THE OLYMPIAD CORNER

No. 276

R.E. Woodrow

We begin this *Corner* with the 19th Korean Mathematical Olympiad, 2006. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for our use.

The 19th Korean Mathematical Olympiad 2006
March 25–26, 2006

First Day

1. Let ABC be a triangle with $\angle B \neq \angle C$. The incircle I of a triangle ABC touches the sides BC, CA, and AB at the points D, E, and F, respectively. Let P be the point on AD and the incircle I that is different from D.

Let Q be the intersection of the line EF and the line passing through P and perpendicular to AD, and let X and Y be the intersections of the line AQ with the lines DE and DF, respectively. Show that the point A is the midpoint of XY.

2. For a positive integer a, let S_a be the set of all primes p for which there exists an odd integer b such that $\left(2^{2^a}\right)^b - 1$ is divisible by p. For any positive integer a, prove that there exist infinitely many primes that are not in S_a.

3. Three schools A, B, and C participate in a chess tournament with five students from each school. Let

$$a_1, a_2, \ldots, a_5;\ b_1, b_2, \ldots, b_5;\ c_1, c_2, \ldots, c_5$$

be the list of the players from schools A, B, and C, respectively. Let P_A, P_B, and P_C be the respective scores that schools A, B, and C receive when the tournament is over, following the rules described below. Find the remainder of the number of possible triples (P_A, P_B, P_C) when it is divided by 8.

(a) Players from each school have matches in order from the first student, and if a player loses one match, then he or she is eliminated from the tournament. The first match is between players a_1 and b_1.

(b) If y_j from school Y defeats x_i from school X, then y_j plays the next available player in the remaining school Z (different from X, Y). If all players from school Z have been eliminated, then y_j plays a match with x_{i+1}. The tournament is over when two schools are eliminated.

(c) If x_i wins a match, then school X adds 10^{i-1} points to its score.

Second Day

4. Given three distinct real numbers a_1, a_2, and a_3, define three real numbers b_1, b_2, and b_3 as follows

$$b_j \;=\; \left(1+\frac{a_j a_i}{a_j - a_i}\right)\left(1+\frac{a_j a_k}{a_j - a_k}\right), \qquad \text{where} \qquad \{i,j,k\} = \{1,2,3\}\,.$$

Prove that

$$1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \;\le\; (1+|a_1|)(1+|a_2|)(1+|a_3|)\,.$$

When does equality hold?

5. In a convex hexagon $ABCDEF$ the triangles ABC, CDE, and EFA are similar. Find conditions on these three triangles under which triangle ACE is equilateral if and only if triangle BDF is equilateral.

6. A positive integer N is said to be an n-*good number* if it has the following two properties:

(a) N is divisible by at least n distinct primes, and

(b) there exist distinct positive divisors 1, x_2, x_3, ..., x_n of N such that $1 + x_2 + \cdots + x_n = N$.

Show that there exists an n-good number for each $n \ge 6$.

Now we give the problems of the 55th Czech and Slovak Mathematical Olympiad 2006. We again thank Robert Morewood, Canadian Team Leader to the IMO in Slovenia 2006, for collecting them for the *Corner*.

55th Czech and Slovak Mathematical Olympiad 2006
Third Round, Litoměřice, March 26–29, 2006

1. (P. Novotný) A sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers is defined for $n \ge 1$ by $a_{n+1} = a_n + b_n$, where b_n is obtained from a_n by reversing its digits (the number b_n may start with zeroes). For instance if $a_1 = 170$, then $a_2 = 241$, $a_3 = 383$, $a_4 = 766$, ... Decide whether a_7 can be a prime number.

2. (J. Šimša) Let m and n be positive integers such that

$$(x+m)(x+n) \;=\; x+m+n$$

has at least one integer solution. Prove that $\frac{1}{2} < \frac{m}{n} < 2$.

3. (T. Jurík) Triangle ABC is not equilateral, and the angle bisectors at A and B intersect the sides BC and AC at K and L, respectively. Let S be the incentre, O be the circumcentre, and V be the orthocentre of triangle ABC. Prove that the following statements are equivalent

(a) The line KL is tangent to the circumcircles of triangles ALS, BVS, and BKS.

(b) The points A, B, K, L, and O are concyclic.

4. (J. Švrček) A segment AB is given in the plane. Find the locus of the centroids of all acute triangles ABC for which the following holds: the vertices A and B, the orthocentre V, and the centre S of the incircle of the triangle ABC are concyclic.

5. (M. Panák) Find all triples (p,q,r) of distinct prime numbers such that

$$p|(q+r)\,, \qquad q|(r+2p)\,, \qquad r|(p+3q)\,.$$

6. (J. Švrček, P. Calábek) Solve in real numbers the system of equations

$$\begin{aligned}\tan^2 x + 2\cot^2 2y &= 1\,,\\ \tan^2 y + 2\cot^2 2z &= 1\,,\\ \tan^2 z + 2\cot^2 2x &= 1\,.\end{aligned}$$

As a final set of problems for this number we give the Olympiade suisse de mathématiques, Tour final, 2005. Thanks again go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for obtaining them for our use.

Olympiade suisse de mathématiques 2005
Tour final

1. Soit ABC un triangle et D, E, F les milieux des côtés BC, CA, AB respectivements. Les médianes AD, BE et CF se coupent en S, le centre de gravité. Au moins deux des quadrilatères

$$AFSE\,, \qquad BDSF\,, \qquad CESD$$

sont des quadrilatères inscrits. Montrer que le triangle ABC est équilatéral.

2. Soient $4n$ points alignés tels que $2n$ d'entre eux sont blancs et $2n$ sont noirs. Montrer qu'il y a parmi eux une suite de $2n$ points consécutifs avec n points noirs et n points blancs.

3. Pour tout $a_1, \ldots, a_n > 0$, prouver l'inégalité suivante et déterminer tous les cas d'égalité

$$\sum_{k=1}^{n} k a_k \;\leq\; \binom{n}{2} \;+\; \sum_{k=1}^{n} a_k^k\,.$$

4. Déterminer tous les ensembles M de nombres naturels ayant la propriété suivante : Si a et b (non nécessairement distincts) sont des éléments de M, alors $\dfrac{a+b}{\gcd(a,b)}$ se trouve également dans M.

5. "Tailler" un n-gone convexe consiste à choisir deux côtés adjacents AB et BC et à les remplacer par les segments AM, MN, NC, où $M \in AB$ et $N \in BC$ sont des points arbitraires à l'intérieur des segments. Autrement dit, on coup un sommet et on obtient un $(n+1)$-gone.

On part d'un hexagone régulier P_6 d'aire 1 et on le taille pour obtenir une suite de polygones convexes P_6, P_7, P_8, ... Montrer que l'aire de P_n est plus grand que $\frac{1}{2}$ pour tout $n \geq 6$, indépendamment de la façon dont on a taillé.

6. Soient a, b, c des nombres réels positifs avec $abc = 1$. Déterminer toutes les valeurs que peut prendre la somme

$$\frac{1+a}{1+a+ab} \;+\; \frac{1+b}{1+b+bc} \;+\; \frac{1+c}{1+c+ca}\,.$$

7. Soit $n \geq 1$ un nombre naturel. Déterminer toutes les solutions entières positives de l'équation

$$7 \cdot 4^n \;=\; a^2 + b^2 + c^2 + d^2\,.$$

8. Soit ABC un triangle acutangle. Soient M et N des points arbitraires sur les côtés AB et AC respectivement. Les cercles de diamètre BN et CM se coupent en P et Q. Montrer que les points P, Q et l'orthocentre du triangle ABC se trouvent sur une même droite.

9. Trouver toutes les fonctions $f : \mathbb{R}^+ \to \mathbb{R}^+$, vérifiant la condition suivante pour tout $x, y > 0$:

$$f\big(yf(x)\big)(x+y) \;=\; x^2\big(f(x)+f(y)\big)\,.$$

10. Un tournoi de football rassemble n ($n > 10$) équipes. Chaque équipe joue une seule fois contre toutes les autres. Une victoire vaut 2 points, un nul rapporte 1 point et une défaite aucun point. Le tournoi terminé, on constate que chaque équipe a gagné la moitié de ses points contre les 10 plus mauvaises équipes (en particulier chacune de ces dernières a fait la moitié de ses points contre les 9 autres). Trouver toutes les valeurs possibles pour n et, pour chacune d'elles, donner un exemple d'un tel tournoi.

Next we give a solution to a problem of the XXXI Russian Olympiad which was given at [2008 : 20] and for which one other problem was solved in the December Corner at [2008 : 466-467].

1. *(I. Rubanov)* Let $\{a_1, a_2, \ldots, a_{50}, b_1, b_2, \ldots, b_{50}\}$ be a set of 100 real numbers. Suppose that the equation

$$|x-a_1|+\cdots+|x-a_{50}| \;=\; |x-b_1|+\cdots+|x-b_{50}|$$

has N solutions (N is finite). Find the maximal value of N.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Let

$$f(x) \;=\; |x-a_1|+\cdots+|x-a_{50}|-|x-b_1|-\cdots-|x-b_{50}|$$

and $\{c_1, \ldots, c_{100}\} = \{a_1, \ldots, a_{50}, b_1, \ldots, b_{50}\}$ with $c_1 < \cdots < c_{100}$. Then we have $f(x) = \epsilon_1|x-c_1| + \epsilon_2|x-c_2| + \cdots + \epsilon_{100}|x-c_{100}|$, where $\epsilon_i = 1$ if $c_i = a_j$ for some j and $\epsilon_i = -1$ if $c_i = b_k$ for some k. Thus, $\epsilon_1+\epsilon_2+\cdots+\epsilon_{100} = 0$, and we have

$$\begin{aligned} f(c_{k+1})-f(c_k) &= \epsilon_1(c_{k+1}-c_k)+\cdots+\epsilon_k(c_{k+1}-c_k) \\ &\quad - \epsilon_{k+1}(c_{k+1}-c_k)-\cdots-\epsilon_{100}(c_{k+1}-c_k) \\ &= (\epsilon_1+\cdots+\epsilon_k-\epsilon_{k+1}-\cdots-\epsilon_{100})(c_{k+1}-c_k) \\ &= 2(\epsilon_1+\cdots+\epsilon_k)(c_{k+1}-c_k) \end{aligned}$$

For $x \le c_1$ we have $f(x) = (a_1+\cdots+a_{50})-(b_1+\cdots+b_{50})$ while for $x \ge c_{100}$, we have$f(x) = (b_1+\cdots+b_{50})-(a_1+\cdots+a_{50})$ So f is constant on each of the intervals $(-\infty, c_1]$ and $[c_{100}, +\infty)$. Since N is finite, $f(x)$ is nonzero in these intervals with opposite sign. Without loss of generality, we suppose that $f(x) > 0$ for $x \le c_1$. The graph of f in $[c_1, c_{100}]$ consists of line segments connecting the points $(c_1, f(c_1))$, $(c_2, f(c_2))$, $\ldots$, $(c_{100}, f(c_{100}))$, so f has at most one root in each interval $[c_1, c_2]$, $\ldots$, $[c_{99}, c_{100}]$. The function f cannot have a root in each of the intervals $[c_k, c_{k+1}]$ and $[c_{k+1}, c_{k+2}]$. Otherwise, $f(c_{k+1})$ will have a different sign from $f(c_k)$ and $f(c_{k+2})$. If for example $f(c_k) > 0$, $f(c_{k+2}) > 0$, and $f(c_{k+1}) < 0$ then

$$\begin{aligned} f(c_{k+1})-f(c_k) &= 2(\epsilon_1+\cdots+\epsilon_k)(c_{k+1}-c_k) \;<\; 0\,; \\ f(c_{k+2})-f(c_{k+1}) &= 2(\epsilon_1+\cdots+\epsilon_k+\epsilon_{k+1})(c_{k+2}-c_k) \;>\; 0\,, \end{aligned}$$

hence the numbers $\epsilon_1+\cdots+\epsilon_k$ and $\epsilon_1+\cdots+\epsilon_k+\epsilon_{k+1}$ have opposite sign. However, this is impossible, since they are both integers differing by ± 1.

Therefore, f has at most 50 roots. We will prove that 50 roots for f is impossible. Suppose on the contrary that f has 50 roots. In that case f will have the following "sign diagram"

x		c_1		c_2		c_3		c_4		c_5		$\cdots$		c_{97}		c_{98}		c_{99}		c_{100}	
$f(x)$	$+$		$\downarrow$		$-$		$\uparrow$		$+$			$\cdots$			$\downarrow$		$-$		$\uparrow$		$+$

where f is constant in each interval where it is positive or negative. Thus, if f has 50 roots, then f has the same sign in $(-\infty, c_1]$ and $[c_{100}, +\infty)$, a contradiction. An example of f with 49 roots is

$$\begin{aligned} f(x) &= \big(|x-1| - |x-2| - |x-3| + |x-4|\big) \\ &\quad + \big(|x-5| - |x-6| - |x-7| + |x-8|\big) \\ &\quad + \cdots + \big(|x-97| - |x-98| - |x-99| + |x-99.5|\big)\,, \end{aligned}$$

with the corresponding values

$$\begin{array}{llll} f(1) = -0.5\,, & f(2) = 1.5\,, & f(3) = 1.5\,, & f(4) = -0.5\,, \\ f(5) = -0.5\,, & f(6) = 1.5\,, & f(7) = 1.5\,, & f(8) = -0.4\,, \\ \ldots & & & \\ f(97) = -0.5\,, & f(98) = 1.5\,, & f(99) = 1.5\,, & f(99.5) = 0.5\,, \end{array}$$

so that f has 49 roots, one in each of the intervals $[1,2]$, $[3,4]$, $\ldots$, $[97,98]$.

Therefore, the maximal value of N is 49.

Next we give readers' solutions to the Hungarian National Olympiad 2004-2005, Specialized Mathematical Classes, First and Final Rounds, given at [2008 : 147]

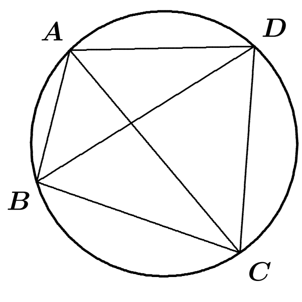
First Round

1. The quadrilateral $ABCD$ is cyclic. Prove that

$$\frac{AC}{BD} = \frac{DA \cdot AB + BC \cdot CD}{AB \cdot BC + CD \cdot DA}\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's write up.

This is Ptolemy's theorem about cyclic quadrilaterals. Here is a proof:



We denote by $[XY\cdots Z]$ the area of the polygon $XY\cdots Z$. Using the known fact that $[ABC] = (AB \cdot BC \cdot CA)/4R$, where R is the circumradius, we have the two equations

$$\begin{aligned} [ABCD] &= [ABC] + [ADC] \\ &= \frac{AB \cdot BC \cdot CA}{4R} + \frac{AD \cdot DC \cdot AC}{4R}\,; \end{aligned}$$

$$[ABCD] = [ABD] + [BCD] = \frac{AB \cdot BD \cdot AD}{4R} + \frac{BC \cdot CD \cdot BD}{4R}\,.$$

It follows that

$$AC(AB \cdot BC + CD \cdot DA) = BD(DA \cdot AB + BC \cdot CD)\,.$$

2. How many real numbers x are there in the interval $0 < x < 2004$ such that $x + \lfloor x^2 \rfloor = x^2 + \lfloor x \rfloor$? (Here $\lfloor c \rfloor$ denotes the greatest integer k such that $k \le c$.)

Solved by Oliver Geupel, Brühl, NRW, Germany; and Pavlos Maragoudakis, Pireas, Greece. We give Geupel's solution, modified by the editor.

We will count the number of roots of $f(x) = x - \lfloor x \rfloor + \lfloor x^2 \rfloor - x^2$ in each of the intervals $I_0 = (0,1)$ and $I_n = \big[\sqrt{n}, \sqrt{n+1}\big)$ for integers n with $1 \le n < 2004^2$.

For each $x \in I_0$ we have $f(x) = x - x^2 > 0$, so f has no roots in I_0.

Let n be a perfect square, say $n = N^2$ for a positive integer N. Then $f(x) = x - N + N^2 - x^2$ on I_n, the function f is decreasing on I_n, and $f\big(\sqrt{n}\big) = 0$. Hence, in this case, f has exactly one root in I_n.

Now let $n+1$ be a perfect square, say $n + 1 = N^2$ for an integer $N > 1$. If $x \in I_n$, then $x^2 \in [n, n+1) = \big[N^2 - 1, N\big)$ and also it follows that $x \in \big[\sqrt{n}, \sqrt{n+1}\big) \subset [N-1, N)$. Thus,

$$f(x) \;=\; x - (N-1) + (N^2 - 1) - x^2 \;=\; N^2 - N + x - x^2$$

for $x \in I_n$. Since $x - x^2$ is decreasing for $x > 1$ and $1 < x < \sqrt{n+1}$ for any $x \in I_n$, it follows that $f(x) > N^2 - N + \sqrt{n+1} - (n+1) = 0$ for any $x \in I_n$. Therefore, in this case, f has no root in I_n.

Finally, assume that neither n nor $n+1$ are perfect squares, that is, assume that $N^2 < n < n+1 < (N+1)^2$. Then $f(x) = n - N + x - x^2$ for $x \in I_n$ and we see that f is decreasing on I_n, so f has at most one root in I_n. At the left endpoint of I_n the function takes the value

$$f\big(\sqrt{n}\big) \;=\; n - N + \sqrt{n} - n \;=\; \sqrt{n} - N \;>\; 0\,.$$

The limit of $f(x)$ from the left towards the right boundary of I_n is negative:

$$\begin{aligned} \lim_{x \to (\sqrt{n+1})^-} f(x) &= \sqrt{n+1} - N + n - (n+1) \\ &= \sqrt{n+1} - (N+1) \;<\; 0\,. \end{aligned}$$

Hence, there is an $x_0 \in I_n$ such that $f(x_0) < 0$. Since f is continuous on $\big[\sqrt{n}, x_0\big]$, by the Intermediate Value Theorem f has at least one root in I_n. Therefore, in this case, f has exactly one root in I_n.

In summary, we have 2004^2 intervals I_0, I_1, $\ldots$, I_{2004^2-1}. The 44 intervals I_{N^2-1}, where $N = 1, 2, \ldots, 44$, contain no root of f. The other intervals each contain just one root. Consequently, the number of solutions to the original equation is $2004^2 - 44 = 4015972$.

3. Let $s(n)$ be the sum of those positive divisors of n that are less than n. A triple of three integers, (a, b, c), is a *friendly* triple if $1 < a \le b \le c$ and $s(a) + s(b) = c$, $s(b) + s(c) = a$, and $s(c) + s(a) = b$. Determine all friendly triples (a, b, c) where c is even.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Let (a, b, c) be a friendly triple with c even. If $c > 2$, then $\frac{c}{2}$ is a divisor of c with $\frac{c}{2} \neq 1$, so $s(c) > \frac{c}{2} + 1$. Since $s(a) + s(b) + 2s(c) = a + b$ and $s(a) + s(b) = c$ we have $s(c) = \frac{a + b - c}{2}$, hence $\frac{a + b - c}{2} \geq \frac{c}{2} + 1$ and $a + b \geq 2c + 2$, a contradiction because $a \leq c$ and $b \leq c$. Thus, $c = 2$ and $1 < a \leq b \leq 2$, so necessarily $(a, b, c) = (2, 2, 2)$, which is a friendly triple.

4. The set A of positive integers has k elements. If the positive integers x and y are not in A, then $2x$, $2y$, and $x + y$ are also not in A. The sum of the elements in A is s. Find the maximum possible value of s.

Solved by Oliver Geupel, Brühl, NRW, Germany; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Pavlos Maragoudakis, Pireas, Greece. We give Maragoudakis' solution.

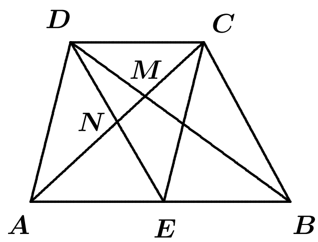
Let $A = \{a_1, a_2, a_3, \dots, a_k\}$, with $a_1 < a_2 < \dots < a_k$. If $a_i \geq 2i$ for some i, then a_i can be written as $x + y$ or $2x$ with x and y positive integers in at least i different ways. Since $a_i \in A$, we have that $x \in A$ or $y \in A$ for each of these ways. Hence, there are at least i positive integers in A less than a_i, a contradiction. Therefore, $a_i \leq 2i - 1$ for each i.

The set $\{1, 3, \dots, 2k - 1\}$ has the desired property and the maximum sum, which is $1 + 3 + \dots + 2k - 1 = k^2$.

Final Round

1. Let $ABCD$ be a trapezoid with parallel sides AB and CD. Let E be a point on the side AB such that EC and AD are parallel. Further, let the area of the triangle determined by the lines AC, BD, and DE be t, and the area of ABC be T. Determine the ratio $AB : CD$, if $t : T$ is maximal.

Solved by Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's write up.



Let M be the intersection of diagonals AC and BD, and N be the intersection of AC and DE.

Since $EC||AD$ and $AE||DC$, the quadrilateral $AECD$ is a parallelogram, hence N is the midpoint of AC.

Let $a = AB$, $b = CD$. Since $\triangle MCD$ is similar to $\triangle MAB$ we obtain

$$\frac{MC}{MA} = \frac{b}{a} \Leftrightarrow \frac{NC - MN}{NA + MN} = \frac{b}{a} \Leftrightarrow \frac{AC - 2MN}{AC + 2MN} = \frac{b}{a}.$$

By the last equality and some algebra we obtain $\frac{MN}{AC} = \frac{a - b}{2(a + b)}$. If h_t is the altitude of $\triangle DMN$ from D, and h_T is the altitude of $\triangle ABC$ from B, we have

$$\frac{h_t}{h_T} = \frac{DM}{MB} = \frac{b}{a}.$$

It follows that

$$\frac{t}{T} = \frac{MN \cdot h_t}{AC \cdot h_T} = \frac{b}{a} \cdot \frac{a-b}{2(a+b)} = \frac{\frac{a}{b} - 1}{2\frac{a}{b}(\frac{a}{b}+1)},$$

and we have to find $\frac{a}{b} = x$ when $\frac{x-1}{x(x+1)}$ takes its maximum value.

Setting $f(x) = \frac{x-1}{x(x+1)}$, we have $f'(x) = \frac{-x^2+2x+1}{x^2(x+1)^2}$, hence f takes its maximum value for $x = 1 + \sqrt{2}$. Hence, $t : T$ is maximized when $AB : CD = 1 + \sqrt{2}$.

3. Haydn and Beethoven celebrate the birthday of Mozart with a game. They take numbers alternately according to the following rules. First Haydn takes the number 2. The next player can take the sum or the product of any two numbers which were taken earlier (it is possible to choose just one number twice, thus taking the square of it). The numbers which are taken must be distinct and smaller than 1757. The winner is the player who takes the number 1756. Which player has a winning strategy?

Solution by Oliver Geupel, Brühl, NRW, Germany.

Beethoven can win. This remains true if we replace 1756 by $N = 4p$, where p is any odd prime ($1756 = 4 \cdot 439$, and 439 is prime). To prove this, let $\{h_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ denote the sequences of numbers taken by Haydn and Beethoven, H and B, respectively. We have $h_1 = 2$, $b_1 = 4$, and we denote by $T = \{h_1, b_1, h_2, b_2, \ldots, h_n\}$ the set of all numbers taken after n moves by H and $n-1$ moves by B. Let $P_k = \{2k, N-2k\}$ for $1 \leq k \leq p-1$, and consider the following strategy for B's n^{th} move, where $n \leq (p-1)/2$:

(i) If there is an index k with $1 \leq k \leq p$ and such that $P_k \subseteq T$ or if $a_n = 2p$, then B takes $b_n = N$ and he wins.

(ii) Otherwise, B takes $b_n = 2m$, where $m = \min\{k : P_k \cap T = \emptyset\}$.

It suffices to prove the following statements.

(a) By the rules of the game, B may actually choose b_n as described in (i) and (ii).

(b) Subsequently, $h_{n+1} \neq N$; hence, H cannot win.

(c) The game is over for some $n \leq (p-1)/2$.

Proof of (a). If (i) is the case, then clearly B can take $b_n = N$ and win. If B is faced with case (ii), then $|T| = 2(n-1)+1 < p-1$; thus, m is well defined. Then at least half of the $m-3$ sets $P_3, \ldots, P_{m-1}$ have nonempty intersection with the set $\{b_1, b_2, \ldots, b_{n-1}\}$ (this follows from B's strategy by an easy induction). Hence, the set $U = \{2, 4, 6, \ldots, 2(m-1)\} \cap T$ has at least $2 + \lceil (m-3)/2 \rceil = \lceil (m+1)/2 \rceil$ elements. We define $\lceil (m+1)/2 \rceil$ pairs $Q_k = \{2k, 2m-2k\}$ for $1 \le k \le \lceil (m-1)/2 \rceil$. By the Pigeonhole Principle, for an appropriate index k with $(2 \le k \le \lceil (m-1)/2 \rceil)$, we have $|Q_k \cap U| = 2$. Thus, $b_n = 2m = 2k + (2m-2k)$, where $2k$ and $2m-2k$ are in $Q_k \cap U \subseteq T$.

Proof of (b). Assume on the contrary that there are $c, d \in T \cup \{b_n\}$ such that $N = c + d$ or $N = cd$. If $N = c + d$, then there exists an integer k with $\{c, d\} = P_k$, $1 \le k \le p-1$. However, this is impossible since by (ii) B could not have taken both c and d. If $N = cd$, then $\{c, d\} = \{2, 2p\}$, which is also impossible since the number $2p$ was not taken earlier.

Proof of (c). We consider the number $d = |\{k : P_k \cap T = \emptyset; 1 \le k \le p-1\}|$. Initially $d = p-1$ and d decreases by 1 each time B moves as in (ii), hence, after at most $(p-1)/2$ numbers are taken by B we shall have $d = 0$, and then B wins by moving as in (i).

Next we present our readers' solutions to problems of the Hungarian National Olympiad 2004-2005, Grades 11-12, Second and Final Rounds, given at [2008 : 148-149].

Second Round

1. Find all real solutions to the following system of equations:

$$\begin{aligned} \sqrt{x+y} + \sqrt{x-y} &= 10\,, \\ x^2 - y^2 - z^2 &= 476\,, \\ 2^{(\log|y| - \log z)} &= 1\,. \end{aligned}$$

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give Wang's solution.

Label the equations by (1), (2), and (3), in the order they are given.

From (3) we see that $z = |y|$, so $z^2 = y^2$. Substituting this into (2) we have

$$x^2 - 2y^2 = 476\,. \tag{4}$$

Squaring each side of (1), we have $2x + 2\sqrt{x^2 - y^2} = 100$, so that

$$x^2 - y^2 = (50-x)^2 = x^2 - 100x + 2500\,,$$

and hence $y^2 = 100x - 2500$. Substituting this into (4) and simplifying, we have $x^2 - 200x + 4524 = 0$ or $(x-26)(x-174) = 0$. Hence, $x = 26$ or $x = 174$.

However, if $x = 174$, then regardless of the sign of y, we would have $\sqrt{x+y} + \sqrt{x-y} > \sqrt{174} > 10$, contradicting (1). Therefore, $x = 26$, and from (4) we deduce that $y = \pm 10$.

Finally, it is straightforward to check that $(x, y, z) = (26, \pm 10, 10)$ satisfy the given system.

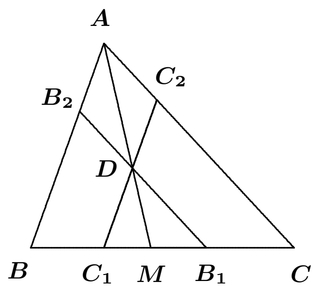
2. In triangle ABC, the points B_1 and C_1 are on BC, point B_2 is on AB, and point C_2 is on AC such that the segment B_1B_2 is parallel to AC and the segment C_1C_2 is parallel to AB. Let the lines B_1B_2 and C_1C_2 meet at D. Denote the areas of triangles BB_1B_2 and CC_1C_2 by b and c, respectively.

(a) Prove that if $b = c$, then the centroid of ABC is on the line AD.

(b) Find the ratio $b : c$ if D is the incentre of ABC and $AB = 4$, $BC = 5$, and $CA = 6$.

Solution by Titu Zvonaru, Comănești, Romania.

(a) Let F denote the area of $\triangle ABC$ and let $M = \overline{AD} \cap \overline{BC}$. For convenience, replace b and c by F_b and F_c, respectively. Since $B_1B_2 || AC$ and $C_1C_2 || AB$, we have



$$\frac{F_B}{F} = \left(\frac{BB_1}{BC}\right)^2 ; \qquad \frac{F_C}{F} = \left(\frac{CC_1}{BC}\right)^2 ,$$

hence $BB_1 = CC_1$ (and $BC_1 = CB_1$) since $F_B = F_C$. By Menelaus' theorem applied to $\triangle ABM$ and the traverse B_1DB_2 we obtain

$$\frac{B_1M}{B_1B} \cdot \frac{B_2B}{B_2A} \cdot \frac{DA}{DM} = 1 .$$

Similarly, by applying Menelaus' theorem to $\triangle ACM$ and the traverse C_1DC_2 we obtain

$$\frac{C_1M}{C_1C} \cdot \frac{C_2C}{C_2A} \cdot \frac{DA}{DM} = 1 .$$

The last two equations imply that

$$B_1M \cdot \frac{B_2B}{B_2A} = C_1M \cdot \frac{C_2C}{C_2A} .$$

Since $\dfrac{B_2B}{B_2A} = \dfrac{B_1B}{B_1C} = \dfrac{C_1C}{C_1B} = \dfrac{C_2C}{C_2A}$, it follows that $B_1M = C_1M$, hence M is the midpoint of BC (recall that $BC_1 = CB_1$). It follows that the centroid of ABC is on the line AD.

(b) Let $a = BC$, $b = CA$, $c = AB$, r be the inradius of $\triangle ABC$, and h_b and h_c be the altitudes of $\triangle ABC$ from B and C, respectively. Since

$\triangle BB_1B$ and $\triangle CC_1C_2$ are similar, we have

$$\frac{F_B}{F_C} = \left(\frac{BB_1}{CC_1}\right)^2 = \left(\frac{BB_1}{BC}\right)^2\left(\frac{BC}{CC_1}\right)^2 = \left(\frac{h_b - r}{h_b}\cdot\frac{h_c}{h_c - r}\right)^2 .$$

By substituting $h_b = \frac{2F}{b}$, $h_c = \frac{2F}{c}$, and $r = \frac{2F}{a+b+c}$ in the above we deduce that

$$\begin{aligned} \frac{F_B}{F_C} &= \left(\frac{\frac{2F}{b} - \frac{2F}{a+b+c}}{\frac{2F}{b}}\cdot\frac{\frac{2F}{c}}{\frac{2F}{c} - \frac{2F}{a+b+c}}\right)^2 \\ &= \left(\frac{a+c}{a+b+c}\cdot\frac{a+b+c}{a+b}\right)^2 = \left(\frac{a+c}{a+b}\right)^2 . \end{aligned}$$

Since $a = 5$, $b = 6$, and $c = 4$ were given, we obtain $\frac{F_B}{F_C} = \frac{81}{121}$.

4. The divisors of n are $d_1 < d_2 < \cdots < d_8$, where $d_1 = 1$ and $d_8 = n$. It is known that $20 \le d_6 \le 25$. Find all possible values of n.

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's version.

The only such n are 66, 88, 105, 110, and 154. Let the prime power decomposition of n be $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where each α_i is a positive integer and the p_i's are primes such that $p_1 < p_2 < \cdots < p_k$.

By the well-known formula for the number of divisors of n in terms of its prime power decomposition, $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1) = 8$. Hence, there are three possible cases to consider.

Case 1: $k = 1$ and $\alpha_1 = 7$.
Case 2: $k = 2$, $\alpha_1 = 1$, $\alpha_2 = 3$; or $\alpha_1 = 3$, $\alpha_2 = 1$.
Case 3: $k = 3$ and $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

In Case 1 we have $n = p^7$ for some prime p. Then clearly $d_6 = p^5$. Hence, we have $20 \le p^5 \le 25$, which implies that $p < 2$, a contradiction.

In Case 2 we have either (2a) $n = p_1p_2^3$ or (2b) $n = p_1^3p_2$ where p_1 and p_2 are primes such that $p_1 < p_2$.

In subcase (2a), $d_6 = p_1p_2^2$ since

$$1 < p_1 < p_2 < p_1p_2 < p_2^2 < p_1p_2^2 < p_2^3 < p_1p_2^3 .$$

Hence, $20 \le p_1p_2^2 \le 25$. Clearly, $p_2 \le 3$, so $p_1 = 2$ and $p_2 = 3$ yielding $p_1p_2^2 = 18$, a contradiction. Thus, there are no solutions in subcase (2a).

In subcase (2b), we first identify d_6. If $p_2 < p_1^2$, then $d_6 = p_1^3$ since

$$1 < p_1 < p_2 < p_1^2 < p_1p_2 < p_1^3 < p_1^2p_2 < p_1^3p_2 .$$

Hence, $20 \le p_1^3 \le 25$, which has no solutions. If $p_2 > p_1^2$, then $d_6 = p_1p_2$ since 1, p_1, p_2, p_1^2, and p_1^3 are each less than p_1p_2 while $p_1p_2 < p_1^2p_2 < p_1^3p_2$.

Hence, $20 \le p_1p_2 \le 25$ and $(p_1, p_2) = (2, 11)$ or $(p_1, p_2) = (3, 7)$. The first pair yields $n = 88$ while the last pair is discarded since $3^2 > 7$.

Finally, in Case 3 we have $d_6 = p_1p_3$, since $p_1p_3 < p_2p_3 < p_1p_2p_3$ and each of 1, p_1, p_2, p_3, and p_1p_2 are less than p_1p_3. Hence, $20 \le p_1p_3 \le 25$ and $(p_1, p_3) = (2, 11)$ or $(p_1, p_3) = (3, 7)$ by the result in subcase (2b). We then have $(p_1, p_2, p_3) = (2, 3, 11)$, $(2, 5, 11)$, $(2, 7, 11)$, or $(3, 5, 7)$. The corresponding values of n are 66, 110, 154, and 105.

Therefore, $n = 66$, 88, 105, 110, or 154, as claimed.

Final Round

1. A positive integer n is *charming* if there are integers a_1, a_2, $\dots$, a_n (not necessarily distinct) such that $a_1 + a_2 + \cdots + a_n = a_1a_2\cdots a_n = n$. Find all charming integers.

Solution by Oliver Geupel, Brühl, NRW, Germany.

One easily checks that $n = 4$ is not charming. If $n = 1$ or if $n > 4$ and n is congruent to 0 or 1 modulo 4, then the table below shows that n is charming:

n	values of a_k	multiplicity
$n = 8m$	$4m$	1
$(m \ge 1)$	2	1
	1	$6m - 2$
	-1	$2m$
$n = 4(4m-1)$	$4m - 1$	1
$(m \ge 1)$	2	2
	1	$14m - 7$
	-1	$2m$
$n = 4(4m+1)$	$4m + 1$	1
$(m \ge 1)$	2	1
	1	$14m + 2$
	-1	$2m - 1$
	-2	1
$n = 4m + 1$	$4m + 1$	1
$(m \ge 0)$	1	$2m$
	-1	$2m$

Let $n \equiv 2 \pmod 4$. If $n = \prod\limits_{k=1}^{n} a_k$, then exactly one of the numbers a_k is even. Therefore, $\sum\limits_{k=1}^{n} a_k$ is odd, and consequently n is not charming.

Finally, consider $n \equiv -1 \pmod 4$, say $n = 4m - 1$. Let $n = \prod\limits_{k=1}^{n} a_k$, and let q of the numbers a_k be congruent to -1 modulo 4 with the other $4m - 1 - q$ numbers a_k being congruent to 1 modulo 4. Then q is odd, say

$q = 2s + 1$, hence $\sum_{k=1}^{n} a_k \equiv (-2s - 1) + (4m - 1 - 2s - 1) \equiv 1 \pmod 4$. Thus, $n \neq \sum_{k=1}^{n} a_k$ and n is not charming.

Therefore, a positive integer n is charming if and only if $n = 1$ or $n > 4$ and n is congruent to 0 or 1 modulo 4.

2. Let a, b, and c be positive real numbers.

(a) Prove that

$$\sqrt{\frac{a^2+b^2}{2}} + \frac{2}{\frac{1}{a}+\frac{1}{b}} \geq \frac{a+b}{2} + \sqrt{ab}\,.$$

(b) Is it true always that

$$\sqrt{\frac{a^2+b^2+c^2}{3}} + \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \frac{a+b+c}{3} + \sqrt[3]{abc}\,?$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

(a) If $a = b$, then the equality holds. Assume $a \neq b$, then the following inequalites are successively equivalent:

$$\begin{aligned}
\sqrt{\frac{a^2+b^2}{2}} + \frac{2}{\frac{1}{a}+\frac{1}{b}} &\geq \frac{a+b}{2} + \sqrt{ab}\,;\\
\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab} &\geq \frac{a+b}{2} - \frac{2ab}{a+b}\,;\\
\frac{\left(\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\right)}{\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}} &\geq \frac{(a+b)^2-4ab}{2(a+b)}\,;\\
\frac{(a-b)^2}{2\left(\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\right)} &\geq \frac{(a-b)^2}{2(a+b)}\,;\\
a+b &\geq \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\,;
\end{aligned}$$

$$\begin{aligned}\frac{a^2+b^2}{2}+ab+2\sqrt{ab}\sqrt{\frac{a^2+b^2}{2}} &\leq a^2+b^2+2ab\,;\\ \frac{a^2+b^2}{2}+ab-2\sqrt{ab}\sqrt{\frac{a^2+b^2}{2}} &\geq 0\,;\\ \left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)^2 &\geq 0\,;\end{aligned}$$

and the last inequality is true.

(b) Taking $a = 1$, $b = 2$, and $c = 3$ the inequality becomes

$$\sqrt{\frac{14}{3}}+\frac{18}{11} \geq 2+\sqrt[3]{6}\,,$$

which is false, as one may verify by hand calculation that $\sqrt{\frac{14}{3}}+\frac{18}{11} < \frac{19}{5}$ and $\frac{19}{5} < 2+\sqrt[3]{6}$.

[*Ed.:* Michel Bataille, Rouen, France comments that by coincidence, this problem is similar to problem 3266 proposed in the October 2007 issue.]

3. Triangle ABC is acute angled, $\angle BAC = 60^\circ$, $AB = c$, and $AC = b$ with $b > c$. The orthocentre and the circumcentre of ABC are M and O, respectively. The line OM intersects AB and CA at X and Y, respectively.

(a) Prove that the perimeter of triangle AXY is $b + c$.

(b) Prove that $OM = b - c$.

Solved by Michel Bataille, Rouen, France; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comănești, Romania. We give the solution of Smeenk.

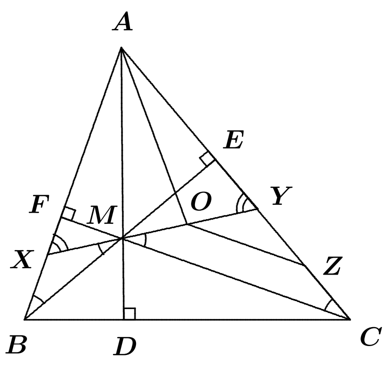
(a) Let α and β be the angles at A and B, respectively. Since

$$AM = 2R\cos\alpha = R = AO\,,$$

we have that $\angle AMO = \angle AOM$. Also, since

$$\angle XAM = \angle YAO = 90^\circ - \beta\,,$$

it follows that $\triangle XAM \sim \triangle YAO$. Therefore, $\angle AXY = \angle AYX = 60^\circ$ and $\triangle AXY$ is equilateral.



Now $\angle ABM = 30^\circ$, hence $\angle XMB = 30^\circ$ and $XB = XM$. In the same way we can prove that $YC = YM$. It now follows that the perimeter of $\triangle AXY$ is $b + c$.

(b) Let Z be the point on AC such that $AZ = AB = c$. Then we have $ZC = b - c$, and from part (a)

$$YZ = BX = MX = OY .$$

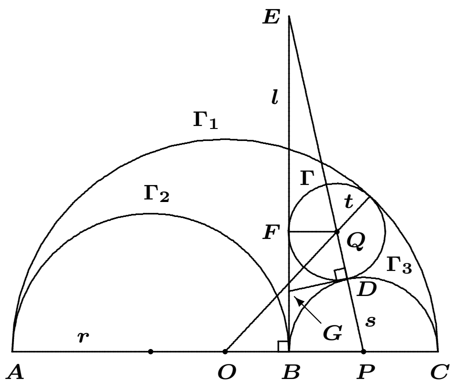
Therefore, $\triangle YOZ$ is isosceles, and $\angle YOZ = \angle YZO = 30°$. Since we have $\angle MCZ = \angle CMO = 30°$, we conclude that $MCZO$ is an isosceles trapezoid. This establishes that $OM = ZC = b - c$, as desired.

Now we turn to solutions to problems of the Indian Team Selection Test to the IMO 2002, given at [2008 : 149-151].

1. Let A, B, and C be three points on a line with B between A and C. Let Γ_1, Γ_2, and Γ_3 be semicircles, all on the same side of AC, and with AC, AB, and BC as diameters, respectively. Let l be the line perpendicular to AC through B. Let Γ be the circle which is tangent to the line l, tangent to Γ_1 internally, and tangent to Γ_3 externally. Let D be the point of contact of Γ and Γ_3. The diameter of Γ through D meets l in E. Show that $AB = DE$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let O, P, and Q, be the centres of Γ_1, Γ_3, and Γ, respectively. Let r, s, and t be the radii of Γ_2, Γ_3, and Γ, respectively. Let Γ meet l at the point F. The tangent to Γ at D meets l at the point G. Let u denote the common length of the tangent segments GB, GD, and GF. Let $\alpha = \angle BPD$.



By the Law of Cosines in $\triangle OPQ$ we have

$$\begin{aligned}(r+s-t)^2 &= OQ^2 = OP^2 + PQ^2 - 2OP \cdot PQ \cos\angle OPQ\,,\\ &= r^2 + (s+t)^2 - 2r(s+t)\cos\alpha\,;\end{aligned}$$

hence

$$\cos\alpha = \frac{2st + rt - rs}{rs + rt}\,. \tag{1}$$

Applying the Law of Cosines to the triangles BPD, BGD, DGF, and DQF we have

$$\begin{aligned} 2s^2(1-\cos\alpha) &= BD^2 = 2u^2(1+\cos\alpha)\,, \\ 2u^2(1-\cos\alpha) &= FD^2 = 2t^2(1+\cos\alpha)\,; \end{aligned}$$

thus, $(1+\cos\alpha)t = (1-\cos\alpha)s$. We substitute for $\cos\alpha$ the expression in equation (1) and simplify successively to obtain

$$\begin{aligned} \left(\frac{rt+st}{rs+rt}\right)t &= \left(\frac{rs-st}{rs+st}\right)s\,, \\ (r+s)t^2+s^2t-rs^2 &= 0\,, \\ \big((r+s)t-rs\big)(t+s) &= 0\,. \end{aligned}$$

Since $t>0$, we have $t=\dfrac{rs}{r+s}$.

Let $d=ED$. We observe that the triangles EFQ and EBP are homothetic. Hence,

$$d-t \;=\; EQ \;=\; \frac{FQ\cdot EP}{BP} \;=\; \frac{t(d+s)}{s}\,,$$

so that $s(d-t)=t(d+s)$.

Therefore,

$$DE \;=\; d \;=\; \frac{2st}{s-t} \;=\; \frac{2s\cdot\dfrac{rs}{r+s}}{s-\dfrac{rs}{r+s}} \;=\; \frac{2rs^2}{s^2} \;=\; 2r \;=\; AB\,,$$

and the proof is complete.

4. Let ABC be an acute triangle with orthocentre H and circumcentre O. Show that there are points D, E, and F on BC, CA, and AB, respectively, such that AD, BE, and CF are concurrent and

$$DO+DH \;=\; EO+EH \;=\; FO+FH\,.$$

Solved by Oliver Geupel, Brühl, NRW, Germany. Remark by Michel Bataille, Rouen, France.

This problem has already been solved in this *Corner*. See Vol. 31, No. 5 (September 2005) p. 295.

5. Let a, b, and c be positive real numbers such that $a^2+b^2+c^2=3abc$. Prove that

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2} \;\geq\; \frac{9}{a+b+c}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Apostolopoulos' solution.

Applying the Cauchy–Schwarz Inequality we have

$$\begin{aligned}&(a+b+c)\cdot\left(\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\right)\\ \geq\ &\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)^2 \;=\; \left(\frac{a^2+b^2+c^2}{abc}\right)^2 \;=\; \left(\frac{3abc}{abc}\right)^2 \;=\; 9\,,\end{aligned}$$

because $a^2+b^2+c^2=3abc$.

Therefore, $\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq\frac{9}{a+b+c}$.

(*Ed.:* Zvonaru remarks that this problem is #2859 at [2003 : 318] with a solution at [2004 : 314].)

11. Let ABC be a triangle and let P be an exterior point in the plane of the triangle. Let AP, BP, and CP meet the (possibly extended) sides BC, CA, and AB in D, E, and F, respectively. If the areas of the triangles PBD, PCE, and PAF are all equal, prove that their common area is equal to the area of the triangle ABC.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $[XYZ]$ denote the area of $\triangle XYZ$. We assume without loss of generality that $[ABC]=1$. Let (a,b,c) be the barycentric coordinates of P with respect to $\triangle ABC$. The oriented (signed) area of $\triangle PBD$ is then

$$[PBD] \;=\; \left(\begin{array}{ccc} a & b & c\\ 0 & 1 & 0\\ 0 & \frac{b}{b+c} & \frac{c}{b+c}\end{array}\right) \;=\; \frac{ac}{b+c}\,.$$

Similarly, we have $[PCE]=\frac{ab}{a+c}$ and $[PAF]=\frac{ab}{a+c}$. The assumption $|[PBD]|=|[PCE]|=|[PAF]|$ implies $|a(a+b)|=|b(b+c)|=|c(c+a)|$, where at least two of the real numbers $a(a+b)$, $b(b+c)$, $c(c+a)$ are equal. Without loss of generality assume that $a(a+b)=c(c+a)$. We can scale the barycentric coordinates so that $a=1$. Therefore, $1(1+b)=c(c+1)$, that is

$$b \;=\; c^2+c-1\,. \qquad (1)$$

The equation $|b(b+c)|=|a(a+b)|=|1+b|$ leads to two cases.

Case 1. $b(b+c)=1+b$. We subsitute for b as in (1) and obtain

$$\begin{aligned}\left(c^2+c-1\right)\left(c^2+2c-1\right) &= c^2+c\,;\\ c^4+2c^3-c^2-4c+1 &= (c-1)\left(c^3+4c^2+3c-1\right) \;=\; 0\,.\end{aligned}$$

If $c = 1$ then also $b = a = 1$, which contradicts the hypothesis that P is an exterior point of $\triangle ABC$. Consequently, $c^3 + 4c^2 + 3c - 1 = 0$. It follows from (1) that

$$\begin{aligned}[PBD] &= \frac{c}{(c^2+2c)(c^2+2c-1)} \\ &= \frac{1}{c^3+4c^2+3c-2} \;=\; -1 \;=\; -[ABC]\,,\end{aligned}$$

which is the desired conclusion.

Case 2. $b(b+c) = -1 - b$. We substitute according to (1) to obtain

$$\begin{aligned}(c^2+c-1)\,(c^2+2c-1) &= -c^2-c\,; \\ c^4+c^3+c^2-2c+1 &= 0\,.\end{aligned} \qquad (2)$$

We will prove that $f(x) = x^4 + 3x^3 + x^2 - 2x + 1$ is positive for $x \in \mathbb{R}$, thus proving that (3) is impossible. Since $f(x) = x^4 + 3x^3 + (x-1)^2$, we see that $f(x) > 0$ for $x \in (-\infty, -3] \cup [0, \infty)$.

It suffices to show that $f(x) > 0$ for $x \in [-3, 0]$. Let $t = -x$. The AM–GM Inequality yields

$$\begin{aligned}f(x) &= -4t\left(\frac{t}{2}\right)\left(\frac{t}{2}\right)(3-t) \;+\; (t+1)^2 \\ &\geq -4t\left(\frac{1}{3}\left(\frac{t}{2}+\frac{t}{2}+3-t\right)\right)^3 \;+\; (t+1)^2 \;=\; (t-1)^2 \;\geq\; 0\,,\end{aligned}$$

where at least one of the last two inequalities is strict. This completes the proof.

15. Let x_1, x_2, ..., x_n be real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2} \;<\; \sqrt{n}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give the solution of Díaz-Barrero.

For vectors $\vec{u} = (a_1, a_2, \ldots, a_n)$ and $\vec{v} = (1, 1, \ldots, 1)$ an application of the Cauchy–Schwartz Inequality yields

$$a_1 + a_2 + \cdots + a_n \;\leq\; \sqrt{n}\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\,.$$

Setting $a_i = \dfrac{x_i}{1+x_1^2+\cdots+x_i^2}$ for $1 \leq i \leq n$, it suffices to prove that

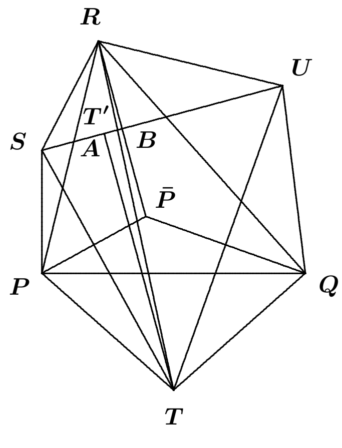
$$\left(\frac{x_1}{1+x_1^2}\right)^2 + \left(\frac{x_2}{1+x_1^2+x_2^2}\right)^2 + \cdots + \left(\frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2}\right)^2 \;<\; 1\,.$$

We have
$$\frac{x_1^2}{\left(1+x_1^2\right)^2} \;\le\; \frac{x_1^2}{\left(1+x_1^2\right)} \;=\; 1-\frac{1}{1+x_1^2}\,,$$
and for $2 \le i \le n$ we can assert that
$$\begin{aligned}\frac{x_i^2}{\left(1+x_1^2+\cdots+x_i^2\right)^2} \;&\le\; \frac{x_i^2}{\left(1+x_1^2+\cdots+x_{i-1}^2\right)\left(1+x_1^2+\cdots+x_i^2\right)}\\ &=\; \frac{1}{1+x_1^2+\cdots+x_{i-1}^2}-\frac{1}{1+x_1^2+\cdots+x_i^2}\,.\end{aligned}$$
Adding the preceding expressions, we obtain
$$\sum_{i=1}^{n}\frac{x_i^2}{\left(1+x_1^2+\cdots+x_i^2\right)^2} \;\le\; 1-\frac{1}{1+x_1^2+\cdots+x_n^2} \;<\; 1\,,$$
and we are done.

19. Let PQR be an acute triangle. Let SRP, TPQ, and UQR be isosceles triangles exterior to PQR, with $SP = SR$, $TP = TQ$, and $UQ = UR$, such that $\angle PSR = 2\angle QPR$, $\angle QTP = 2\angle RQP$, and $\angle RUQ = 2\angle PRQ$. Let S', T', and U' be the points of intersection of SQ and TU, TR and US, and UP and ST, respectively. Determine the value of
$$\frac{SQ}{SS'}+\frac{TR}{TT'}+\frac{UP}{UU'}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $\angle RPQ = \alpha$, $\angle PQR = \beta$, and $\angle QRP = \gamma$. Let $\bar{P}$, $\bar{Q}$, and $\bar{R}$ be the reflections of P, Q, and R, with respect to the axes ST, TU, and US, respectively. Since $PT = \bar{P}T$, $\bar{P}$ is on the circle Γ_T with centre T and radius TP. Since $PS = \bar{P}S$, $\bar{P}$ is on the circle Γ_S with centre S and radius SP. Therefore, $\angle P\bar{P}Q = 180^\circ = \beta$ and $\angle R\bar{P}P = 180^\circ - \alpha$. We now have that $\angle Q\bar{P}R = 360^\circ - (180^\circ - \alpha) - (180^\circ - \beta) = 180^\circ - \gamma$. Hence, $\bar{P}$ is also on the circle Γ_U with centre U and radius UR.



We conclude that the three circles Γ_S, Γ_T, and Γ_U intersect at $\bar{P}$. By symmetry, $\bar{P}$ coincides with $\bar{Q}$ and with $\bar{R}$, so we have $\triangle PST \cong \triangle \bar{P}ST$, $\triangle QTU \cong \triangle \bar{P}TU$, and $\triangle RUS \cong \triangle \bar{P}US$.

Let A and B be the feet of the perpendiculars from T and R to SU, respectively. Then we have

$$\frac{RT'}{TT'} = \frac{RB}{TA} = \frac{[RUS]}{[STU]} = \frac{[\bar{P}US]}{[STU]},$$

thus

$$\frac{TR}{TT'} = 1 + \frac{[\bar{P}US]}{[STU]}.$$

We finally obtain

$$\begin{aligned} &\frac{SQ}{SS'} + \frac{TR}{TT'} + \frac{UP}{UU'} \\ &= \left(1 + \frac{[\bar{P}TU]}{[STU]}\right) + \left(1 + \frac{[\bar{P}US]}{[STU]}\right) + \left(1 + \frac{[\bar{P}ST]}{[STU]}\right) = 4. \end{aligned}$$

20. Let a, b, and c be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

Solved by Arkady Alt, San Jose, CA, USA; and George Apostolopoulos, Messolonghi, Greece. We give Alt's solution.

Due to the cyclic symmetry, we can suppose that $c = \min\{a, b, c\}$.

Let x, y, and z be nonzero real numbers. Since

$$x^2z + y^2x + z^2y = 3xyz + z(x-y)^2 + y(x-z)(y-z),$$

we obtain

$$\begin{aligned} \frac{x}{y} + \frac{y}{z} + \frac{z}{x} &= \frac{x^2z + y^2x + z^2y}{xyz} \\ &= 3 + \frac{(x-y)^2}{xy} + \frac{(x-z)(y-z)}{xz}. \end{aligned}$$

Setting $x = a$, $y = b$, $z = c$ and then $x = c+a$, $y = c+b$, $z = a+b$, we obtain (respectively) the two equations

$$\begin{aligned} \frac{a}{b} + \frac{b}{c} + \frac{c}{a} &= 3 + \frac{(a-b)^2}{ab} + \frac{(a-c)(b-c)}{ac}, \\ \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a} &= 3 + \frac{(a-b)^2}{(c+a)(c+b)} + \frac{(a-c)(b-c)}{(c+a)(a+b)}. \end{aligned}$$

Comparing the last two equations gives the result, since $(c+a)(c+b) > ab$, $(a+b)(c+a) > ac$, $(a-b)^2 \geq 0$, and $(b-c)(a-c) \geq 0$.

21. Given a prime p, show that there is a positive integer n such that the decimal representation of p^n has a block of 2002 consecutive zeros.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove the following generalization: Given a prime p and a positive integer N, there are infinitely many natural numbers n such that the decimal representation of p^n has a block of N consecutive zeroes. To prove this we distinguish three cases.

Case 1. $p \notin \{2, 5\}$. Let $k > N$, $k \in \mathbb{Z}$, and let $n = \phi\left(10^k\right) = \frac{2}{5} \cdot 10^k$, where ϕ is Euler's totient function. We have $p^n > 1$ and, by Euler's theorem, $p^n \equiv 1 \pmod{10^k}$; hence n has the desired property.

Case 2. $p = 2$. Let $k > 2N$, $k \in \mathbb{Z}$, and let $n = \phi\left(5^k\right) + k = 4 \cdot 5^{k-1} + k$. We see that $2^n > 2^k$ and, by Euler's theorem, that $2^{\phi(5^k)} \equiv 1 \pmod{5^k}$. Therefore, $2^n = 2^{\phi(5^k)} \cdot 2^k \equiv 2^k \pmod{10^k}$. Since $k > 2N > \log_5 10 \cdot N$, we have $\dfrac{10^k}{2^k} = 5^k > 10^N$. Consequently, the number 2^n contains a block of N consecutive zeros to the left of the rightmost 2^k digits.

Case 3. $p = 5$. Let $k > 4N$, $k \in \mathbb{Z}$, and let $n = \phi\left(2^k\right) + k = 2^{k-1} + k$. We note that $5^n > 5^k$, and again by Euler's theorem, that $5^{\phi(2^k)} \equiv 1 \pmod{2^k}$. Therefore, $5^n = 5^{\phi(2^k)} \cdot 5^k \equiv 5^k \pmod{10^k}$. Since $k > 4N > \log_2 10 \cdot N$, we have $\dfrac{10^k}{5^k} = 2^k > 10^N$. Thus, the number 5^n contains a block of N consecutive zeroes to the left of the rightmost 5^k digits.

That completes the material for this number of the *Corner*. Please send me Olympiad contests as well as your nice solutions and generalizations.