(PQ, PR),$$

so that $\angle(AB, AD) = \angle(CB, CD)$ if and only if $\angle(QP, QS) = \angle(PQ, PR)$ mod $\frac{\pi}{2}$, that is, if and only if $QS \perp PR$ (since QS and PR are not parallel). ∎

We are guided to the next theorem by considering the circumcircle to be the dual of the incircle, and recalling the duality between a tangent to a circle and its point of tangency.

External Theorem A bicentric quadrilateral is uniquely determined by three of its vertices lying on a given circle.

As above, the proof rests upon a characterization of the bicentric quadrilateral and will provide a construction of this quadrilateral.

We begin by showing that the conjunction of two simple properties distinguishes bicentric quadrilaterals among all quadrilaterals inscribed in a given circle. The following lemma is inspired by (and is an extension of) a problem posed in this journal (see [2], [3]).

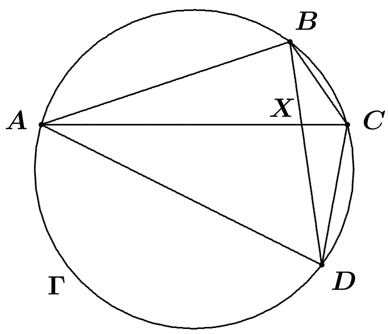
Lemma Let $ABCD$ be a convex quadrilateral inscribed in a circle Γ with diagonals AC and BD meeting at X. Then $ABCD$ is bicentric if and only if

(a) B and D are on the same side of the perpendicular bisector of AC, and

(b) $BX = \left(\cos^2\frac{B}{2}\right) BD$.

Proof: First, we assume that $ABCD$ is bicentric. Then,

$$BA - BC = DA - DC \qquad (1)$$



so that $BA - BC$ and $DA - DC$ certainly have the same sign and (a) holds.
Let us denote area by $[\cdot]$ and for simplicity let $AB = a$, $BC = b$, $CD = c$, $DA = d$, and $AC = e$. Observing that $D = \angle ADC = \pi - B$, we have

$$\frac{BX}{BD} = \frac{[ABX]}{[ABD]} = \frac{[CBX]}{[CBD]} = \frac{[ABC]}{[ABCD]} = \frac{ab\sin B}{ab\sin B + cd\sin D} = \frac{ab}{ab+cd}\,.$$

Furthermore, from the Law of Cosines we deduce that

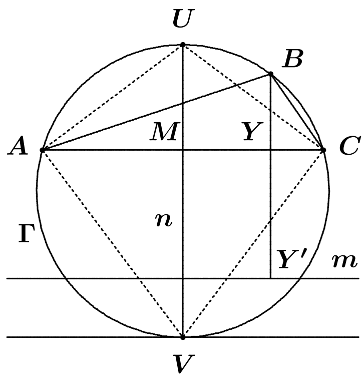
$$\cos B = \frac{a^2+b^2-e^2}{2ab} = \frac{e^2-c^2-d^2}{2cd} = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\,.$$

Squaring each side of (1), we obtain $a^2+b^2-c^2-d^2 = 2(ab-cd)$ and so $\cos B = \dfrac{ab-cd}{ab+cd} = 2\cdot\dfrac{BX}{BD} - 1$ and (b) follows.

Conversely, if (b) holds, then $(a-b)^2 = (c-d)^2$ follows readily from the above calculations. Since $a-b$ and $d-c$ have the same sign by part (a), we have $a-b = d-c$, that is, $AB + CD = BC + DA$, which implies that $ABCD$ has an incircle (see [1]). ∎

For a proof of the external theorem, consider a triangle ABC with circumcircle Γ. We are looking for a point D on Γ such that $ABCD$ is bicentric. Let Y be the foot of the perpendicular from B to AC, and let Y' be the point such that $\overrightarrow{BY} = \left(\cos^2\frac{B}{2}\right)\overrightarrow{BY'}$; equivalently, $\overrightarrow{YY'} = \left(\tan^2\frac{B}{2}\right)\overrightarrow{BY}$.

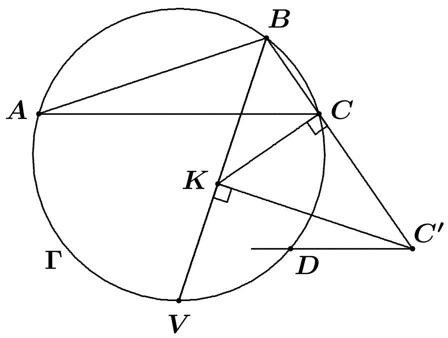
The lemma implies that the desired point D must lie both on Γ and on the line m through Y' perpendicular to BY and on the same side as B of the perpendicular bisector n of AC. It remains to show that m necessarily contains a point of Γ. To this end, let n intersect AC at M and Γ at U and V, where U and B are on the same arc AC.



Now $\angle AUC = B$ and $AVCU$ is cyclic, hence $\angle AVC = \pi - B$ and we have the relations $UM = \frac{1}{2}\left(\cot\frac{B}{2}\right)AC$ and $VM = \frac{1}{2}\left(\tan\frac{B}{2}\right)AC$. Observing that $BY \le UM$, we obtain

$$YY' \le \left(\tan^2\frac{B}{2}\right)\cdot\frac{1}{2}\left(\cot\frac{B}{2}\right)AC = \frac{1}{2}\left(\tan\frac{B}{2}\right)AC = VM\,.$$

It follows that m lies between AC and the tangent to Γ at V. In other words, except if $B = U$, m intersects Γ in two points, one of which is the desired point D. The lemma then ensures that $ABCD$ is bicentric.



To conclude, the figure at right shows how to quickly construct D using K on the bisector BV of $\angle ABC$ with KC perpendicular to BC, since $\frac{BC}{BK} = \cos\frac{B}{2} = \frac{BK}{BC'}$ implies that $\frac{BC}{BC'} = \cos^2\frac{B}{2}$.

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References

[1] N. Altshiller-Court, *College Geometry*, Dover reprint (2007), p. 135.

[2] D.J. Smeenk, Problem 2027, ***Crux Mathematicorum***, **21** (1995) p. 90; Solution in **22** (1996), p. 94.

[3] Problem 3211, ***Crux Mathematicorum with Mathematical Mayhem***, **33** (2007) p. 43; Solution in **34** (2008), p. 61.

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