Problem of the Month

Ian VanderBurgh

Et tu, Brute force?

Problem (2008 Cayley Contest) The average value of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2$$

over all possible arrangements (a, b, c, d, e, f, g) of the seven numbers 1, 2, 3, 11, 12, 13, 14 is

(A) 398 (B) 400 (C) 396 (D) 392 (E) 394

We learn how to calculate averages early on in our mathematics careers – add up all of the values and divide by the number of values. This isn't so hard when you're trying to calculate the average of your six marks at school, but can be a real pain if there are significantly more values to consider.

In this problem there are $7! = 7(6)(5)(4)(3)(2)(1) = 5040$ possible arrangements of the seven numbers 1, 2, 3, 11, 12, 13, 14. We could try to calculate the 5040 required values of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2\,,$$

add them up, and divide by 5040. I think that you will agree that, in principle, we could do this calculation (the hard way!) by hand. Yes, it would take a very long time. Yes, we would be liable to make a whole bunch of arithmetic mistakes. Yes, it would be extremely annoying. But, yes, we could do it, as the underlying mathematics is not that hard. There must be a better way!

One approach would be to try to add the 5040 values without actually having to calculate the 5040 values. If we did this, we could then divide the total by 5040 and get our answer.

Put another way, we can think of the brute force approach of computing the value of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2 \qquad (*)$$

for each of the 5040 arrangements and then adding these values as "adding across then adding down". (In other words, compute each value and then add up the column of values.) But addition is commutative, that is, we don't have to add the values in the given order to get the correct total, so we could even break up the values into components and add these in separately. Let's give this a try.

Solution To determine the average value of the expression in $(*)$ we determine the sum of the values of this expression over all possible arrangements, and then divide by the number of arrangements. We determine the sum of all of the values of $(*)$ by examining the contribution of each possible term.

Let x and y be 2 of the 7 given numbers. In how many of these arrangements are x and y adjacent? Treat x and y as a single unit (xy) with 5 other numbers to be placed on either side of, but not between, xy. This gives 6 things (xy as a single unit and the 5 remaining numbers) to arrange, which can be done in $6(5)(4)(3)(2)(1)$ or $6!$ ways. But y could be followed by x, so there are $2(6!)$ arrangements with x and y adjacent, since there are the same number of arrangements with x followed by y as there are with y followed by x.

Since we want the sum of all of the possible values of $(*)$, we can calculate the total contribution of each possible term $(x-y)^2$ and add up these contributions. When we add up the values of $(*)$ over all possible arrangements, the term $(x-y)^2$ (which is equal to $(y-x)^2$) will occur $2(6!)$ times. This is true for any pair x and y. Thus, the sum of all of the possible values of $(*)$ must be equal to $2(6!)$ times the sum of all possible values of $(x-y)^2$.

The sum of all possible values of $(x-y)^2$ is

$$\begin{array}{ccccccccccccc}
1^2 & + & 2^2 & + & 10^2 & + & 11^2 & + & 12^2 & + & 13^2 & & \\
 & + & 1^2 & + & 9^2 & + & 10^2 & + & 11^2 & + & 12^2 & & \\
 & & & + & 8^2 & + & 9^2 & + & 10^2 & + & 11^2 & & \\
 & & & & & + & 1^2 & + & 2^2 & + & 3^2 & & \\
 & & & & & & & + & 1^2 & + & 2^2 & & \\
 & & & & & & & & & + & 1^2 & = & 1372\,.
\end{array}$$

Here, we have paired 1 with each of the 6 larger numbers, then 2 with each of the 5 larger numbers, and so on. We only need to pair each number with all of the larger numbers because we have accounted for the reversed pairs in our method above.

Therefore, $2(6!)$ times the sum of $(x-y)^2$ over all choices of x and y with $x<y$, divided by $7!$ is the average value. This average value is

$$\frac{2(6!)(1372)}{7!} = \frac{2(1372)}{7} = 392\,.$$

This is one of the powerful things about mathematics – being able to turn a problem that looks as if it is difficult to solve in a short period of time into one that has a reasonably quick solution. We'll see another such problem in a couple of months.

There is an interesting footnote to this problem. When creating a problem, but especially a multiple choice problem, it's not good to be able to get the right answer for the wrong reason. As the CEMC was developing this problem, the fact it was multiple choice meant fiddling the actual numbers to avoid this issue. Try redoing this problem with 1, 2, 3, 4, 5, 6, 7, 8 and a suitably modified expression. You should get the answer 84. This is also the answer you'd get by assuming that the average value of any one of the squared terms is the average of $(5-4)^2$, $(6-3)^2$, $(7-2)^2$, and $(8-1)^2$. This would be a curious wrong way to get the right answer.