Problem of the Month

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Have you ever looked at two things and tried to figure out which is taller? Maybe two friends or two trees? This isn’t so hard if the two things are right next to each other, but to compare that maple tree in your backyard with the the oak tree in your front yard isn’t that easy, since moving the trees is difficult, unless of course you’re in a production of “the Scottish play”. With two friends, you could get them to stand “back-to-back” and compare them, but what if the objects are not easily moveable? Some of us might be tempted to use trigonometry or some other advanced techniques.

There is another good way to do this – compare them to a “standard”. This could be your house, a long stick that you have, or maybe another tree part way in between – basically, against anything that is easy to compare to each of them.

How does this relate to mathematics?

**Problem 1** (2002 UK Intermediate Challenge). Given that \( x = \frac{111110}{111111} \), \( y = \frac{222221}{222223} \), \( z = \frac{333331}{333334} \), which of the following statements is correct?

(A) \( x < y < z \)  
(B) \( x < z < y \)  
(C) \( y < z < x \)  
(D) \( z < x < y \)  
(E) \( y < x < z \)

Now wait – no calculators allowed! What could we do? We could try some long division. We could guess wildly. We could try comparing one of these fractions to another of these and do some arithmetical manipulations.

Or, we could compare them to a common standard. Can you see a “nice” number that is close to each of \( x, y, \) and \( z \)?

**Solution.** Each of \( x, y, \) and \( z \) is close to 1, so let’s see how far each is from 1 and compare them this way:

\[
x = 1 - \frac{1}{111111}; \quad y = 1 - \frac{2}{222223}; \quad z = 1 - \frac{3}{333334}.
\]

So we’ve done an initial comparison of each of \( x, y, \) and \( z \) to 1. Can you tell which is the biggest now and which is the smallest?

Perhaps my brother (the smart one in the family!) could tell, but I’m not that quick. We’ve compared each to a common standard (that is, to 1) but now what about these differences? Again, there are lots of ways to do this, but let’s try a variation on the common standard approach.

If we wrote each of these differences as a fraction with numerator 1, we could compare them relatively easily by comparing the sizes of the denominators. Let’s try this. First, we rewrite these as

\[
x = 1 - \frac{1}{111111}; \quad y = 1 - \frac{1}{\left(\frac{222223}{2}\right)}; \quad z = 1 - \frac{1}{\left(\frac{333334}{3}\right)};
\]
and then we convert each of the denominators to a mixed fraction:

\[ x = 1 - \frac{1}{111111} ; \quad y = 1 - \frac{1}{111111\frac{1}{2}} ; \quad z = 1 - \frac{1}{111111\frac{1}{3}}. \]

Now, can you compare the denominators? With a little bit of thought, we can see that \(111111 < 111111\frac{1}{2} < 111111\frac{1}{3}\).

This means that \(\frac{1}{111111} > \frac{1}{111111\frac{1}{2}} > \frac{1}{111111\frac{1}{3}}\). So \(x\) is the furthest away from 1, since its difference with 1 is the largest. Similarly, \(y\) is the closest to 1, since its difference with 1 is the smallest. This tells us that \(x < z < y\), so answer (B) is correct.

It’s always satisfying to be able to answer this type of problem without using either a calculator or any algebra. Here’s another problem that can use this “common standard” approach.

**Problem 2** (1999 Pascal Contest). If \(w = 2^{129} \cdot 3^{81} \cdot 5^{128}, x = 2^{127} \cdot 3^{81} \cdot 5^{128}, y = 2^{126} \cdot 3^{82} \cdot 5^{128},\) and \(z = 2^{125} \cdot 3^{82} \cdot 5^{129},\) then the order from smallest to largest is

(A) \(w, x, y, z\) \quad (B) \(x, w, y, z\) \quad (C) \(x, y, z, w\)

(D) \(z, y, x, w\) \quad (E) \(x, w, z, y\)

Here, your calculator wouldn’t do you much good, as these numbers are likely way too big for your calculator to handle. So let’s again try the “common standard” technique. But what is our common standard going to be?

**Solution.** We pick a common standard, \(N\), to be the product of the smallest power of each of 2, 3, and 5 that occurs in each of the four original numbers. (Some of you may recognize \(N\) as the greatest common divisor of \(w, x, y,\) and \(z\).) Among the four numbers, the smallest power of 2 that occurs is \(2^{125},\) the smallest power of 3 that occurs is \(3^{81},\) and the smallest power of 5 that occurs is \(5^{128}.\) So we define \(N = 2^{125} \cdot 3^{81} \cdot 5^{128}\).

How do we compare \(N\) to each of \(w, x, y\) and \(z\)? Should we use subtraction again? It actually makes more sense to use multiplication (or division, depending on your perspective).

Let’s first compare \(N\) to \(w.\) Since \(N\) contains 125 factors of 2 and \(w\) contains 129 factors of 2, then we need to multiply \(N\) by \(2^4\) to get the correct number of factors of 2 for \(w.\) Since \(N\) contains 81 factors of 3 and \(w\) contains 81 factors of 3, then \(N\) already gives us the correct number of factors of 3 for \(w.\) Since \(N\) contains 128 factors of 5 and \(w\) contains 128 factors of 5, then \(N\) already gives us the correct number of factors of 5 for \(w.\) Thus, \(w = (2^{125} \cdot 3^{81} \cdot 5^{128}) \cdot 2^4 = N \cdot 2^4.\)

Similarly, \(x = N \cdot 2^2\) and \(y = N \cdot 2^1 \cdot 3^1\) and \(z = N \cdot 3^1 \cdot 5^1.\)

Put another way, \(w = 16N, x = 4N, y = 6N,\) and \(z = 15N.\) Since \(N\) is positive, \(4N < 6N < 15N < 16N,\) or \(x < y < z < w,\) so answer (C) is correct.