SKOLIAD No. 118

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Please send your solutions to problems in this Skoliad by ***1 December, 2009***. A copy of Crux will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

This special issue of ***CRUX with MAYHEM*** features a selection from a contest that Jim Totten played a pivotal role in developing. The Cariboo College High School Mathematics Contest began in 1973 and has since developed into the British Columbia Secondary School Math Contest. In 1992 a compilation of the problems was published in a book edited by Jim Totten. In the preface he wrote: "One person above all must receive our thanks for all his time spent proof-reading and offering suggestions and encouragement; John Ciriani continues to guide and inspire all of us in this department with his dedication to mathematics and teaching."

We therefore asked John Ciriani to select the problems for this Skoliad. John Ciriani replied: "It was difficult to select a contest which was memorable in terms of Jim's early contributions. In the end I chose the Junior Final 1990. This final contains a problem about golfers in a hurry to get to the course and reflects Jim's love of the game of golf."

Our thanks go to John Grant McLoughlin, University of New Brunswick, for suggesting John Ciriani and for contacting him for us; to John Ciriani, Kamloops, British Columbia, for his choice of contest and the reasons behind the choice; and to Rolland Gaudet, University College of St. Boniface, Winnipeg, MB, for the translation.

Cariboo College High School Mathematics Contest 1990

Junior Final, Part B

1. A boy on a bicycle coasts down from the top of a hill. He covers 4 metres in the first second and in each succeeding second covers 5 metres more than in the previous second. He reaches the bottom of the hill in 11 seconds.

(a) How long is the hill?

(b) What is the boy's average speed in metres per second?

(c) What distance did he cover in the last second?

2. Two golfers, on their way to the course, reached a railway crossing just as a 2.5 km train arrived. Rather than waiting, they decided to go on to the next crossing 1 km along in the direction the train was going. They travelled at 50 km/h while the train travelled at 70 km/h.

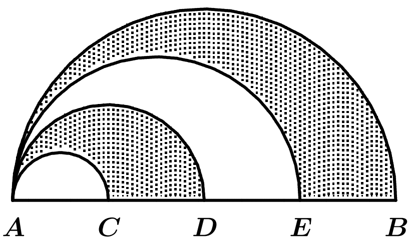
(a) How long did they have to wait for the train to clear the crossing?

(b) Rather than travelling at 50 km/h, how fast would they have had to travel to reach the crossing just as the train was clearing the crossing?

3. A student asks you to choose a number from 1 to 9 and multiply it by 109, then asks you to find the sum of the digits in the product. Knowing the sum of the digits, the student is able to tell you the number with which you began. Explain how this can be done.

4. Suppose you throw 5 darts at a round board with a radius of $25\sqrt{2}$ cm. If all 5 darts stick in the board, show that at least two of them must be within 50 cm of each other.

5. The diameter, AB, of a circle is divided into 4 equal parts by the points C, D, and E. Semicircles are drawn on AC, AD, AE, and AB as shown. Find the ratio of the area of the shaded parts to the area of the unshaded parts.



Concours mathématique du Collège Cariboo 1990
Niveau secondaire, finale junior, partie B

1. Un garçon roule sans pédaler, à partir du haut d'une colline. Il couvre 4 mètres pendant la première seconde ; chaque seconde subséquente, il couvre 5 mètres de plus que pendant la seconde précédente. Il arrive au bas de la colline après 11 secondes.

(a) Quelle est la distance totale parcourue ?

(b) Quelle est la vitesse moyenne du garçon, en mètres à la seconde ?

(c) Quelle est la distance parcourue pendant la dernière seconde ?

2. Deux golfeurs arrivent à un passage à niveau juste au moment où un train de longueur $2{,}5$ kilomètres s'y présente. Au lieu d'attendre, ils décident de se rendre au prochain passage à niveau, à une distance de 1 kilomètre du premier passage à niveau, dans la direction où le train se dirige. Ils se déplacent à 50 kilomètres à l'heure tandis que le train va à 70 kilomètres à l'heure.

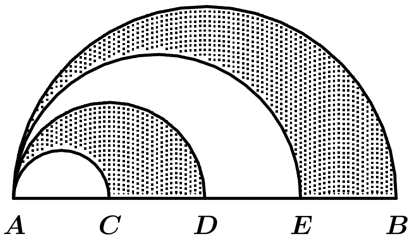
(a) Combien de temps ont-ils à attendre au deuxième passage à niveau, avant que le train ait dégagé le passage ?

(b) Au lieu de se déplacer à 50 kilomètres à l'heure, à quelle vitesse les deux golfeurs auraient-ils besoin de se déplacer afin d'arriver au deuxième passage à niveau justement au moment où le train dégage le passage ?

3. Un étudiant vous demande de choisir un entier de 1 à 9, puis de le multiplier par 109 ; il vous demande alors de lui fournir la somme des chiffres dans ce produit. Connaissant cette somme, l'étudiant peut alors vous dire avec quel entier vous avez commencé. Expliquer comment il le fait.

4. Vous lancez 5 fléchettes vers une cible de rayon $25\sqrt{2}$ centimètres. Les 5 fléchettes frappent la cible. Montrer qu'au moins deux d'entre elles doivent se retrouver à une distance d'au plus 50 centimètres l'une de l'autre.

5. Des demi cercles sont tracés avec AC, AD, AE et AB comme diamètres, où le segment AB est divisé en 4 parties égales par les points C, D et E. Pour le schéma à droite, déterminer le ratio des parties colorées aux parties non colorées.



Next follow the solutions to the British Columbia Secondary School Mathematics Contest, 2008, Junior Final, Part A [2008 : 321–324]. Note that problems 7 and 10 below have been adjusted so that the length of a segment is given as $|XY|$ instead of $\overline{XY}$.

1. Jeeves the valet was promised a salary of \$$8000$ and a car for a year of service. Jeeves left the job after 7 months of service and received the car and \$$1600$ as his correctly prorated salary. The dollar value of the car was:

(A) 6400 (B) 7200 (C) 7360 (D) 8000 (E) 15360

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let x be the cost of the car. The total value of everything that Jeeves receives after seven months equals $\frac{7}{12}$ of what he would have received had he worked for a whole year. Thus $x + 1600 = \frac{7}{12}(x + 8000)$. It follows that $12x + 19200 = 7x + 56000$, so $5x = 36800$. Hence $x = 7360$ and the answer is (C).

Also solved by JOCHEM VAN GAALEN, student, Medway High School, Arva, ON.

2. Recall that $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. The maximum value of the integer x such that 3^x divides $30!$ is:

(A) 30 (B) 14 (C) 13 (D) 10 (E) 4

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

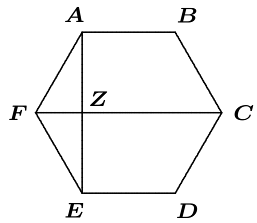
The following table lists the multiples of 3 that are less than or equal to 30 along with the number of times each is divisible by 3.

Multiple	3	6	9	12	15	18	21	24	27	30
Times	1	1	2	1	1	2	1	1	3	1

Thus $30!$ is divisible by three $1+1+2+1+1+2+1+1+3+1$ or fourteen times, and the answer is (B).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

3. In the diagram, $ABCDEF$ is a regular hexagon. Line segments AE and FC meet at Z. The ratio of the area of triangle FZE to the area of the quadrilateral $ABCZ$ is:



(A) $1:5$ (B) $1:4$ (C) $4:1$
(D) $5:1$ (E) $1:6$

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

Assume without loss of generality that the side length of the hexagon is 2.

Angles inside a regular hexagon are $120°$, so $\angle FEZ = 120°-90° = 30°$ and $\angle EFZ = \frac{1}{2}(120°) = 60°$. Thus $\triangle FZE$ is a $30°$–$60°$–$90°$ triangle with sides 1, 2, and $\sqrt{3}$, namely $|FZ| = 1$, $|EF| = 2$, and $|EZ| = \sqrt{3}$. Thus, $\triangle FZE$ has area $\frac{\sqrt{3}}{2}$.

Now draw a line segment from B to D intersecting CF at the point G. Then $\triangle CGB \cong \triangle FZE$, so the area of $\triangle CGB$ is also $\frac{\sqrt{3}}{2}$. Since $ABGZ$ is a rectangle with sides 2 and $\sqrt{3}$, it has area $2\sqrt{3}$. Hence the area of trapezoid $ABCZ$ is $\frac{\sqrt{3}}{2}+2\sqrt{3} = \frac{5\sqrt{3}}{2}$.

Therefore, the desired ratio is $\frac{\sqrt{3}}{2} : \frac{5\sqrt{3}}{2} = 1:5$, and the answer is (A).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

4. Define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. For example, $\lfloor 7 \rfloor = 7$, $\lfloor 7.2 \rfloor = 7$, and $\lfloor -5.5 \rfloor = -6$. If z is a real number that is not an integer, then the value of $\lfloor z \rfloor + \lfloor 1-z \rfloor$ is:

(A) -1 (B) 0 (C) 1 (D) 2 (E) z

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let $x = \lfloor z \rfloor$ and $\alpha = z-x$. Then x is an integer and $0 < \alpha < 1$ (since z is not an integer, $\alpha \neq 0$). Evidently, $\lfloor 1-z \rfloor = \lfloor 1-x-\alpha \rfloor = \lfloor -x+(1-\alpha) \rfloor$. Since $0 < 1-\alpha < 1$, you have that $\lfloor -x+(1-\alpha) \rfloor = -x$. Therefore, $\lfloor z \rfloor + \lfloor 1-z \rfloor = x+(-x) = 0$ and the answer is (B).

As our solver points out, the question implies that the value of $\lfloor z \rfloor + \lfloor 1-z \rfloor$ is independent of z (as long as z is not an integer). Thus you can find the answer by simply substituting some value, say 1.5, for z. Of course this does not prove *anything but relies on trust in the formulation of the question.*

5. Examinations in each of three subjects, Anatomy, Biology, and Chemistry, were taken by a group of 41 students. The following table shows how many students failed in each subject, as well as in the various combinations:

subject	A	B	C	AB	AC	BC	ABC
# failed	12	5	8	2	6	3	1

(For instance, 5 students failed in Biology, among whom there were 3 who failed both Biology and Chemistry, and just 1 of the 3 who failed all three subjects.) The number of students who passed all three subjects is:

(A) 4 (B) 16 (C) 21 (D) 24 (E) 26

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The two students who fail Anatomy and Biology include the single student who fails all three subjects. Therefore just one student fails Anatomy and Biology but not Chemistry. Likewise, five students fail Anatomy and Chemistry but not Biology, and two students fail Biology and Chemistry but not Anatomy.

The 12 students who fail Anatomy include

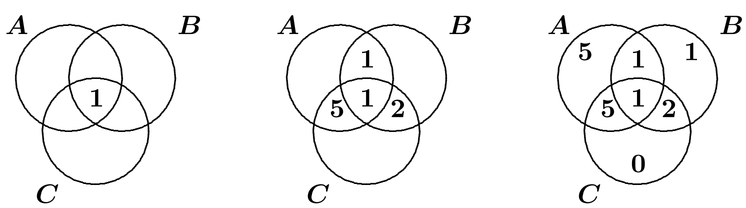
- the single student who fails Anatomy and Biology but not Chemistry; and
- the five students who fail Anatomy and Chemistry but not Biology; and
- the single student who fails all three subjects.

Therefore the number of students who fail Anatomy but pass the other two subjects is $12-1-5-1=5$. Likewise, one student fails Biology and passes the other two subjects, and zero students fail Chemistry but pass the other two subjects.

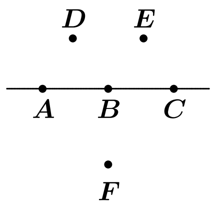
Therefore the total number of students who fail at least one subject is $5+1+0+1+5+2+1=15$, whence the number of students who pass all three subjects is $41-15=26$, and the answer is (E).

Also solved by JOCHEM VAN GAALEN, student, Medway High School, Arva, ON.

This type of question is most easily solved by means of a Venn Diagram. The diagrams below show how you may sequentially enter information into the Venn Diagram.



6. Six points, A, B, C, D, E, and F are arranged in the formation shown in the diagram, with A, B, and C on a straight line. Three of these six points are selected to form a triangle. The number of such triangles that can be formed is:



(A) 12 (B) 14 (C) 16

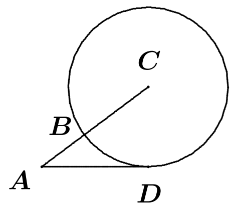
(D) 19 (E) 20

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

There are $\binom{6}{3} = {}_6C_3 = \dfrac{6!}{3!(6-3)!} = 20$ ways to choose three points from the given six. One of these, $\{A, B, C\}$, yields no triangle as A, B, C are collinear. Thus, you can make 19 triangles, and the answer is (D).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

7. In the diagram, C is the centre of the circle and AD is tangent to the circle at D. AC is a straight line. If $|AD|=10$ and $|AB|=7$, the length of BC is:



(A) $\dfrac{\sqrt{151}-7}{2}$ (B) $\sqrt{14}$ (C) $\dfrac{51}{14}$

(D) $\dfrac{\sqrt{51}}{2}$ (E) $\dfrac{7}{2}$

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since AD is tangent to the circle, $\angle CDA = 90^\circ$. Let r be the radius of the circle. Then $|CD| = |BC| = r$ and $|AC| = |AB| + |BC| = 7 + r$. By the Pythagorean Theorem, $|AD|^2 + |CD|^2 = |AC|^2$, so $10^2 + r^2 = (7+r)^2$. Thus $100 + r^2 = 49 + 14r + r^2$, whence $51 = 14r$, and $r = \dfrac{51}{14}$. Then the answer is (C).

Also solved by JOCHEM VAN GAALEN, student, Medway High School, Arva, ON.

8. When 2008^{2008} is multiplied out, the units digit in the final product is:

(A) 8 (B) 6 (C) 4 (D) 2 (E) 0

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

Since the units digit of a product depends only on the units digits of the factors, we only need to consider the units digit of powers of 8. The table shows the first of these.

Power	8^1	8^2	8^3	8^4	8^5	8^6	8^7
Units digit	8	4	2	6	8	4	2

Note that the pattern 8, 4, 2, 6 repeats itself after 8^4 as it must, since only the units digits matter.

Now $2008 \bmod 4 = 0$, that is, the remainder when dividing 2008 by 4 is 0. Thus the units digit of 2008^{2008} is 6 and the answer is (B).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

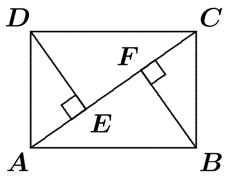
9. Recall that a prime number is an integer greater than one that is divisible only by one and itself. Consider the set of two-digit numbers less than 40 that are either prime or divisible by only one prime number. From this set select those for which the sum of the digits is a prime number, and the positive difference between the digits is another prime number. The sum of the values of the numbers selected is:

(A) 29 (B) 41 (C) 54 (D) 70 (E) 93

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The two-digit primes or powers of primes less than 40 are 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, and 37. Requiring the digit sum to be a prime narrows the list to 11, 16, 23, 25, 29, and 32. Requiring that the difference of the digits be a prime further narrows the list to 16, 25, and 29. The sum of these three numbers is 70, and the answer is (D).

10. In the diagram $ABCD$ is a rectangle with $|AD| = 1$, and both DE and BF perpendicular to the diagonal AC. Further, $|AE| = |EF| = |FC|$. The length of the side AB is:



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2

(D) $\sqrt{5}$ (E) 3

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let x be the common length of AE, EF, and FC. Then $|EC| = 2x$. By the Pythagorean Theorem, $|AE|^2 + |DE|^2 = |AD|^2$, so $x^2 + |DE|^2 = 1^2$, whence $|DE| = \sqrt{1-x^2}$.

Note that $\angle ECD = 90^\circ - \angle EDC = \angle ADE$, so $\triangle DEA \sim \triangle CED$. Therefore,

$$\frac{|EA|}{|DE|} = \frac{|ED|}{|CE|} \iff \frac{x}{\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}}{2x}\,.$$

Thus $2x^2 = 1 - x^2$, so $x^2 = \frac{1}{3}$. Finally, $|AB| = |CD|$ and CD has length $\sqrt{|DE|^2 + |CE|^2} = \sqrt{1 - x^2 + 4x^2} = \sqrt{1 + 3x^2} = \sqrt{1 + 3/3} = \sqrt{2}$, and the answer is (A).

That completes another *Skoliad*. This issue's prize for the best solutions goes to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON. We look forward to receiving more solutions from more readers.