THE OLYMPIAD CORNER

No. 279

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After a much needed break, Joanne Canape has agreed to tackle transforming my scribbles into LaTeX again. Welcome back!

We begin this number with problems of the 37th Austrian Mathematical Olympiad of 2006. Thanks go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for our use.

37th AUSTRIAN MATHEMATICAL OLYMPIAD
Regional Competition (Qualifying Round)
April 27, 2006

1. Let $0 < x < y$ be real numbers and

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad \text{and} \quad Q = \sqrt{\frac{x^2+y^2}{2}}$$

be the harmonic, geometric, arithmetic, and quadratic means of x and y, respectively. It is well known that $H < G < A < Q$ holds. Order the intervals $[H,G]$, $[G,A]$, and $[A,Q]$ by length.

2. Let $n > 1$ be an integer and a a real number. Determine all real solutions $(x_1, x_2, \ldots, x_n)$ of the following system of equations:

$$\begin{aligned} x_1 + ax_2 &= 0, \\ x_2 + a^2x_3 &= 0, \\ x_3 + a^3x_4 &= 0, \\ &\vdots \\ x_{n-1} + a^{n-1}x_n &= 0, \\ x_n + a^nx_1 &= 0. \end{aligned}$$

3. In a nonisosceles triangle ABC, w is the bisector of the external angle at C. The extension of AB intersects w in D. Let k_A be the circumcircle of the triangle ADC, and k_B the circumcircle of the triangle BDC. Furthermore, let t_A be the tangent of k_A at A, t_B be the tangent of k_B at B, and P be the point in which these two tangents intersect.

Assume that the two points A and B are given. Determine the set of all points $P = P(C)$, such that ABC is acute-angled but not isosceles.

4. Let $\{h_n\}_{n=1}^{\infty}$ be a harmonic sequence of positive rational numbers. In other words, each h_n is the harmonic mean of its neighbours:

$$h_n \;=\; \frac{2h_{n-1}h_{n+1}}{h_{n-1}+h_{n+1}}\,.$$

Prove that if some term h_j of the sequence is the square of a rational number, then the sequence contains an infinite number of terms h_k that are each squares of rational numbers.

National Competition (Final Round, Part 1)

May 21, 2006

1. Let k and n be integers with $k \geq 2$, $n > 10^k$, and the decimal expansion of n ending with exactly k zeros. Give the best possible lower bound (in terms of $k = k(n) \geq 2$) for the number of ways to represent n as the difference of squares of two nonnegative integers.

2. Prove that the sequence $\left\{\dfrac{(n+1)^n n^{2-n}}{7n^2+1}\right\}_{n=0}^{\infty}$ is strictly increasing.

3. The incircle of triangle ABC touches the lines BC and AC at D and E, respectively. Prove that if AD and BE are of the same length, then the triangle is isosceles.

4. Let $\lfloor u \rfloor$ denote the greatest integer less than or equal to the real number u and let $\{u\} = u - \lfloor u \rfloor$. Let $f(x) = \lfloor x^2 \rfloor + \{x\}$ for all positive real numbers x. Find an infinite arithmetic progression of distinct positive rational numbers with denominator 3 (after cancellation) which do not lie in the image of f.

National Competition (Final Round, Part 2)

Day 1 (May 31, 2006)

1. Find the number of nonnegative integers $n \leq N$ with the property that the decimal expansion of some multiple of n contains only the digits 2 and 6 (not necessarily the same number of each).

2. Prove that

$$3(a+b+c) \;\geq\; 8\sqrt[3]{abc} \;+\; \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

for all positive real numbers a, b, and c. Determine when equality holds.

3. Given triangle ABC, let point R be on the extension of AB beyond B with $BR = BC$, and let point S be on the extension of AC beyond C with $CS = CB$. Let the diagonals of $BRSC$ intersect in the point A', and construct the points B' and C' similarly. Prove that the area of the hexagon $AC'BA'CB'$ is the sum of the areas of triangles ABC and $A'B'C'$.

Day 2 (June 1, 2006)

4. Determine all rational numbers x such that $1 + 105 \cdot 2^x$ is the square of a rational number.

5. Find all monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(-f(x)) = f(f(x)) = f(x)^2 .$$

(A function f is monotonic if either $f(a) \leq f(b)$ for all $a < b$ or $f(a) \geq f(b)$ for all $a < b$.)

6. Let A be a nonzero integer. Find all integer solutions of the following system of equations:

$$\begin{aligned} x + y^2 + z^3 &= A , \\ \frac{1}{x} + \frac{1}{y^2} + \frac{1}{z^3} &= \frac{1}{A} , \\ xy^2z^3 &= A^2 . \end{aligned}$$

Next we give the Brazilian Mathematical Olympiad 2005. Thanks again go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for our use.

Brazilian Mathematical Olympiad 2005
First Day (October 22, 2005)

1. A positive integer is a *palindrome* if reversing its digits leaves it unchanged (for example, 481184, 131, and 2 are palindromes). Find all pairs (m, n) of positive integers such that $\underbrace{111\ldots1}_{m \text{ ones}} \times \underbrace{111\ldots1}_{n \text{ ones}}$ is a palindrome.

2. Determine the smallest real number C such that

$$C\left(x_1^{2005} + x_2^{2005} + \cdots + x_5^{2005}\right) \geq x_1x_2x_3x_4x_5\left(x_1^{125} + x_2^{125} + \cdots + x_5^{125}\right)^{16}$$

for all positive real numbers x_1, x_2, x_3, x_4, and x_5.

3. A square is contained in a cube if each of its points is on a face or in the interior of the cube. Determine the largest ℓ such that there exists a square of side ℓ contained in a cube of edge length 1.

Second Day (October 23, 2005)

4. We have four charged batteries, four uncharged batteries, and a radio which needs two charged batteries to work. We do not know which batteries are charged and which ones are uncharged. What is the least number of attempts that suffices to make sure the radio will work? (An attempt consists of putting two batteries in the radio and checking if the radio works or not).

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5. Let ABC be an acute triangle and let F be its Fermat point, that is, the interior point of ABC such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each of the triangles ABF, BCF, and CAF, draw its Euler line, that is, the line connecting its circumcentre and its centroid.

Prove that these three lines are concurrent.

6. Let b be an integer and let a and c be positive integers. Prove that there exists a positive integer x such that

$$a^x + x \equiv b \pmod{c} ,$$

that is, prove there exists a positive integer x such that c divides $a^x + x - b$.

Next we look at the problems of the 4^{th} Grade Croatian Mathematical Olympiad, written April 26–29, 2006. Thanks again go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for our use.

Croatian Mathematical Olympiad 2006

National Competition

4^{th} Grade

1. Prove that three tangents to a parabola always form the sides of a triangle whose altitudes intersect on the directrix of the parabola.

2. Let k and n be positive integers. Prove that

$$\left(n^4 - 1\right)\left(n^3 - n^2 + n - 1\right)^k \; + \; (n+1)n^{4k-1}$$

is divisible by $n^5 + 1$.

3. The circles Γ_1 and Γ_2 intersect at the points A and B. The tangent line to Γ_2 through the point A meets Γ_1 again at C and the tangent line to Γ_1 through A meets Γ_2 again at D. A half-line through A, interior to the angle $\angle CAD$, meets Γ_1 at M, meets Γ_2 at N, and meets the circumcircle of $\triangle ACD$ at P. Prove that $|AM| = |NP|$.

4. Six islands are connected by the Ferryboat and the Hydrofoil Boat companies. Any pair of islands is connected, in both directions, by exactly one of these two companies. Prove that it is possible to tour four of the islands using exactly one company. That is, prove that there are four islands A, B, C, and D and one company whose boats sail on the lines $A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, $D \leftrightarrow A$.

Next we give the problems of the Balkan Mathematical Olympiad 2006 written at Nicosia, Cyprus. Again we thank Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for the *Corner*.

Balkan Mathematical Olympiad 2006
Nicosia, Cyprus

1. *(Greece)* Let a, b, and c be real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}\,.$$

2. *(Greece)* Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The lines through points A, B, C and parallel to the line m intersect the circumcircle of triangle ABC again at the points A_1, B_1, C_1, respectively. Prove that the lines A_1E, B_1F, and C_1D are concurrent.

3. *(Romania)* Find all triples of positive rational numbers (m, n, p) such that each of the numbers

$$m + \frac{1}{np}\,, \qquad n + \frac{1}{pm}\,, \qquad p + \frac{1}{mn}$$

is an integer.

4. *(Bulgaria)* Let m be a fixed positive integer. For each positive integer a let the sequence $\{a_n\}_{n=0}^{\infty}$ be defined by $a_0 = a$ and for $n \geq 0$ the recursion

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even}\,, \\ a_n + m & \text{otherwise}\,. \end{cases}$$

Find all values of a such that the sequence is periodic.

As a final set of problems for this number we give the problems of the Finnish Mathematical Olympiad 2006, Final Round. Thanks again go to Robert Morewood, Canadian Team Leader at the IMO in Slovenia for collecting them.

Finnish Mathematical Olympiad 2006
Final Round (February 3, 2006)

1. Determine all pairs (x, y) of positive integers such that

$$x + y + xy = 2006 .$$

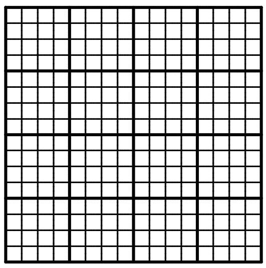
2. For all real numbers a, prove that

$$3\left(1 + a^2 + a^4\right) \geq \left(1 + a + a^2\right)^2$$

3. The numbers p, $4p^2 + 1$, and $6p^2 + 1$ are primes. Determine p.

4. Prove that if two medians of a triangle are perpendicular, then the triangle whose sides are congruent to the medians of the original triangle is a right triangle.

5. The game of *Nelipe* is played on a 16×16 grid. At each turn a player picks a number from $\{1, 2, \ldots, 16\}$ and writes it in one of the squares of the grid. At each stage of the game the numbers in each row, column, and in every one of the 16 four by four subsquares must be different. A player loses if he or she has no legal move. Which player wins, if both play with an optimal strategy?

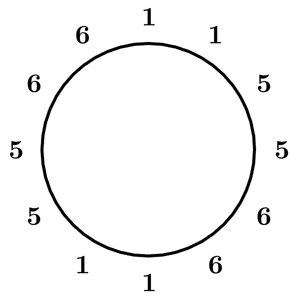


Next we turn to solutions from our readers to problems from the September 2008 number of the *Corner*, from the 19th Lithuanian Team Contest in Mathematics written October 2, 2004 [2008: 282–284].

1. Twelve numbers – four 1's, four 5's, and four 6's – are written in some order around a circle. Does there always exist a three-digit number comprised of three neighbouring numbers (its digits can be taken clockwise or counterclockwise) that is divisible by 3?

Solved by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Titu Zvonaru, Comănești, Romania. We give the response of Grant McLoughlin.

The arrangement at right shows that there is **not** always a three digit number comprised of three neighbouring numbers that is divisible by 3.



2. Solve the equation $2\cos(2\pi x) + \cos(3\pi x) = 0$.

Solved by George Apostolopoulos, Messolonghi, Greece; and George Tsapakidis, Agrinio, Greece. We give the write-up of Tsapakidis.

Using the double and triple angle formulas, the equation can be written equivalently as

$$\begin{aligned} 4\cos^3(\pi x) + 4\cos^2(\pi x) - 3\cos(\pi x) - 2 &= 0\,, \\ \big[2\cos(\pi x) + 1\big]\big[2\cos^2(\pi x) + \cos(\pi x) - 2\big] &= 0\,. \end{aligned}$$

Therefore $\cos(\pi x) = -\frac{1}{2}$ or $2\cos^2(\pi x) + \cos(\pi x) - 2 = 0$, hence, either $\cos(\pi x) = -\frac{1}{2}$ or $\cos(\pi x) = \frac{\sqrt{17}-1}{4}$. So, either $\cos(\pi x) = \cos\frac{2\pi}{3}$ or $\cos(\pi x) = \cos\theta$, where $\theta = \arccos\left(\frac{\sqrt{17}-1}{4}\right)$. Thus,

$$\pi x = 2k\pi \pm \frac{2\pi}{3} \qquad \text{or} \qquad \pi x = 2k\pi \pm \arccos\left(\frac{\sqrt{17}-1}{4}\right)\,, \quad k \in \mathbb{Z}\,,$$

and hence

$$x = 2k \pm \frac{2}{3} \qquad \text{or} \qquad x = 2k \pm \frac{1}{\pi}\arccos\left(\frac{\sqrt{17}-1}{4}\right)\,, \quad k \in \mathbb{Z}\,.$$

3. Solve the equation $3x^{\lfloor x\rfloor} = 13$, where $\lfloor x\rfloor$ denotes the integer part of the number x.

Solved by Pavlos Maragoudakis, Pireas, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write-up.

We show that the only solution is $x = \frac{\sqrt{39}}{3}$.

Since 0^0 is undefined, $x \neq 0$.

If $x < 0$, then $\lfloor x\rfloor = -n$ for some integer $n \geq 1$. Thus, $3x^{\lfloor x\rfloor} = \frac{3}{x^n}$. If n is odd, then $\frac{3}{x^n} < 0$. If n is even, then $n \geq 2$ implies $x < -1$ or $-x > 1$. Thus, $\frac{3}{x^n} = \frac{3}{(-x)^n} < 3$. In either case, $3x^{\lfloor x\rfloor} \neq 13$. Hence, it remains to consider $x > 0$.

If $x \leq 2$, then $x^{\lfloor x\rfloor} \leq 4$ and if $x \geq 3$, then $x^{\lfloor x\rfloor} \geq 27$. Hence, $2 < x < 3$. Let $x = 2 + r$ where $0 < r < 1$. Then $3x^{\lfloor x\rfloor} = 13$ becomes $3(2+r)^2 = 3\big(r^2+4r+4\big) = 13$, and we have $3r^2+12r-1 = 0$. Solving, we obtain $r = \frac{-12 \pm \sqrt{156}}{6} = -2\pm\frac{\sqrt{39}}{3}$. Since $r > 0$, we reject $r = -2-\frac{\sqrt{39}}{3}$. Hence, $r = -2 + \frac{\sqrt{39}}{3}$ and it follows that $x = \frac{\sqrt{39}}{3}$.

4. Let $0 \leq a \leq 1$ and $0 \leq b \leq 1$. Prove the inequality

$$\frac{a}{\sqrt{2b^2+5}} + \frac{b}{\sqrt{2a^2+5}} \leq \frac{2}{\sqrt{7}}\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comǎnești, Romania. We give the solution of Bataille.

Since $a^2 \leq 1$, we have $2b^2 + 5 \geq 2b^2 + 2a^2 + 3$. Similarly, we have $2a^2 + 5 \geq 2b^2 + 2a^2 + 3$ and it follows that

$$\frac{a}{\sqrt{2b^2+5}} + \frac{b}{\sqrt{2a^2+5}} \leq \frac{a+b}{\sqrt{2b^2+2a^2+3}}.$$

Therefore, it suffices to show that $\sqrt{7}(a+b) \leq 2\sqrt{2b^2+2a^2+3}$. Squaring and rearranging terms, this is equivalent to $12ab \leq (a-b)^2 + 12$, which is certainly true since $12ab \leq 12$ and $(a-b)^2 + 12 \geq 12$. The result follows. Clearly, equality holds if and only if $a = b = 1$.

5. If a, b, and c are nonzero real numbers, what values can be taken by the expression

$$\frac{a^2-b^2}{a^2+b^2} + \frac{b^2-c^2}{b^2+c^2} + \frac{c^2-a^2}{c^2+a^2}\ ?$$

Solution by Titu Zvonaru, Comǎnești, Romania.

We have $-1 < \dfrac{a^2-b^2}{a^2+b^2} < 1$, since $-2a^2 < 0 < 2b^2$.

Setting $\alpha = a^2b^2 + b^2c^2 + c^2a^2$, we obtain

$$\begin{aligned}
&\left(a^2-b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right) + \left(b^2-c^2\right)\left(a^2+b^2\right)\left(c^2+a^2\right)\\
&\qquad + \left(c^2-a^2\right)\left(a^2+b^2\right)\left(b^2+c^2\right)\\
&= \left(a^2-b^2\right)\left(c^4+\alpha\right) + \left(b^2-c^2\right)\left(a^4+\alpha\right) + \left(c^2-a^2\right)\left(b^4+\alpha\right)\\
&= \left(a^2-b^2\right)c^4 + \left(b^2-c^2\right)a^4 + \left(c^2-a^2\right)b^4\\
&= a^4b^2 - a^2b^4 - a^4c^2 + b^4c^2 + a^2c^4 - b^2c^4\\
&= a^2b^2\left(a^2-b^2\right) - c^2\left(a^4-b^4\right) + c^4\left(a^2-b^2\right)\\
&= \left(a^2-b^2\right)\left(a^2b^2 - a^2c^2 - b^2c^2 + c^4\right)\\
&= \left(a^2-b^2\right)\left[a^2\left(b^2-c^2\right) - c^2\left(b^2-c^2\right)\right]\\
&= -\left(a^2-b^2\right)\left(b^2-c^2\right)\left(c^2-a^2\right).
\end{aligned}$$

It follows that

$$\left|\frac{a^2-b^2}{a^2+b^2} + \frac{b^2-c^2}{b^2+c^2} + \frac{c^2-a^2}{c^2+a^2}\right| = \left|\frac{a^2-b^2}{a^2+b^2}\cdot\frac{b^2-c^2}{b^2+c^2}\cdot\frac{c^2-a^2}{c^2+a^2}\right| < 1.$$

Let $k \in (-1,1)$ be a real number. Then, setting $a^2 = b^2\left(\dfrac{1+k}{1-k}\right)$, we obtain

$$\lim_{c\to 0}\left(\frac{a^2-b^2}{a^2+b^2} + \frac{b^2-c^2}{b^2+c^2} + \frac{c^2-a^2}{c^2+a^2}\right) = k+1-1 = k.$$

It follows that $\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{b^2-c^2}{b^2+c^2}+\dfrac{c^2-a^2}{c^2+a^2}$ takes all real values in $(-1,1)$.

[*Ed.: It seems the solver assumes a basic fact about connectedness, namely, that the image of a connected set under a continuous, real-valued function is a connected subset of the real line, that is, an interval. Thus, the expression in a, b, and c achieves all values between any two of its given values, and since the image contains points arbitrarily close to 1 and -1 the conclusion follows.*]

6. Determine all pairs of real numbers (x,y) such that

$$\begin{aligned} x^6 &= y^4+18\,, \\ y^6 &= x^4+18\,. \end{aligned}$$

Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; George Tsapakidis, Agrinio, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give the write-up of Apostolopoulos.

Observe that $x\neq 0$ and $y\neq 0$. Since $x^6-y^4=18$ and $y^6-x^4=18$, we have

$$\begin{aligned} x^6-y^6+x^4-y^4 &= 0\,, \\ \left(x^2\right)^3-\left(y^2\right)^3+\left(x^2\right)^2-\left(y^2\right)^2 &= 0\,, \\ \left(x^2-y^2\right)\left(x^4+x^2y^2+y^4\right)+\left(x^2-y^2\right)\left(x^2+y^2\right) &= 0\,, \\ \left(x^2-y^2\right)\left(x^4+x^2y^2+y^4+x^2+y^2\right) &= 0\,. \end{aligned}$$

However, $x^4+x^2y^2+y^4+x^2+y^2>0$, hence $x^2=y^2$ and $x^6-x^4-18=0$. We put $x^2=w$, then the equation in x becomes $w^3-w^2-18=0$, or $(w-3)\left(w^2+2w+6\right)=0$, hence $w=3$ since $w^2+2w+6=x^4+2x^2+6$ is positive for all real numbers x.

Now, $x^2=w=3$, hence $x=\pm\sqrt{3}$ and there are four solutions $(x,y)=(\pm\sqrt{3},\pm\sqrt{3})$.

7. Find all triples (m,n,r) of positive integers such that

$$2001^m + 4003^n = 2002^r\,.$$

Solution by Titu Zvonaru, Cománeşti, Romania.

We will find all triples of *nonnegative* integers satisfying the equation.

Since $3^{2k}=9^k\equiv 1 \pmod 8$ and $3^{2k+1}=9^k\cdot 3\equiv 3 \pmod 8$, we have that

$$2001^m+4003^n \equiv 1+3^n \equiv 2,4 \pmod 8\ .$$

If $r\geq 3$, then $2002^n\equiv 0 \pmod 8$, hence $r\leq 2$ and we have $0\leq n<r\leq 2$.

If $r = 2$ and $n = 0$, then the equation becomes $2001^m + 1 = 2002^2$, hence $2001^m = 2002^2 - 1$, hence $2001^m = 2001 \cdot 2003$, hence we have $2001^{m-1} = 2003$ and we see that there are no solutions in this case.

If $r = 2$ and $n = 1$, then we obtain $2001^m + 4003 = 2002^2$, hence $2001^m = (2001+1)^2 - (2 \cdot 2001 + 1)$, hence $2001^m = 2001^2$ and $m = 2$.

If $r = 1$ and $n = 0$, then $2001^m + 1 = 2002$, hence $m = 1$.

Therefore, the solutions are $(m, n, r) \in \{(1, 0, 1), (2, 1, 2)\}$, of which only $(m, n, r) = (2, 1, 2)$ consists entirely of positive integers.

8. Assume that m and n are positive integers. Prove that, if $mn - 23$ is divisible by 24, then $m^3 + n^3$ is divisible by 72.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Wang's version.

By assumption, $mn = 24k + 23$, $k \in \mathbb{Z}$; hence $mn \equiv 7 \pmod 8$.

Clearly, m and n are both odd. Thus, $m \equiv 1$, 3, 5, or $7 \pmod 8$ and similarly for n. All of the solutions to $mn \equiv 7 \pmod 8$ are given by $(m, n) \equiv (1, 7)$, $(7, 1)$, $(3, 5)$, or $(5, 3) \pmod 8$, where the congruence $(m, n) \equiv (a, b) \pmod d$ means that $m \equiv a \pmod d$ and $n \equiv b \pmod d$. In the first two cases, we have $m^3 + n^3 \equiv 1 + 343 = 344 \equiv 0 \pmod 8$ and in the other two cases, we have $m^3 + n^3 \equiv 27 + 125 = 152 \equiv 0 \pmod 8$. Hence,

$$8 \mid \left(m^3 + n^3\right) . \tag{1}$$

It remains to show that

$$9 \mid \left(m^3 + n^3\right) . \tag{2}$$

Since $mn = 24k + 23$, we have $mn \equiv 2 \pmod 3$, which implies that $(m, n) \equiv (2, 1)$ or $(1, 2) \pmod 3$. Thus, $m + n \equiv 0 \pmod 3$. Since $m^3 + n^3 = (m+n)\big((m+n)^2 - 3mn\big)$, the relation (2) follows.

The conclusion now follows from (1) and (2).

9. Is it possible that, for some a, both expressions $\dfrac{1 - 2a\sqrt{35}}{a^2}$ and $a + \sqrt{35}$ are integers?

Solved by Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Geupel's write-up.

Yes, this is possible. Let $m = \dfrac{1 - 2a\sqrt{35}}{a^2}$ and $n = a + \sqrt{35}$; then $m = 1$ and $n = \pm 6$ if $a = \pm 6 - \sqrt{35}$.

We prove additionally that there are no other solutions.

If $m = 0$, then $a = \dfrac{\sqrt{35}}{70}$ and $n = \dfrac{71\sqrt{35}}{70}$ is not an integer. Hence, $|m| \geq 1$ because m is an integer, and then $|1 - 2a\sqrt{35}| \geq a^2$.

If $1 - 2a\sqrt{35} \geq a^2$, then $n^2 = \left(a + \sqrt{35}\right)^2 \leq 36$ and thus $|n| \leq 6$. Otherwise, $-\left(1 - 2a\sqrt{35}\right) \geq a^2$ and $a^2 - 2\sqrt{35}a + 1 \leq 0$. Therefore, $2\sqrt{35} - \sqrt{34} \leq n \leq 2\sqrt{35} + \sqrt{34}$, hence $7 \leq n \leq 17$. Altogether, we have $-6 \leq n \leq 17$. The 24 cases can now be eliminated using a calculator.

[*Ed.: Substitute* $a = n - \sqrt{35}$ *into* $ma^2 = 1 - 2a\sqrt{35}$ *and simplify to obtain* $m\left(n^2 + 35\right) - 71 = (m-1)2n\sqrt{35}$. *Since each side is a rational number,* $n = 0$ *or* $m = 1$, *which quickly leads to the unique solution.*]

11. What is the greatest value that a product of positive integers can take if their sum is equal to 2004?

Solved by George Apostolopoulos, Messolonghi, Greece; and Oliver Geupel, Brühl, NRW, Germany. We first give Apostolopoulos' solution.

Let $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2004$, where each α_j is a positive integer. We want to maximize the product $\alpha_1\alpha_2\cdots\alpha_n$.

If $\alpha_k \geq 4$ for some k, then replacing α_k by 2 and $\alpha_k - 2$ leaves the sum the same and can only increase the product, since $2(\alpha_k - 2) \geq a_k$ for $a_k \geq 4$. Therefore, the maximum product is achieved when each a_k is either 2 or 3. Now, $2 + 2 + 2 = 3 + 3$ but $2^3 < 3^2$, so the product is maximized by taking as many 3's in the sum as possible while leaving at most two 2's. Since $2004 = 3 \cdot 668 + 0 \cdot 2$, the largest possible product is 3^{668}.

Next we give Geupel's solution to a different reading of the problem.

We prove that the greatest value a product of two positive integers x and y can take if their sum is equal to a fixed positive integer n, is $\lfloor \frac{n^2}{4} \rfloor$. In the special case $n = 2004$ the result is $1002^2 = 1004004$. Indeed, we have $xy = x(n-x) = -\left(x - \frac{n}{2}\right)^2 + \frac{n^2}{4}$. For even n, we obtain $xy \leq \frac{n^2}{4}$, where equality holds if and only if $x = y = \frac{n}{2}$. For odd n we have $xy \leq \frac{n^2-1}{4}$, where equality holds if and only if $\{x, y\} = \left\{\frac{n-1}{2}, \frac{n+1}{2}\right\}$.

12. Positive integers a, b, c, u, v, and w satisfy the system of equations

$$\begin{aligned} a + u &= 21\,, \\ b + v &= 31\,, \\ c + w &= 667\,. \end{aligned}$$

Can abc be equal to uvw?

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write-up.

Yes. If $a = 2{\cdot}7$, $u = 7$, $b = 3{\cdot}5$, $v = 2^4$, $c = \left(2^3\right){\cdot}29$, and $w = 3{\cdot}5{\cdot}29$, then the equations are satisfied and $abc = uvw = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 29 = 48720$.

13. Let u be the real root of the equation $x^3 - 3x^2 + 5x - 17 = 0$, and let v be the real root of the equation $x^3 - 3x^2 + 5x + 11 = 0$. Find $u + v$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Díaz-Barrero.

For real numbers x and t let $f(x) = x^3 - 3x^2 + 5x$ and $g(t) = t^3 + 2t$. Then $f(x) = (x-1)^3 + 2(x-1) + 3$ and g is an odd, increasing, and bijective function, as can be easily checked. Furthermore,

$$\begin{aligned} g(u-1) &= (u-1)^3 + 2(u-1) = f(u) - 3 = 17 - 3 = 14\,, \\ g(v-1) &= (v-1)^3 + 2(v-1) = f(v) - 3 = -11 - 3 = -14\,. \end{aligned}$$

We have $g(u-1) = -g(v-1) = g(1-v)$ because g is odd, and moreover since g is bijective, we deduce that $u - 1 = 1 - v$, hence $u + v = 2$.

15. Does there exist a polynomial, $P(x)$, with integer coefficients such that for all x in the interval $\left[\frac{4}{10}, \frac{9}{10}\right]$ the inequality $|P(x) - \frac{2}{3}| < 10^{-10}$ is valid?

Solution by Oliver Geupel, Brühl, NRW, Germany.

We give such a polynomial explicitly. Let $P(x) = \dfrac{2}{3}\left[1 - \left(1 - 3x^4\right)^n\right]$ and note that $P(x)$ has integer coefficients. For $0.4 \le x \le 0.9$ we have

$$-0.9683 = 1 - 3(0.9)^4 \le 1 - 3x^4 \le 1 - 3(0.4)^4 = 0.9232\,,$$

hence $\left|1 - 3x^4\right| \le 0.9683$. Therefore, the condition $\left|P(x) - \dfrac{2}{3}\right| < 10^{-10}$ is satisfied for $0.4 < x < 0.9$ if $\dfrac{2}{3}(0.9683)^n < 10^{-10}$. It now suffices to choose $n = \left\lfloor \dfrac{1 - \log_{10} 3 - \log_{10} 2}{1 - \log_{10} 9.683} \right\rfloor + 1$. (Using a calculator, we find that $n = 59$.)

16. Does there exist a positive number a_0 such that all the members of the infinite sequence $a_0, a_1, a_2, \ldots$, defined by the recurrence formula $a_n = \sqrt{a_{n-1} + 1}$, $n \ge 1$, are rational numbers?

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove by contradiction that there is no such a_0. For each $n \ge 0$ let $a_n = \frac{p_n}{q_n}$, where p_n and q_n are coprime positive integers. From

$$\frac{p_n^2}{q_n^2} = a_n^2 = a_{n-1} + 1 = \frac{p_{n-1} + q_{n-1}}{q_{n-1}}\,,$$

it follows that

$$q_{n-1} p_n^2 = q_n^2 \left(p_{n-1} + q_{n-1}\right)\,. \qquad (1)$$

Then $q_{n-1} \mid q_n^2(p_{n-1}+q_{n-1})$ and the numbers q_{n-1} and $p_{n-1}+q_{n-1}$ are coprime, hence $q_{n-1} \mid q_n^2$.

On the other hand, (1) yields $q_n^2 \mid q_{n-1}p_n^2$ and p_n and q_n are coprime, hence $q_n^2 \mid q_{n-1}$.

It follows that $q_n^2 = q_{n-1}$ for all $n \geq 0$. We conclude that $q_n = 1$ for each n. The a_n are therefore positive integers. It is readily checked that $a_n > 1$ and $a_n > a_{n+1}$ for all $n > 0$. This contradiction completes the proof.

17. Let a, b, and c be the sides of a triangle and let x, y, and z be real numbers such that $x+y+z=0$. Prove that

$$a^2yz + b^2zx + c^2xy \;\leq\; 0\,.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the write-up of Amengual Covas.

More generally, let α, β, γ, x, y, and z be real numbers and suppose that $x+y+z=0$.

For real numbers u and v we have $(u+v)^2 \leq 2(u^2+v^2)$, hence

$$\begin{aligned}
&(\beta+\gamma)^2yz + (\gamma+\alpha)^2zx + (\alpha+\beta)^2xy \\
&\quad\leq\; 2\big[(\beta^2+\gamma^2)yz + (\gamma^2+\alpha^2)zx + (\alpha^2+\beta^2)xy\big] \\
&\quad=\; 2\left[\alpha^2x(y+z) + \beta^2y(z+x) + \gamma^2z(x+y)\right] \\
&\quad=\; -2\left(\alpha^2x^2+\beta^2y^2+\gamma^2z^2\right) \;\leq\; 0\,.
\end{aligned}$$

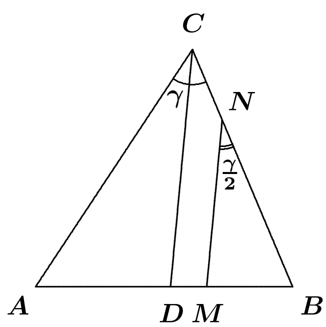
Equality occurs only if $\alpha=\beta=\gamma$ and $\alpha x=\beta y=\gamma z=0$.

To establish the proposed inequality, we make use of the linear transformation $a=\beta+\gamma$, $b=\gamma+\alpha$, and $c=\alpha+\beta$, where α, β, and γ are uniquely determined positive numbers and apply the above inequality.

Equality holds only if the triangle is equilateral and $x=y=z=0$.

18. Points M and N are on the sides AB and BC of the triangle ABC, respectively. It is given that $\frac{AM}{MB} = \frac{BN}{NC} = 2$ and $\angle ACB = 2\angle MNB$. Prove that ABC is an isosceles triangle.

Solved by George Apostolopoulos, Messolonghi, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Smeenk.

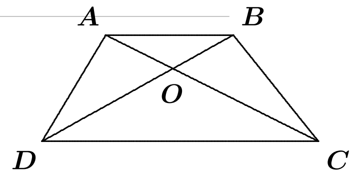


Let $\gamma = \angle ACB$, so that $\angle MNB = \frac{1}{2}\gamma$. The line through C parallel to NM meets AB at D. Then $2:3 = BN:BC = BM:BD$, hence $BD = \frac{3}{2}BM = \frac{1}{2}AB$. Since the bisector of $\angle ACB$ is also a median, $CA=CB$.

19. The two diagonals of a trapezoid divide it into four triangles. The areas of three of them are 1, 2, and 4 square units. What values can the area of the fourth triangle have?

Solved by George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Tsapakidis.

Let $[XYZ]$ be the area of triangle XYZ. Then $[ABD] = [ABC]$, as the two triangles have equal altitudes from their common base AB. Therefore,



$$[OAD] + [OAB] \;=\; [OBC] + [OAB]\,,$$

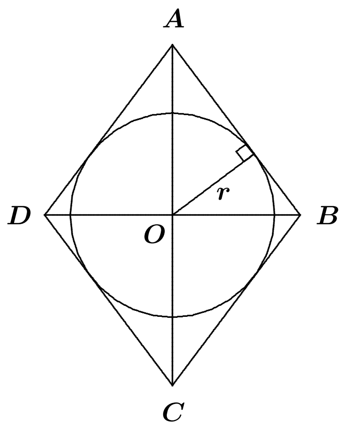
hence $[OAD] = [OBC]$.

Triangles OAB and OCD are similar, so we have $\dfrac{OA}{OC} = \dfrac{OB}{OD}$, that is $\dfrac{OA}{OC} \cdot \dfrac{OD}{OB} = 1$, and hence $\dfrac{[OAB]}{[OBC]} \cdot \dfrac{[ODC]}{[OBC]} = 1$, so $[OBC]^2 = [OAB][OBC]$. It follows that $[OBC] = 2 = [OAD]$, $[OAB] = 1$, and $[ODC] = 4$, that is, the fourth triangle has area 2 square units.

20. The ratio of the lengths of the diagonals of a rhombus is $a : b$. Find the ratio of the area of the rhombus to the area of an inscribed circle.

Solved by George Apostolopoulos, Messolonghi, Greece; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Tsapakidis.

Let r be the radius of the circle inscribed in the rhombus. The required ratio is then



$$\begin{aligned} \frac{4\cdot\frac{1}{2}OA\cdot OB}{\pi r^2} &= \frac{2}{\pi}\cdot OA\cdot OB\cdot\frac{1}{r^2} \\ &= \frac{2}{\pi}\cdot OA\cdot OB\left(\frac{1}{OA^2}+\frac{1}{OB^2}\right) \\ &= \frac{2}{\pi}\left(\frac{OB}{OA}+\frac{OA}{OB}\right) \;=\; \frac{2}{\pi}\left(\frac{b}{a}+\frac{a}{b}\right) \\ &= \frac{2(a^2+b^2)}{\pi ab}\,, \end{aligned}$$

where $\frac{1}{r^2} = \frac{1}{OA^2} + \frac{1}{OB^2}$ holds since r is the altitude of the right triangle OAB.

That completes the *Corner* for this special issue in honour of Jim Totten. The Editorial Board decided to stay with "business as usual" for the *Olympiad Corner*. Having worked with Jim over many years, I suspect that is what he would have opted for.