SKOLIAD No. 115

Lily Yen and Mogens Hansen

Please send solutions to problems in this Skoliad by **September 1, 2009**. Solutions should be sent to Lily Yen and Mogens Hansen at the address inside the back cover. A copy of ***MATHEMATICAL MAYHEM*** will be presented to the pre-university reader who sends in the best solution(s) before the deadline. The decision of the editors is final.

The Skoliad is in transition and, unfortunately, some submitted solutions have been lost. Our apologies for this inconvenience. Please resubmit any solutions to contests appearing in Skoliad in or after the March 2008 issue of ***CRUX***.

Our featured contest for this month is the British Columbia Secondary School Mathematics Contest 2007, Final Round, Part B. Our thanks go to Clint Lee, Okanagan College, Vernon, BC, for permission to use it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

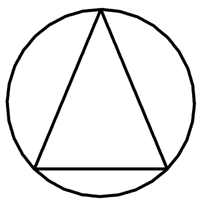
Concours mathématique des écoles secondaires de la Colombie-Britannique 2007

Ronde finale, partie B

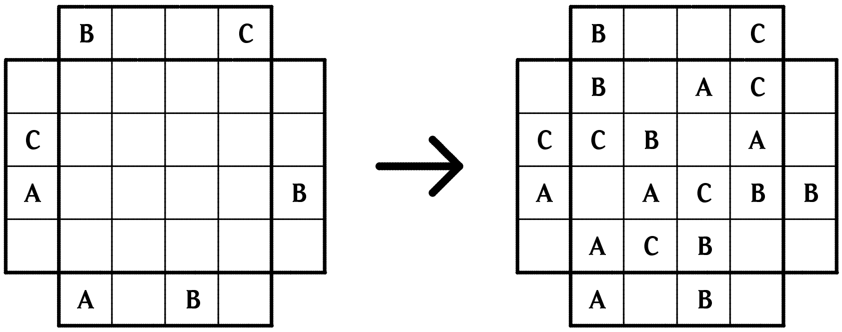
1. Jeanne a une collection de pièces de monnaie de $5¢$, de $10¢$ et de $25¢$, dont la valeur totale est de $2,00\$$. Si les pièces de $5¢$ étaient de $10¢$ et les pièces de $10¢$ étaient de $5¢$, alors la valeur de la collection serait de $1,70\$$. Déterminer toutes les possibilités de nombres de pièces de $5¢$, de $10¢$ et de $25¢$ dans la collection de Jeanne.

2. Un cube $3 \times 3 \times 3$ est formé en empilant des cubes $1 \times 1 \times 1$. Déterminer le nombre total de cubes, dont les côtés sont de taille entière, contenus dans le cube $3 \times 3 \times 3$.

3. Les longueurs des côtés d'un triangle sont 13, 13 et 10. Le cercle circonscrit de ce triangle est le cercle passant par les trois sommets du triangle et ayant ici son centre à l'intérieur du triangle (voir le diagramme à droite). Déterminer le rayon du cercle circonscrit.



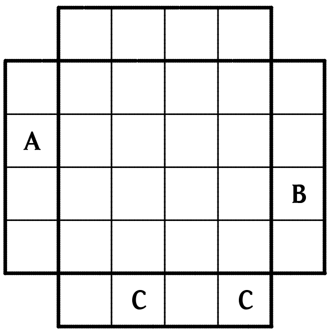
4. Le jeu de Latin se joue sur un tableau formé d'une grille quatre par quatre, avec une rangée additionnelle en haut comme en bas, puis une colonne additionnelle à gauche comme à droite. Les lettres A, B et C sont placées dans les cases de la grille quatre par quatre, de façon à ce que chaque lettre se retrouve exactement une fois dans chaque rangée et exactement une fois dans chaque colonne. En conséquence chaque rangée, de même que chaque colonne, aura exactement une case vide. Des lettres sont placées, comme indices de solution, à certains endroits dans les rangées et colonnes additionnelles ; elles indiquent la lettre la plus près se retrouvant dans la rangée ou colonne de la grille contenant l'indice. Le diagramme suivant donne une position de départ du jeu de Latin et la solution qui en résulte.



Debut **Solution**

Le diagramme d'un autre jeu de Latin est fourni à droite.

Compléter ce tableau, en fournissant une solution complète. Donner une justification des étapes permettant d'obtenir votre solution.



5. Déterminer toutes les solutions x et y, entières et positives, à l'équation

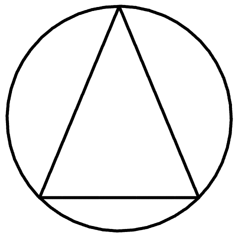
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}\,.$$

British Columbia Secondary School Mathematics Contest 2007
Final Round, Part B

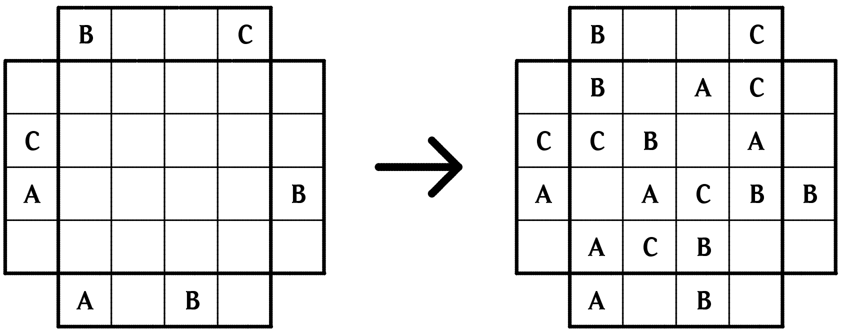
1. Joan has a collection of nickels, dimes, and quarters worth \$2.00. If the nickels were dimes and the dimes were nickels, the value of the coins would be \$1.70. Determine all of the possibilities for the number of nickels, dimes, and quarters that Joan could have.

2. A $3 \times 3 \times 3$ cube is formed by stacking $1 \times 1 \times 1$ cubes. Determine the total number of cubes with sides of integral length that are contained in the $3 \times 3 \times 3$ cube.

3. The lengths of the sides of a triangle are 13, 13, and 10. The circumscribed circle of a triangle is the circle that goes through each of the three vertices of the triangle and here has its centre inside the triangle (see the diagram at right). Find the radius of the circumscribed circle.



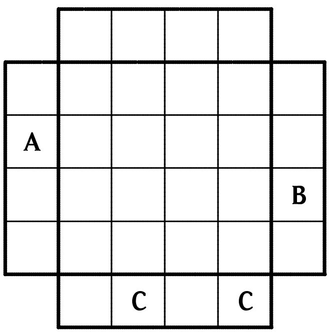
4. The game of End View consists of a tableau with a four by four grid, one additional row at the top and at the bottom, and one additional column on the right and on the left. The letters A, B, and C are placed in the four by four grid in such a way that every letter appears exactly once in each row and each column. This means that there will be exactly one empty square in each row and each column. Letters are placed in the additional rows and columns as hints, at the end of some rows and columns of the four by four grid, to indicate the nearest letter that can be found by reading that row or column of the grid. The diagram below shows the starting tableau and the resulting solution tableau for a game of End View.



Start **Solution**

The diagram for another game of End View is shown at right.

Fill in this tableau with the complete solution. Give a justification of the steps that you used to find the solution.



5. Determine all of the positive integer solutions, x and y, to the equation

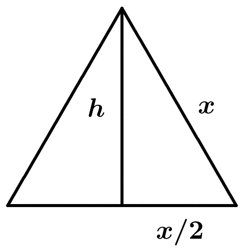
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$$

Now we will give solutions to the British Columbia Secondary School Mathematics Contest 2006, Junior Final Round, Part B [2008 : 129–130]. This time all the solutions are based on in-class work by students of the editors. Unfortunately, the solutions below were already typeset when we received a batch of (recovered) solutions from our readers. Thus, we could not feature any reader's solution this time.

1. Equilateral triangles I, II, III, and IV are such that the altitude of triangle I is the side of triangle II, the altitude of triangle II is the side of triangle III, and the altitude of triangle III is the side of triangle IV. If the area of triangle I is 2, find the area of triangle IV.

Solution.

If the side length of triangle I is x_1, then by the Pythagorean Theorem the height, h_1, of triangle I is $\frac{\sqrt{3}}{2}x_1$. But then the side length, x_2, of triangle II is $\frac{\sqrt{3}}{2}x_1$ and the height, h_2, of triangle II is $\frac{\sqrt{3}}{2}h_1$. Therefore the area, A_2, of triangle II is $\frac{x_2h_2}{2} = \left(\frac{\sqrt{3}}{2}\right)^2 \frac{x_1h_1}{2} = \frac{3}{4}A_1$, where A_1 is the area of triangle I. Similarly, $A_3 = \frac{3}{4}A_2$ and $A_4 = \frac{3}{4}A_3$, where A_3 and A_4 are the areas of triangle III and triangle IV, respectively. Since $A_1 = 2$, we finally obtain $A_4 = \left(\frac{3}{4}\right)^3 A_1 = \frac{27}{32}$.



Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; KARTHIK NATARAJAN; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

2. A square has an area of 3 square units, and a cube has a volume of 5 cubic units. Which is larger, the edge length of the square or the edge length of the cube? Justify your answer using the exact values of the two quantities.

Solution.

Let s be the side length of the square, and let c be the side length of the cube. Then $s^2 = 3$ and $c^3 = 5$, so $s^6 = 3^3 = 27$ and $c^6 = 5^2 = 25$, whence $s > c$.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; KARTHIK NATARAJAN; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

3. A certain positive integer has "6" as its last (rightmost) digit. This number is transformed into a new number by moving the "6" to the beginning of the number (leftmost position). For example, the number 1236 would be transformed to 6123, while 51476 becomes 65147. What is the smallest such positive integer for which this transformation increases the value of the number by a factor of 4?

Solution.

Say the integer is $\dots A6$, that is the rightmost two digits are A and 6. Then $4(\dots A6) = 6\dots A$. But since $4 \cdot 6 = 24$, the unit digit of $4(\dots A6)$ must be 4. Thus $A = 4$ and the integer must be $\dots B46$. Since $4 \cdot 46 = 184$ and $4(\dots B46) = 6\dots B4$, you have that $B = 8$ and the integer must be $\dots C846$. Now, $4 \cdot 846 = 3384$ and $4(\dots C846) = 6\dots C84$, so $C = 3$ and the integer is $\dots D3846$. Again, $4 \cdot 3846 = 15384$ and so we have $4(\dots D3846) = 6\dots D384$, so $D = 5$ and the integer is $\dots E53846$. Mustering all our patience, $4 \cdot 53846 = 215384$ and $4(\dots E53846) = 6\dots E5384$, so $E = 1$ and the integer is $\dots F153846$. Since $4 \cdot 153846 = 615384$—that's it, the rightmost six digits of the required integer must be 153846. Hence, the smallest such integer is 153846.

Also solved by KARTHIK NATARAJAN.

4. The members of a committee sit at a circular table so that each committee member has two neighbours. Each member of the committee has a certain number of dollars in his or her wallet. The chairperson of the committee has one more dollar than the vice chairperson, who sits on his right and has one more dollar than the member on her right, who has one more dollar than the person on his right, and so on, until the member on the chair's left is reached. The chairperson now gives one dollar to the vice chair, who gives two dollars to the member on her right, who gives three dollars to the member on his right, and so on, until the member on the chair's left is reached. There are then two neighbours, one of whom has four times as much as the other.

(a) What is the smallest possible number of members of the committee? In this case, how much did the poorest member of the committee have at first?

(b) If there are at least 12 members of the committee, what is the smallest possible number of members of the committee? In this case, how much did the poorest member of the committee have at first?

Solution.

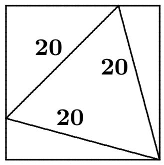
Say the committee has n members and the poorest member has x dollars. Then the chair has $x+n-1$ dollars, the vice chair has $x+n-2$ dollars, and so on. Each person, except the poorest, receives one fewer dollar than he or she gives away, so each person, except the poorest, loses one dollar. Consequently, the poorest person gains $n - 1$ dollars. The chair therefore now has $x+n-2$ dollars, the vice chair has $x+n-3$ dollars, and so on until the person who used to be the second poorest now has x dollars, while the person who used to be poorest finds him- or herself with $x+n-1$ dollars. Thus the dollar amounts have simply shifted by one person around the table. The difference in dollar amounts between neighbours is therefore always one dollar, except between the poorest and the wealthiest person. A difference of one dollar cannot quadruple an integer amount of dollars, so $x+n-1=4x$. But then $n-1=3x$, so $n-1$ is a multiple of 3.

For part (a), you can use $n=4$ and $x=1$, so the smallest possible committee has four members and the poorest member has one dollar.

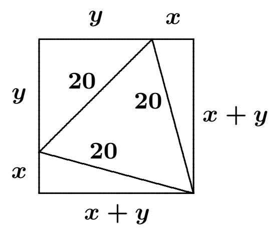
For part (b), you can use $n=13$ and $x=4$, so the smallest committee with 12 or more members has 13 members.

Also solved by KARTHIK NATARAJAN.

5. An equilateral triangle, 20 centimetres on a side, is inscribed in a square, as shown in the diagram. Find the length of the side of the square.



Solution.



The two smaller triangles in the figure both have a right angle, a hypotenuse of length 20, and a side shared with the square. Therefore the third sides of those triangles are equal, and you may label the pieces as shown.

Using the Pythagorean Theorem twice, $2y^2 = 20^2$ and $(x+y)^2 + x^2 = 20^2$, so $y = \pm 10\sqrt{2}$ and $2x^2 + 2xy + y^2 = 400$. Since y is a length, $y = 10\sqrt{2}$ and $x^2 + xy - 100 = 0$. The quadratic formula now yields

$$x = \frac{-y \pm \sqrt{y^2+400}}{2} = \frac{-10\sqrt{2} \pm 10\sqrt{6}}{2} = -5\sqrt{2} \pm 5\sqrt{6}\,.$$

Since x is also a length, $x = 5\sqrt{6} - 5\sqrt{2}$, and therefore the side length of the square is $x + y = 5\sqrt{6} + 5\sqrt{2}$.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; KARTHIK NATARAJAN; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

We have also recovered a correct solution to problem #4 of the Mathematics Association of Quebec Contest 2006 at [2008 : 66, 68] by Luyun Zhong-Qiao, Columbia International College, Hamilton, ON.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 juin 2009**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M382. *Proposé par l'Équipe de Mayhem.*

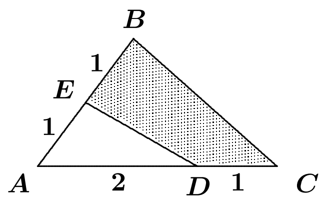
Déterminer toutes les paires (x, y) d'entiers tels que $4x^2 - y^2 = 480$.

M383. *Proposé par l'Équipe de Mayhem.*

Dans un rectangle $ABCD$, P est sur BC et Q est sur DC de sorte que $BP = 1$, $AP = PQ = 2$ et l'angle $APQ = 90°$. Déterminer la longueur de QD.

M384. *Proposé par Kunal Singh, étudiant, Kendriya Vidyalaya School, Shillong, Inde.*

Dans la figure ci-contre, le point E est sur AB et le point D est sur AC de sorte que $AE = EB = DC = 1$ et $AD = 2$. Déterminer le rapport de l'aire du quadrilatère $BCDE$ à celle du triangle ABC.



M385. *Proposé par Mihály Bencze, Brasov, Roumanie.*

En base 10, l'entier $N = 1 \cdots 114 \cdots 44$ commence avec 2009 chiffres 1 consécutifs suivis de 4018 chiffres 4 consécutifs. Montrer que N n'est pas un carré parfait.

M386. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Déterminer tous les nombres réels x pour lesquels

$$\sqrt{2+4x-2x^2} + \sqrt{6+6x-3x^2} = x^2-2x+6 .$$

M387. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

On peut mesurer la température en degrés Fahrenheit (F) ou en degrés Celsius (C). Les deux échelles sont reliées par la formule $F = 1.8C + 32$. Lorsqu'on convertit en Fahrenheit une température exprimée en Celsius par un nombre de deux chiffres, on constate parfois que, une fois les Fahrenheit arrondis à l'entier le plus proche, les chiffres des dizaines et des unités ont été permutés. Trouver toutes les valeurs entières de deux chiffres en C pour lesquelles ceci arrive.

. . . .

M382. *Proposed by the Mayhem Staff.*

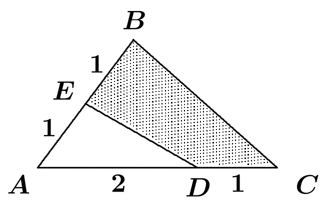
Determine all pairs (x, y) of integers for which $4x^2 - y^2 = 480$.

M383. *Proposed by the Mayhem Staff.*

In rectangle $ABCD$, P is on side BC and Q is on side DC so that $BP = 1$, $AP = PQ = 2$ and $\angle APQ = 90^\circ$. Determine the length of QD.

M384. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

In the diagram at right, the point E is on AB and the point D is on AC such that $AE = EB = DC = 1$ and $AD = 2$. Determine the ratio of the area of quadrilateral $BCDE$ to the area of triangle ABC.



M385. *Proposed by Mihály Bencze, Brasov, Romania.*

The base 10 integer $N = 1\cdots 114\cdots 44$ starts off with 2009 consecutive digits 1 followed by 4018 consecutive digits 4. Prove that N is not a perfect square.

M386. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all real numbers x for which

$$\sqrt{2+4x-2x^2} \;+\; \sqrt{6+6x-3x^2} \;=\; x^2-2x+6\,.$$

M387. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Temperature can be measured in degrees Fahrenheit (F) or in degrees Celsius (C). The two scales are related by the formula $F = 1.8C + 32$. When a two-digit integer degree temperature in Celsius is converted to Fahrenheit and rounded to the nearest integer degree, it turns out the ones and tens digits of the original Celsius temperature C sometimes switch places to give the rounded Fahrenheit equivalent. Find all two-digit integer values of C for which this occurs.

Mayhem Solutions

M344. *Proposed by the Mayhem Staff.*

Consider the square array

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

formed by listing the numbers 1 to 9 in order in consecutive rows. The sum of the integers on each diagonal is 15. If a similar array is constructed using the integers 1 to $10\,000$, what is the sum of the numbers on each diagonal?

Solution by José Hernández Santiago, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico, modified by the editor.

Let S_1 represent the sum of the integers on the diagonal that runs from top left to bottom right. Let S_2 represent the sum of the integers on the diagonal that runs from top right to bottom left.

Our array is 100 by 100. When we move one column to the right in the same row, the number increases by 1; when we move one column to the left in the same row, the number decreases by 1. When we move one row down in the same column, the number increases by 100.

On the top left to bottom right diagonal, each number is one column to the right and one row down, so is $1 + 100 = 101$ greater than the previous number on the diagonal. Thus,

$$\begin{aligned} S_1 &= 1 + 102 + 203 + \cdots + 9899 + 10000 \\ &= (100 \cdot 0 + 1) + (100 \cdot 1 + 2) + \cdots + (100 \cdot 98 + 99) \\ &\quad + (100 \cdot 99 + 100) \\ &= 100(0 + 1 + \cdots + 98 + 99) \ + \ (1 + 2 + \cdots + 99 + 100) \\ &= 100\left(\frac{1}{2}(99)(100)\right) \ + \ \frac{1}{2}(100)(101) \\ &= 100(99)(50) + 50(101) \ = \ 495000 + 5050 \ = \ 500050\,. \end{aligned}$$

On the top right to bottom left diagonal, each number is one column to the left and one row down, so is $-1 + 100 = 99$ greater than the previous number on the diagonal. Thus,

$$\begin{aligned} S_2 &= 100 + 199 + 298 + \cdots + 9802 + 9901 \\ &= (100(0) + 100) + (100(1) + 99) + \cdots + (100(98) + 2) \\ &\quad + (100(99) + 1) \\ &= 100(0 + 1 + \cdots + 98 + 99) \ + \ (100 + 99 + \cdots + 2 + 1)\,. \end{aligned}$$

Therefore, $S_2 = S_1 = 50050$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M345. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The area of isosceles $\triangle ABC$ is $q\sqrt{15}$. Given that $AB = 2BC$, express the perimeter of $\triangle ABC$ in terms of q.

Solution by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

Let $BC = x$ and $AB = 2x$. Since $\triangle ABC$ is isosceles, then $AC = BC$ or $AC = AB$. Since $\triangle ABC$ is presumably not a degenerate triangle, we take $AC = AB = 2x$ (otherwise $AC = BC = x$ and $AB = 2x$, which would mean that the triangle was a straight line segment of length $2x$).

Let D be the midpoint of BC. Since $\triangle ABC$ is isosceles, then AD is perpendicular to BC. The length of this altitude is thus

$$\begin{aligned} AD &= \sqrt{AB^2 - \left(\frac{1}{2}BC\right)^2} \ = \ \sqrt{(2x)^2 - \left(\frac{1}{2}x\right)^2} \\ &= \sqrt{4x^2 - \frac{1}{4}x^2} \ = \ \sqrt{\frac{15}{4}x^2} \ = \ \frac{\sqrt{15}}{2}x\,. \end{aligned}$$

The area of $\triangle ABC$ is $q\sqrt{15}$ and also equal to $\frac{1}{2}(BC)(AD) = \frac{1}{2}x\left(\frac{\sqrt{15}}{2}x\right)$, so that $q\sqrt{15} = \frac{\sqrt{15}}{4}x^2$, or $x^2 = 4q$ and hence $x = 2\sqrt{q}$.

Thus, $AB + BC + CA = 2x + x + 2x = 5x = 10\sqrt{q}$ is the perimeter of the triangle.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 2 incorrect solutions submitted.

M346. *Proposed by the Mayhem Staff.*

Without using a calculator, find the number of digits in the integer 2^{80}.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

We determine the positive integer n for which $10^n \leq 2^{80} < 10^{n+1}$. This will imply that the integer 2^{80} has $n + 1$ digits.

Since $1000 = 10^3 < 2^{10} = 1024$, raising both sides to the exponent 8 we obtain $10^{24} < 2^{80}$.

Also, $65536 = 2^{16} < 10^5 = 100000$, hence $2^{80} < 10^{25}$ upon raising both sides to the exponent 5.

Thus, $10^{24} < 2^{80} < 10^{25}$, whence 2^{80} has 25 digits.

Also solved by RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 5 incomplete solutions submitted.

Singh submitted a solution that included more of a thought process that led to the same outcome as that of Ajanovic.

M347. *Proposed by the Mayhem Staff.*

Four positive integers a, b, c, and d are such that

$$\begin{aligned} (a+b+c)d &= 420\,, \\ (a+c+d)b &= 403\,, \\ (a+b+d)c &= 363\,, \\ (b+c+d)a &= 228\,. \end{aligned}$$

Find the four integers.

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

First, we write out the prime factorizations of the right sides of the equations:

$$\begin{aligned} 420 &= 2^2 \cdot 3 \cdot 5 \cdot 7\,, & 403 &= 13 \cdot 31\,, \\ 363 &= 3 \cdot 11^2\,, & 228 &= 2^2 \cdot 3 \cdot 19\,. \end{aligned}$$

Next, we subtract the second equation from the first equation to obtain $(a+c)d + bd - (a+c)b - bd = 420 - 403$, or $(a+c)(d-b) = 17$. Since a, b, c, and d are positive integers, then $a + c > 1$. Since $(a+c)$ is a divisor of 17, which is a prime number, then $a + c = 17$ and so $d - b = 1$.

From the third equation, c is odd since it is a divisor of 363. Since $a + c = 17$, then $a = 17 - c$ so a is even. Also, $1 \le c \le 15$ and $2 \le a \le 16$.

Since a is an even divisor of $228 = 2^2 \cdot 3 \cdot 19$ between 2 and 16 inclusive, then a could be 2, 4, 6, or 12. Since c is an odd divisor of $363 = 3 \cdot 11^2$ between 1 and 15 inclusive, then c could be 1, 3, or 11. Since $a + c = 17$, then a must equal 6 and c must equal 11.

So we know that $a = 6$, $c = 11$, and $d = b + 1$. Substituting these into the second equation, we obtain

$$\begin{aligned} (6 + 11 + b + 1)b &= 403\,; \\ (b+18)b &= 403\,; \\ b^2 + 18b - 403 &= 0\,; \\ (b-13)(b+31) &= 0\,. \end{aligned}$$

Since b is a positive integer, we have $b = 13$, whence $d = 14$.

Therefore, $(a, b, c, d) = (6, 13, 11, 14)$, which we can check satisfies each of the four equations.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 2 incomplete solutions and 1 incorrect solution submitted.

M348. *Proposed by the Mayhem Staff.*

The perimeter of a sector of a circle is 12 (the perimeter includes the two radii and the arc). Determine the radius of the circle that maximizes the area of the sector.

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia, modified by the editor.

Let r be the radius of the circle, let θ be the central angle of the sector measured in radians, and let A be the area of the sector.

The length of the arc of the sector is $\frac{\theta}{2\pi}(2\pi r) = \theta r$. Since the perimeter of the sector is 12, then the arc length of the sector also equals $12 - 2r$ (by subtracting the total length of the two radii). Therefore, $\theta r = 12 - 2r$.

Also, the area of the sector is $A = \frac{\theta}{2\pi}\left(\pi r^2\right) = \frac{1}{2}\theta r^2$. Thus, we have $A = \frac{1}{2}(\theta r)r = \frac{1}{2}(12 - 2r)r = 6r - r^2$.

The vertex of the parabola defined by this expression corresponds to the maximum value for A. This occurs when $r = -\frac{6}{2(-1)} = 3$. Therefore, the radius of the circle that maximizes the area of the sector is $r = 3$ (which gives a maximum area of $A = 9$).

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M349. *Proposed by the Mayhem Staff.*

(a) Find all ordered pairs of integers (x, y) with $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$.

(b) How many ordered pairs of integers (x, y) are there with

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}?$$

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

First, in (a) note that x cannot equal 0 or 5, and y cannot equal 0 or 5, since neither the denominators nor the individual fractions can equal 0. Solving the equation for y, we find that $\frac{1}{y} = \frac{1}{5} - \frac{1}{x} = \frac{x-5}{5x}$ or

$$y = \frac{5x}{x-5} = \frac{(5x-25)+25}{x-5} = 5 + \frac{25}{x-5}.$$

Since y is an integer, then so is $\frac{25}{x-5}$. Hence, $x - 5$ divides 25 and the possible values of $x-5$ are ±1, ±5, and ±25. Since $x \neq 0$, then $x-5 \neq -5$. The other values of $x - 5$ yield 6, 4, 10, 30, or -20 as possible values of x.

Each of these values for x generates an ordered pair of integers (x, y) which is a solution to the equation. These are $(6, 30)$, $(4, -20)$, $(10, 10)$, $(30, 6)$, and $(-20, 4)$.

In (b), we note that x cannot equal 0 or 1200, and y cannot equal 0 or 1200. Proceeding as in part (a) and solving the equation for y, we find that

$$y = \frac{1200x}{x-1200} = 1200 + \frac{1440000}{x-1200}.$$

Similarly, each divisor of 1440000 other than -1200 gives a possible value for $x - 1200$ and hence for x. Thus, the number of ordered pairs (x, y) satisfying $\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}$ is 1 less than the number of integer divisors of 1440000.

Now $1440000 = 2^8 \cdot 3^2 \cdot 5^4$, so 1440000 has $(8+1)(2+1)(4+1) = 135$ positive integer divisors. The number of integer pairs that are solutions is therefore $270 - 1 = 269$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam (part (a) only); JACLYN CHANG, student, Western Canada High School, Calgary, AB (part (a) only); and NECULAI STANCIU, Saint Mucenic Sava Technological High School, Berca, Romania. There were 4 incorrect solutions submitted.

Problem of the Month

Ian VanderBurgh

Et tu, Brute force?

Problem (2008 Cayley Contest) The average value of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2$$

over all possible arrangements (a,b,c,d,e,f,g) of the seven numbers 1, 2, 3, 11, 12, 13, 14 is

(A) 398 (B) 400 (C) 396 (D) 392 (E) 394

We learn how to calculate averages early on in our mathematics careers – add up all of the values and divide by the number of values. This isn't so hard when you're trying to calculate the average of your six marks at school, but can be a real pain if there are significantly more values to consider.

In this problem there are $7! = 7(6)(5)(4)(3)(2)(1) = 5040$ possible arrangements of the seven numbers 1, 2, 3, 11, 12, 13, 14. We could try to calculate the 5040 required values of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2\,,$$

add them up, and divide by 5040. I think that you will agree that, in principle, we could do this calculation (the hard way!) by hand. Yes, it would take a very long time. Yes, we would be liable to make a whole bunch of arithmetic mistakes. Yes, it would be extremely annoying. But, yes, we could do it, as the underlying mathematics is not that hard. There must be a better way!

One approach would be to try to add the 5040 values without actually having to calculate the 5040 values. If we did this, we could then divide the total by 5040 and get our answer.

Put another way, we can think of the brute force approach of computing the value of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2 \qquad (*)$$

for each of the 5040 arrangements and then adding these values as "adding across then adding down". (In other words, compute each value and then add up the column of values.) But addition is commutative, that is, we don't have to add the values in the given order to get the correct total, so we could even break up the values into components and add these in separately. Let's give this a try.

Solution To determine the average value of the expression in $(*)$ we determine the sum of the values of this expression over all possible arrangements, and then divide by the number of arrangements. We determine the sum of all of the values of $(*)$ by examining the contribution of each possible term.

Let x and y be 2 of the 7 given numbers. In how many of these arrangements are x and y adjacent? Treat x and y as a single unit (xy) with 5 other numbers to be placed on either side of, but not between, xy. This gives 6 things (xy as a single unit and the 5 remaining numbers) to arrange, which can be done in $6(5)(4)(3)(2)(1)$ or $6!$ ways. But y could be followed by x, so there are $2(6!)$ arrangements with x and y adjacent, since there are the same number of arrangements with x followed by y as there are with y followed by x.

Since we want the sum of all of the possible values of $(*)$, we can calculate the total contribution of each possible term $(x-y)^2$ and add up these contributions. When we add up the values of $(*)$ over all possible arrangements, the term $(x-y)^2$ (which is equal to $(y-x)^2$) will occur $2(6!)$ times. This is true for any pair x and y. Thus, the sum of all of the possible values of $(*)$ must be equal to $2(6!)$ times the sum of all possible values of $(x-y)^2$.

The sum of all possible values of $(x-y)^2$ is

$$\begin{array}{ccccccccccccc} 1^2 & + & 2^2 & + & 10^2 & + & 11^2 & + & 12^2 & + & 13^2 & & \\ & + & 1^2 & + & 9^2 & + & 10^2 & + & 11^2 & + & 12^2 & & \\ & & & + & 8^2 & + & 9^2 & + & 10^2 & + & 11^2 & & \\ & & & & & + & 1^2 & + & 2^2 & + & 3^2 & & \\ & & & & & & & + & 1^2 & + & 2^2 & & \\ & & & & & & & & & + & 1^2 & = & 1372\,. \end{array}$$

Here, we have paired 1 with each of the 6 larger numbers, then 2 with each of the 5 larger numbers, and so on. We only need to pair each number with all of the larger numbers because we have accounted for the reversed pairs in our method above.

Therefore, $2(6!)$ times the sum of $(x-y)^2$ over all choices of x and y with $x<y$, divided by $7!$ is the average value. This average value is

$$\frac{2(6!)(1372)}{7!} = \frac{2(1372)}{7} = 392\,.$$

This is one of the powerful things about mathematics – being able to turn a problem that looks as if it is difficult to solve in a short period of time into one that has a reasonably quick solution. We'll see another such problem in a couple of months.

There is an interesting footnote to this problem. When creating a problem, but especially a multiple choice problem, it's not good to be able to get the right answer for the wrong reason. As the CEMC was developing this problem, the fact it was multiple choice meant fiddling the actual numbers to avoid this issue. Try redoing this problem with 1, 2, 3, 4, 5, 6, 7, 8 and a suitably modified expression. You should get the answer 84. This is also the answer you'd get by assuming that the average value of any one of the squared terms is the average of $(5-4)^2$, $(6-3)^2$, $(7-2)^2$, and $(8-1)^2$. This would be a curious wrong way to get the right answer.

THE OLYMPIAD CORNER

No. 276

R.E. Woodrow

We begin this *Corner* with the 19th Korean Mathematical Olympiad, 2006. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for our use.

The 19th Korean Mathematical Olympiad 2006
March 25–26, 2006

First Day

1. Let ABC be a triangle with $\angle B \neq \angle C$. The incircle I of a triangle ABC touches the sides BC, CA, and AB at the points D, E, and F, respectively. Let P be the point on AD and the incircle I that is different from D.

Let Q be the intersection of the line EF and the line passing through P and perpendicular to AD, and let X and Y be the intersections of the line AQ with the lines DE and DF, respectively. Show that the point A is the midpoint of XY.

2. For a positive integer a, let S_a be the set of all primes p for which there exists an odd integer b such that $\left(2^{2^a}\right)^b - 1$ is divisible by p. For any positive integer a, prove that there exist infinitely many primes that are not in S_a.

3. Three schools A, B, and C participate in a chess tournament with five students from each school. Let

$$a_1,\, a_2,\, \ldots,\, a_5;\; b_1,\, b_2,\, \ldots,\, b_5;\; c_1,\, c_2,\, \ldots,\, c_5$$

be the list of the players from schools A, B, and C, respectively. Let P_A, P_B, and P_C be the respective scores that schools A, B, and C receive when the tournament is over, following the rules described below. Find the remainder of the number of possible triples (P_A, P_B, P_C) when it is divided by 8.

(a) Players from each school have matches in order from the first student, and if a player loses one match, then he or she is eliminated from the tournament. The first match is between players a_1 and b_1.

(b) If y_j from school Y defeats x_i from school X, then y_j plays the next available player in the remaining school Z (different from X, Y). If all players from school Z have been eliminated, then y_j plays a match with x_{i+1}. The tournament is over when two schools are eliminated.

(c) If x_i wins a match, then school X adds 10^{i-1} points to its score.

Second Day

4. Given three distinct real numbers a_1, a_2, and a_3, define three real numbers b_1, b_2, and b_3 as follows

$$b_j \;=\; \left(1+\frac{a_j a_i}{a_j - a_i}\right)\left(1+\frac{a_j a_k}{a_j - a_k}\right), \qquad \text{where} \qquad \{i,j,k\} = \{1,2,3\}\,.$$

Prove that

$$1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \;\le\; (1+|a_1|)(1+|a_2|)(1+|a_3|)\,.$$

When does equality hold?

5. In a convex hexagon $ABCDEF$ the triangles ABC, CDE, and EFA are similar. Find conditions on these three triangles under which triangle ACE is equilateral if and only if triangle BDF is equilateral.

6. A positive integer N is said to be an n-*good number* if it has the following two properties:

(a) N is divisible by at least n distinct primes, and

(b) there exist distinct positive divisors 1, x_2, x_3, $\dots$, x_n of N such that $1 + x_2 + \cdots + x_n = N$.

Show that there exists an n-good number for each $n \ge 6$.

Now we give the problems of the 55th Czech and Slovak Mathematical Olympiad 2006. We again thank Robert Morewood, Canadian Team Leader to the IMO in Slovenia 2006, for collecting them for the *Corner*.

55th Czech and Slovak Mathematical Olympiad 2006
Third Round, Litoměřice, March 26–29, 2006

1. (P. Novotný) A sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers is defined for $n \ge 1$ by $a_{n+1} = a_n + b_n$, where b_n is obtained from a_n by reversing its digits (the number b_n may start with zeroes). For instance if $a_1 = 170$, then $a_2 = 241$, $a_3 = 383$, $a_4 = 766$, $\dots$. Decide whether a_7 can be a prime number.

2. (J. Šimša) Let m and n be positive integers such that

$$(x+m)(x+n) \;=\; x+m+n$$

has at least one integer solution. Prove that $\frac{1}{2} < \frac{m}{n} < 2$.

3. (T. Jurík) Triangle ABC is not equilateral, and the angle bisectors at A and B intersect the sides BC and AC at K and L, respectively. Let S be the incentre, O be the circumcentre, and V be the orthocentre of triangle ABC. Prove that the following statements are equivalent

(a) The line KL is tangent to the circumcircles of triangles ALS, BVS, and BKS.

(b) The points A, B, K, L, and O are concyclic.

4. (J. Švrček) A segment AB is given in the plane. Find the locus of the centroids of all acute triangles ABC for which the following holds: the vertices A and B, the orthocentre V, and the centre S of the incircle of the triangle ABC are concyclic.

5. (M. Panák) Find all triples (p,q,r) of distinct prime numbers such that

$$p|(q+r)\,,\qquad q|(r+2p)\,,\qquad r|(p+3q)\,.$$

6. (J. Švrček, P. Calábek) Solve in real numbers the system of equations

$$\begin{aligned}\tan^2 x + 2\cot^2 2y &= 1\,,\\ \tan^2 y + 2\cot^2 2z &= 1\,,\\ \tan^2 z + 2\cot^2 2x &= 1\,.\end{aligned}$$

As a final set of problems for this number we give the Olympiade suisse de mathématiques, Tour final, 2005. Thanks again go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for obtaining them for our use.

Olympiade suisse de mathématiques 2005
Tour final

1. Soit ABC un triangle et D, E, F les milieux des côtés BC, CA, AB respectivements. Les médianes AD, BE et CF se coupent en S, le centre de gravité. Au moins deux des quadrilatères

$$AFSE\,,\qquad BDSF\,,\qquad CESD$$

sont des quadrilatères inscrits. Montrer que le triangle ABC est équilatéral.

2. Soient $4n$ points alignés tels que $2n$ d'entre eux sont blancs et $2n$ sont noirs. Montrer qu'il y a parmi eux une suite de $2n$ points consécutifs avec n points noirs et n points blancs.

3. Pour tout $a_1, \ldots, a_n > 0$, prouver l'inégalité suivante et déterminer tous les cas d'égalité

$$\sum_{k=1}^{n} k a_k \;\leq\; \binom{n}{2} \;+\; \sum_{k=1}^{n} a_k^k\,.$$

4. Déterminer tous les ensembles M de nombres naturels ayant la propriété suivante : Si a et b (non nécessairement distincts) sont des éléments de M, alors $\dfrac{a+b}{\gcd(a,b)}$ se trouve également dans M.

5. "Tailler" un n-gone convexe consiste à choisir deux côtés adjacents AB et BC et à les remplacer par les segments AM, MN, NC, où $M \in AB$ et $N \in BC$ sont des points arbitraires à l'intérieur des segments. Autrement dit, on coup un sommet et on obtient un $(n+1)$-gone.

On part d'un hexagone régulier P_6 d'aire 1 et on le taille pour obtenir une suite de polygones convexes P_6, P_7, P_8, ... Montrer que l'aire de P_n est plus grand que $\frac{1}{2}$ pour tout $n \geq 6$, indépendamment de la façon dont on a taillé.

6. Soient a, b, c des nombres réels positifs avec $abc = 1$. Déterminer toutes les valeurs que peut prendre la somme

$$\frac{1+a}{1+a+ab} \;+\; \frac{1+b}{1+b+bc} \;+\; \frac{1+c}{1+c+ca}\,.$$

7. Soit $n \geq 1$ un nombre naturel. Déterminer toutes les solutions entières positives de l'équation

$$7 \cdot 4^n \;=\; a^2 + b^2 + c^2 + d^2\,.$$

8. Soit ABC un triangle acutangle. Soient M et N des points arbitraires sur les côtés AB et AC respectivement. Les cercles de diamètre BN et CM se coupent en P et Q. Montrer que les points P, Q et l'orthocentre du triangle ABC se trouvent sur une même droite.

9. Trouver toutes les fonctions $f : \mathbb{R}^+ \to \mathbb{R}^+$, vérifiant la condition suivante pour tout $x,y > 0$:

$$f\big(yf(x)\big)(x+y) \;=\; x^2\big(f(x)+f(y)\big)\,.$$

10. Un tournoi de football rassemble n ($n > 10$) équipes. Chaque équipe joue une seule fois contre toutes les autres. Une victoire vaut 2 points, un nul rapporte 1 point et une défaite aucun point. Le tournoi terminé, on constate que chaque équipe a gagné la moitié de ses points contre les 10 plus mauvaises équipes (en particulier chacune de ces dernières a fait la moitié de ses points contre les 9 autres). Trouver toutes les valeurs possibles pour n et, pour chacune d'elles, donner un exemple d'un tel tournoi.

Next we give a solution to a problem of the XXXI Russian Olympiad which was given at [2008 : 20] and for which one other problem was solved in the December Corner at [2008 : 466-467].

1. *(I. Rubanov)* Let $\{a_1, a_2, \ldots, a_{50}, b_1, b_2, \ldots, b_{50}\}$ be a set of 100 real numbers. Suppose that the equation

$$|x - a_1| + \cdots + |x - a_{50}| \;=\; |x - b_1| + \cdots + |x - b_{50}|$$

has N solutions (N is finite). Find the maximal value of N.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Let

$$f(x) \;=\; |x - a_1| + \cdots + |x - a_{50}| - |x - b_1| - \cdots - |x - b_{50}|$$

and $\{c_1, \ldots, c_{100}\} = \{a_1, \ldots, a_{50}, b_1, \ldots, b_{50}\}$ with $c_1 < \cdots < c_{100}$. Then we have $f(x) = \epsilon_1|x - c_1| + \epsilon_2|x - c_2| + \cdots + \epsilon_{100}|x - c_{100}|$, where $\epsilon_i = 1$ if $c_i = a_j$ for some j and $\epsilon_i = -1$ if $c_i = b_k$ for some k. Thus, $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{100} = 0$, and we have

$$\begin{aligned} f(c_{k+1}) - f(c_k) &= \epsilon_1(c_{k+1} - c_k) + \cdots + \epsilon_k(c_{k+1} - c_k) \\ &\quad - \epsilon_{k+1}(c_{k+1} - c_k) - \cdots - \epsilon_{100}(c_{k+1} - c_k) \\ &= (\epsilon_1 + \cdots + \epsilon_k - \epsilon_{k+1} - \cdots - \epsilon_{100})(c_{k+1} - c_k) \\ &= 2(\epsilon_1 + \cdots + \epsilon_k)(c_{k+1} - c_k) \end{aligned}$$

For $x \le c_1$ we have $f(x) = (a_1 + \cdots + a_{50}) - (b_1 + \cdots + b_{50})$ while for $x \ge c_{100}$, we have$f(x) = (b_1 + \cdots + b_{50}) - (a_1 + \cdots + a_{50})$ So f is constant on each of the intervals $(-\infty, c_1]$ and $[c_{100}, +\infty)$. Since N is finite, $f(x)$ is nonzero in these intervals with opposite sign. Without loss of generality, we suppose that $f(x) > 0$ for $x \le c_1$. The graph of f in $[c_1, c_{100}]$ consists of line segments connecting the points $(c_1, f(c_1))$, $(c_2, f(c_2))$, …, $(c_{100}, f(c_{100}))$, so f has at most one root in each interval $[c_1, c_2]$, …, $[c_{99}, c_{100}]$. The function f cannot have a root in each of the intervals $[c_k, c_{k+1}]$ and $[c_{k+1}, c_{k+2}]$. Otherwise, $f(c_{k+1})$ will have a different sign from $f(c_k)$ and $f(c_{k+2})$. If for example $f(c_k) > 0$, $f(c_{k+2}) > 0$, and $f(c_{k+1}) < 0$ then

$$\begin{aligned} f(c_{k+1}) - f(c_k) &= 2(\epsilon_1 + \cdots + \epsilon_k)(c_{k+1} - c_k) \;<\; 0\,; \\ f(c_{k+2}) - f(c_{k+1}) &= 2(\epsilon_1 + \cdots + \epsilon_k + \epsilon_{k+1})(c_{k+2} - c_k) \;>\; 0\,, \end{aligned}$$

hence the numbers $\epsilon_1 + \cdots + \epsilon_k$ and $\epsilon_1 + \cdots + \epsilon_k + \epsilon_{k+1}$ have opposite sign. However, this is impossible, since they are both integers differing by ± 1.

Therefore, f has at most 50 roots. We will prove that 50 roots for f is impossible. Suppose on the contrary that f has 50 roots. In that case f will have the following "sign diagram"

x		c_1		c_2		c_3		c_4		c_5		$\cdots$		c_{97}		c_{98}		c_{99}		c_{100}	
$f(x)$	$+$		$\downarrow$		$-$		$\uparrow$		$+$			$\cdots$			$\downarrow$		$-$		$\uparrow$		$+$

where f is constant in each interval where it is positive or negative. Thus, if f has 50 roots, then f has the same sign in $(-\infty, c_1]$ and $[c_{100}, +\infty)$, a contradiction. An example of f with 49 roots is

$$\begin{aligned} f(x) &= \big(|x-1| - |x-2| - |x-3| + |x-4|\big) \\ &\quad + \big(|x-5| - |x-6| - |x-7| + |x-8|\big) \\ &\quad + \cdots + \big(|x-97| - |x-98| - |x-99| + |x-99.5|\big)\,, \end{aligned}$$

with the corresponding values

$$\begin{array}{llll} f(1) = -0.5\,, & f(2) = 1.5\,, & f(3) = 1.5\,, & f(4) = -0.5\,, \\ f(5) = -0.5\,, & f(6) = 1.5\,, & f(7) = 1.5\,, & f(8) = -0.4\,, \\ \dots & & & \\ f(97) = -0.5\,, & f(98) = 1.5\,, & f(99) = 1.5\,, & f(99.5) = 0.5\,, \end{array}$$

so that f has 49 roots, one in each of the intervals $[1,2]$, $[3,4]$, $\dots$, $[97,98]$.

Therefore, the maximal value of N is 49.

Next we give readers' solutions to the Hungarian National Olympiad 2004-2005, Specialized Mathematical Classes, First and Final Rounds, given at [2008 : 147]

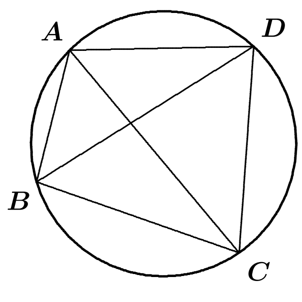
First Round

1. The quadrilateral $ABCD$ is cyclic. Prove that

$$\frac{AC}{BD} = \frac{DA\cdot AB\ +\ BC\cdot CD}{AB\cdot BC\ +\ CD\cdot DA}\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's write up.

This is Ptolemy's theorem about cyclic quadrilaterals. Here is a proof:



We denote by $[XY\cdots Z]$ the area of the polygon $XY\cdots Z$. Using the known fact that $[ABC] = (AB\cdot BC\cdot CA)/4R$, where R is the circumradius, we have the two equations

$$\begin{aligned} [ABCD] &= [ABC] + [ADC] \\ &= \frac{AB\cdot BC\cdot CA}{4R} + \frac{AD\cdot DC\cdot AC}{4R}\,; \end{aligned}$$

$$[ABCD] = [ABD] + [BCD] = \frac{AB\cdot BD\cdot AD}{4R} + \frac{BC\cdot CD\cdot BD}{4R}\,.$$

It follows that

$$AC(AB\cdot BC + CD\cdot DA) = BD(DA\cdot AB + BC\cdot CD)\,.$$

2. How many real numbers x are there in the interval $0 < x < 2004$ such that $x + \lfloor x^2 \rfloor = x^2 + \lfloor x \rfloor$? (Here $\lfloor c \rfloor$ denotes the greatest integer k such that $k \le c$.)

Solved by Oliver Geupel, Brühl, NRW, Germany; and Pavlos Maragoudakis, Pireas, Greece. We give Geupel's solution, modified by the editor.

We will count the number of roots of $f(x) = x - \lfloor x \rfloor + \lfloor x^2 \rfloor - x^2$ in each of the intervals $I_0 = (0,1)$ and $I_n = \left[\sqrt{n}, \sqrt{n+1}\right)$ for integers n with $1 \le n < 2004^2$.

For each $x \in I_0$ we have $f(x) = x - x^2 > 0$, so f has no roots in I_0.

Let n be a perfect square, say $n = N^2$ for a positive integer N. Then $f(x) = x - N + N^2 - x^2$ on I_n, the function f is decreasing on I_n, and $f\left(\sqrt{n}\right) = 0$. Hence, in this case, f has exactly one root in I_n.

Now let $n + 1$ be a perfect square, say $n + 1 = N^2$ for an integer $N > 1$. If $x \in I_n$, then $x^2 \in [n, n+1) = \left[N^2 - 1, N\right)$ and also it follows that $x \in \left[\sqrt{n}, \sqrt{n+1}\right) \subset [N-1, N)$. Thus,

$$f(x) \;=\; x - (N-1) + (N^2 - 1) - x^2 \;=\; N^2 - N + x - x^2$$

for $x \in I_n$. Since $x - x^2$ is decreasing for $x > 1$ and $1 < x < \sqrt{n+1}$ for any $x \in I_n$, it follows that $f(x) > N^2 - N + \sqrt{n+1} - (n+1) = 0$ for any $x \in I_n$. Therefore, in this case, f has no root in I_n.

Finally, assume that neither n nor $n+1$ are perfect squares, that is, assume that $N^2 < n < n+1 < (N+1)^2$. Then $f(x) = n - N + x - x^2$ for $x \in I_n$ and we see that f is decreasing on I_n, so f has at most one root in I_n. At the left endpoint of I_n the function takes the value

$$f\left(\sqrt{n}\right) \;=\; n - N + \sqrt{n} - n \;=\; \sqrt{n} - N \;>\; 0\,.$$

The limit of $f(x)$ from the left towards the right boundary of I_n is negative:

$$\begin{aligned} \lim_{x \to (\sqrt{n+1})^-} f(x) &= \sqrt{n+1} - N + n - (n+1) \\ &= \sqrt{n+1} - (N+1) \;<\; 0\,. \end{aligned}$$

Hence, there is an $x_0 \in I_n$ such that $f(x_0) < 0$. Since f is continuous on $\left[\sqrt{n}, x_0\right]$, by the Intermediate Value Theorem f has at least one root in I_n. Therefore, in this case, f has exactly one root in I_n.

In summary, we have 2004^2 intervals I_0, I_1, $\ldots$, I_{2004^2-1}. The 44 intervals I_{N^2-1}, where $N = 1$, 2, $\ldots$, 44, contain no root of f. The other intervals each contain just one root. Consequently, the number of solutions to the original equation is $2004^2 - 44 = 4015972$.

3. Let $s(n)$ be the sum of those positive divisors of n that are less than n. A triple of three integers, (a,b,c), is a *friendly* triple if $1 < a \le b \le c$ and $s(a) + s(b) = c$, $s(b) + s(c) = a$, and $s(c) + s(a) = b$. Determine all friendly triples (a,b,c) where c is even.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Let (a,b,c) be a friendly triple with c even. If $c > 2$, then $\frac{c}{2}$ is a divisor of c with $\frac{c}{2} \neq 1$, so $s(c) > \frac{c}{2} + 1$. Since $s(a) + s(b) + 2s(c) = a + b$ and $s(a) + s(b) = c$ we have $s(c) = \frac{a+b-c}{2}$, hence $\frac{a+b-c}{2} \geq \frac{c}{2} + 1$ and $a + b \geq 2c + 2$, a contradiction because $a \leq c$ and $b \leq c$. Thus, $c = 2$ and $1 < a \leq b \leq 2$, so necessarily $(a,b,c) = (2,2,2)$, which is a friendly triple.

4. The set A of positive integers has k elements. If the positive integers x and y are not in A, then $2x$, $2y$, and $x+y$ are also not in A. The sum of the elements in A is s. Find the maximum possible value of s.

Solved by Oliver Geupel, Brühl, NRW, Germany; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Pavlos Maragoudakis, Pireas, Greece. We give Maragoudakis' solution.

Let $A = \{a_1, a_2, a_3, \dots, a_k\}$, with $a_1 < a_2 < \cdots < a_k$. If $a_i \geq 2i$ for some i, then a_i can be written as $x + y$ or $2x$ with x and y positive integers in at least i different ways. Since $a_i \in A$, we have that $x \in A$ or $y \in A$ for each of these ways. Hence, there are at least i positive integers in A less than a_i, a contradiction. Therefore, $a_i \leq 2i - 1$ for each i.

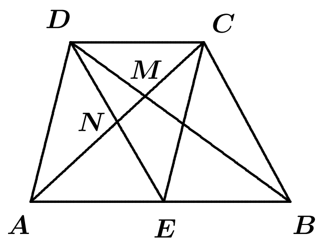
The set $\{1, 3, \dots, 2k-1\}$ has the desired property and the maximum sum, which is $1 + 3 + \cdots + 2k - 1 = k^2$.

Final Round

1. Let $ABCD$ be a trapezoid with parallel sides AB and CD. Let E be a point on the side AB such that EC and AD are parallel. Further, let the area of the triangle determined by the lines AC, BD, and DE be t, and the area of ABC be T. Determine the ratio $AB : CD$, if $t : T$ is maximal.

Solved by Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's write up.

Let M be the intersection of diagonals AC and BD, and N be the intersection of AC and DE.



Since $EC||AD$ and $AE||DC$, the quadrilateral $AECD$ is a parallelogram, hence N is the midpoint of AC.

Let $a = AB$, $b = CD$. Since $\triangle MCD$ is similar to $\triangle MAB$ we obtain

$$\frac{MC}{MA} = \frac{b}{a} \Leftrightarrow \frac{NC - MN}{NA + MN} = \frac{b}{a} \Leftrightarrow \frac{AC - 2MN}{AC + 2MN} = \frac{b}{a}.$$

By the last equality and some algebra we obtain $\frac{MN}{AC} = \frac{a-b}{2(a+b)}$. If h_t is the altitude of $\triangle DMN$ from D, and h_T is the altitude of $\triangle ABC$ from B, we have

$$\frac{h_t}{h_T} = \frac{DM}{MB} = \frac{b}{a}.$$

It follows that

$$\frac{t}{T} = \frac{MN \cdot h_t}{AC \cdot h_T} = \frac{b}{a} \cdot \frac{a-b}{2(a+b)} = \frac{\frac{a}{b}-1}{2\frac{a}{b}(\frac{a}{b}+1)},$$

and we have to find $\frac{a}{b} = x$ when $\frac{x-1}{x(x+1)}$ takes its maximum value.

Setting $f(x) = \frac{x-1}{x(x+1)}$, we have $f'(x) = \frac{-x^2+2x+1}{x^2(x+1)^2}$, hence f takes its maximum value for $x = 1+\sqrt{2}$. Hence, $t : T$ is maximized when $AB : CD = 1+\sqrt{2}$.

3. Haydn and Beethoven celebrate the birthday of Mozart with a game. They take numbers alternately according to the following rules. First Haydn takes the number 2. The next player can take the sum or the product of any two numbers which were taken earlier (it is possible to choose just one number twice, thus taking the square of it). The numbers which are taken must be distinct and smaller than 1757. The winner is the player who takes the number 1756. Which player has a winning strategy?

Solution by Oliver Geupel, Brühl, NRW, Germany.

Beethoven can win. This remains true if we replace 1756 by $N = 4p$, where p is any odd prime ($1756 = 4 \cdot 439$, and 439 is prime). To prove this, let $\{h_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ denote the sequences of numbers taken by Haydn and Beethoven, H and B, respectively. We have $h_1 = 2$, $b_1 = 4$, and we denote by $T = \{h_1, b_1, h_2, b_2, \ldots, h_n\}$ the set of all numbers taken after n moves by H and $n-1$ moves by B. Let $P_k = \{2k, N-2k\}$ for $1 \leq k \leq p-1$, and consider the following strategy for B's n^{th} move, where $n \leq (p-1)/2$:

(i) If there is an index k with $1 \leq k \leq p$ and such that $P_k \subseteq T$ or if $a_n = 2p$, then B takes $b_n = N$ and he wins.

(ii) Otherwise, B takes $b_n = 2m$, where $m = \min\{k : P_k \cap T = \emptyset\}$.

It suffices to prove the following statements.

(a) By the rules of the game, B may actually choose b_n as described in (i) and (ii).

(b) Subsequently, $h_{n+1} \neq N$; hence, H cannot win.

(c) The game is over for some $n \leq (p-1)/2$.

Proof of (a). If (i) is the case, then clearly B can take $b_n = N$ and win. If B is faced with case (ii), then $|T| = 2(n-1)+1 < p-1$; thus, m is well defined. Then at least half of the $m-3$ sets $P_3, \ldots, P_{m-1}$ have nonempty intersection with the set $\{b_1, b_2, \ldots, b_{n-1}\}$ (this follows from B's strategy by an easy induction). Hence, the set $U = \{2, 4, 6, \ldots, 2(m-1)\} \cap T$ has at least $2 + \lceil (m-3)/2 \rceil = \lceil (m+1)/2 \rceil$ elements. We define $\lceil (m+1)/2 \rceil$ pairs $Q_k = \{2k, 2m-2k\}$ for $1 \le k \le \lceil (m-1)/2 \rceil$. By the Pigeonhole Principle, for an appropriate index k with $(2 \le k \le \lceil (m-1)/2 \rceil)$, we have $|Q_k \cap U| = 2$. Thus, $b_n = 2m = 2k + (2m-2k)$, where $2k$ and $2m-2k$ are in $Q_k \cap U \subseteq T$.

Proof of (b). Assume on the contrary that there are $c, d \in T \cup \{b_n\}$ such that $N = c + d$ or $N = cd$. If $N = c + d$, then there exists an integer k with $\{c, d\} = P_k$, $1 \le k \le p-1$. However, this is impossible since by (ii) B could not have taken both c and d. If $N = cd$, then $\{c, d\} = \{2, 2p\}$, which is also impossible since the number $2p$ was not taken earlier.

Proof of (c). We consider the number $d = |\{k : P_k \cap T = \emptyset; 1 \le k \le p-1\}|$. Initially $d = p-1$ and d decreases by 1 each time B moves as in (ii), hence, after at most $(p-1)/2$ numbers are taken by B we shall have $d = 0$, and then B wins by moving as in (i).

Next we present our readers' solutions to problems of the Hungarian National Olympiad 2004-2005, Grades 11-12, Second and Final Rounds, given at [2008 : 148-149].

Second Round

1. Find all real solutions to the following system of equations:

$$\begin{aligned} \sqrt{x+y} + \sqrt{x-y} &= 10\,, \\ x^2 - y^2 - z^2 &= 476\,, \\ 2^{(\log|y| - \log z)} &= 1\,. \end{aligned}$$

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give Wang's solution.

Label the equations by (1), (2), and (3), in the order they are given.

From (3) we see that $z = |y|$, so $z^2 = y^2$. Substituting this into (2) we have

$$x^2 - 2y^2 = 476\,. \qquad (4)$$

Squaring each side of (1), we have $2x + 2\sqrt{x^2 - y^2} = 100$, so that

$$x^2 - y^2 = (50-x)^2 = x^2 - 100x + 2500\,,$$

and hence $y^2 = 100x - 2500$. Substituting this into (4) and simplifying, we have $x^2 - 200x + 4524 = 0$ or $(x-26)(x-174) = 0$. Hence, $x = 26$ or $x = 174$.

However, if $x = 174$, then regardless of the sign of y, we would have $\sqrt{x+y} + \sqrt{x-y} > \sqrt{174} > 10$, contradicting (1). Therefore, $x = 26$, and from (4) we deduce that $y = \pm 10$.

Finally, it is straightforward to check that $(x, y, z) = (26, \pm 10, 10)$ satisfy the given system.

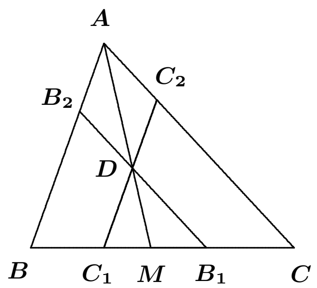
2. In triangle ABC, the points B_1 and C_1 are on BC, point B_2 is on AB, and point C_2 is on AC such that the segment B_1B_2 is parallel to AC and the segment C_1C_2 is parallel to AB. Let the lines B_1B_2 and C_1C_2 meet at D. Denote the areas of triangles BB_1B_2 and CC_1C_2 by b and c, respectively.

(a) Prove that if $b = c$, then the centroid of ABC is on the line AD.

(b) Find the ratio $b : c$ if D is the incentre of ABC and $AB = 4$, $BC = 5$, and $CA = 6$.

Solution by Titu Zvonaru, Cománeşti, Romania.

(a) Let F denote the area of $\triangle ABC$ and let $M = \overline{AD} \cap \overline{BC}$. For convenience, replace b and c by F_b and F_c, respectively. Since $B_1B_2 || AC$ and $C_1C_2 || AB$, we have



$$\frac{F_B}{F} = \left(\frac{BB_1}{BC}\right)^2 ; \qquad \frac{F_C}{F} = \left(\frac{CC_1}{BC}\right)^2 ,$$

hence $BB_1 = CC_1$ (and $BC_1 = CB_1$) since $F_B = F_C$. By Menelaus' theorem applied to $\triangle ABM$ and the traverse B_1DB_2 we obtain

$$\frac{B_1M}{B_1B} \cdot \frac{B_2B}{B_2A} \cdot \frac{DA}{DM} = 1 .$$

Similarly, by applying Menelaus' theorem to $\triangle ACM$ and the traverse C_1DC_2 we obtain

$$\frac{C_1M}{C_1C} \cdot \frac{C_2C}{C_2A} \cdot \frac{DA}{DM} = 1 .$$

The last two equations imply that

$$B_1M \cdot \frac{B_2B}{B_2A} = C_1M \cdot \frac{C_2C}{C_2A} .$$

Since $\frac{B_2B}{B_2A} = \frac{B_1B}{B_1C} = \frac{C_1C}{C_1B} = \frac{C_2C}{C_2A}$, it follows that $B_1M = C_1M$, hence M is the midpoint of BC (recall that $BC_1 = CB_1$). It follows that the centroid of ABC is on the line AD.

(b) Let $a = BC$, $b = CA$, $c = AB$, r be the inradius of $\triangle ABC$, and h_b and h_c be the altitudes of $\triangle ABC$ from B and C, respectively. Since

$\triangle BB_1B$ and $\triangle CC_1C_2$ are similar, we have

$$\frac{F_B}{F_C} = \left(\frac{BB_1}{CC_1}\right)^2 = \left(\frac{BB_1}{BC}\right)^2\left(\frac{BC}{CC_1}\right)^2 = \left(\frac{h_b - r}{h_b}\cdot\frac{h_c}{h_c - r}\right)^2 .$$

By substituting $h_b = \frac{2F}{b}$, $h_c = \frac{2F}{c}$, and $r = \frac{2F}{a+b+c}$ in the above we deduce that

$$\begin{aligned}\frac{F_B}{F_C} &= \left(\frac{\frac{2F}{b} - \frac{2F}{a+b+c}}{\frac{2F}{b}}\cdot\frac{\frac{2F}{c}}{\frac{2F}{c} - \frac{2F}{a+b+c}}\right)^2 \\ &= \left(\frac{a+c}{a+b+c}\cdot\frac{a+b+c}{a+b}\right)^2 = \left(\frac{a+c}{a+b}\right)^2 .\end{aligned}$$

Since $a = 5$, $b = 6$, and $c = 4$ were given, we obtain $\frac{F_B}{F_C} = \frac{81}{121}$.

4. The divisors of n are $d_1 < d_2 < \cdots < d_8$, where $d_1 = 1$ and $d_8 = n$. It is known that $20 \leq d_6 \leq 25$. Find all possible values of n.

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's version.

The only such n are 66, 88, 105, 110, and 154. Let the prime power decomposition of n be $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where each α_i is a positive integer and the p_i's are primes such that $p_1 < p_2 < \cdots < p_k$.

By the well-known formula for the number of divisors of n in terms of its prime power decomposition, $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1) = 8$. Hence, there are three possible cases to consider.

Case 1: $k = 1$ and $\alpha_1 = 7$.
Case 2: $k = 2$, $\alpha_1 = 1$, $\alpha_2 = 3$; or $\alpha_1 = 3$, $\alpha_2 = 1$.
Case 3: $k = 3$ and $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

In Case 1 we have $n = p^7$ for some prime p. Then clearly $d_6 = p^5$. Hence, we have $20 \leq p^5 \leq 25$, which implies that $p < 2$, a contradiction.

In Case 2 we have either (2a) $n = p_1p_2^3$ or (2b) $n = p_1^3p_2$ where p_1 and p_2 are primes such that $p_1 < p_2$.

In subcase (2a), $d_6 = p_1p_2^2$ since

$$1 < p_1 < p_2 < p_1p_2 < p_2^2 < p_1p_2^2 < p_2^3 < p_1p_2^3 .$$

Hence, $20 \leq p_1p_2^2 \leq 25$. Clearly, $p_2 \leq 3$, so $p_1 = 2$ and $p_2 = 3$ yielding $p_1p_2^2 = 18$, a contradiction. Thus, there are no solutions in subcase (2a).

In subcase (2b), we first identify d_6. If $p_2 < p_1^2$, then $d_6 = p_1^3$ since

$$1 < p_1 < p_2 < p_1^2 < p_1p_2 < p_1^3 < p_1^2p_2 < p_1^3p_2 .$$

Hence, $20 \leq p_1^3 \leq 25$, which has no solutions. If $p_2 > p_1^2$, then $d_6 = p_1p_2$ since 1, p_1, p_2, p_1^2, and p_1^3 are each less than p_1p_2 while $p_1p_2 < p_1^2p_2 < p_1^3p_2$.

Hence, $20 \le p_1p_2 \le 25$ and $(p_1, p_2) = (2, 11)$ or $(p_1, p_2) = (3, 7)$. The first pair yields $n = 88$ while the last pair is discarded since $3^2 > 7$.

Finally, in Case 3 we have $d_6 = p_1p_3$, since $p_1p_3 < p_2p_3 < p_1p_2p_3$ and each of 1, p_1, p_2, p_3, and p_1p_2 are less than p_1p_3. Hence, $20 \le p_1p_3 \le 25$ and $(p_1, p_3) = (2, 11)$ or $(p_1, p_3) = (3, 7)$ by the result in subcase (2b). We then have $(p_1, p_2, p_3) = (2, 3, 11)$, $(2, 5, 11)$, $(2, 7, 11)$, or $(3, 5, 7)$. The corresponding values of n are 66, 110, 154, and 105.

Therefore, $n = 66$, 88, 105, 110, or 154, as claimed.

Final Round

1. A positive integer n is *charming* if there are integers a_1, a_2, $\dots$, a_n (not necessarily distinct) such that $a_1 + a_2 + \cdots + a_n = a_1a_2 \cdots a_n = n$. Find all charming integers.

Solution by Oliver Geupel, Brühl, NRW, Germany.

One easily checks that $n = 4$ is not charming. If $n = 1$ or if $n > 4$ and n is congruent to 0 or 1 modulo 4, then the table below shows that n is charming:

n	values of a_k	multiplicity
$n = 8m$	$4m$	1
$(m \ge 1)$	2	1
	1	$6m - 2$
	-1	$2m$
$n = 4(4m - 1)$	$4m - 1$	1
$(m \ge 1)$	2	2
	1	$14m - 7$
	-1	$2m$
$n = 4(4m + 1)$	$4m + 1$	1
$(m \ge 1)$	2	1
	1	$14m + 2$
	-1	$2m - 1$
	-2	1
$n = 4m + 1$	$4m + 1$	1
$(m \ge 0)$	1	$2m$
	-1	$2m$

Let $n \equiv 2 \pmod 4$. If $n = \prod\limits_{k=1}^{n} a_k$, then exactly one of the numbers a_k is even. Therefore, $\sum\limits_{k=1}^{n} a_k$ is odd, and consequently n is not charming.

Finally, consider $n \equiv -1 \pmod 4$, say $n = 4m - 1$. Let $n = \prod\limits_{k=1}^{n} a_k$, and let q of the numbers a_k be congruent to -1 modulo 4 with the other $4m - 1 - q$ numbers a_k being congruent to 1 modulo 4. Then q is odd, say

$q = 2s + 1$, hence $\sum_{k=1}^{n} a_k \equiv (-2s - 1) + (4m - 1 - 2s - 1) \equiv 1 \pmod 4$. Thus, $n \neq \sum_{k=1}^{n} a_k$ and n is not charming.

Therefore, a positive integer n is charming if and only if $n = 1$ or $n > 4$ and n is congruent to 0 or 1 modulo 4.

2. Let a, b, and c be positive real numbers.

(a) Prove that

$$\sqrt{\frac{a^2 + b^2}{2}} + \frac{2}{\frac{1}{a} + \frac{1}{b}} \geq \frac{a + b}{2} + \sqrt{ab}\,.$$

(b) Is it true always that

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} + \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq \frac{a + b + c}{3} + \sqrt[3]{abc}\,?$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

(a) If $a = b$, then the equality holds. Assume $a \neq b$, then the following inequalites are successively equivalent:

$$\begin{aligned}
\sqrt{\frac{a^2 + b^2}{2}} + \frac{2}{\frac{1}{a} + \frac{1}{b}} &\geq \frac{a + b}{2} + \sqrt{ab}\,; \\
\sqrt{\frac{a^2 + b^2}{2}} - \sqrt{ab} &\geq \frac{a + b}{2} - \frac{2ab}{a + b}\,; \\
\frac{\left(\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}\right)\left(\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\right)}{\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}} &\geq \frac{(a + b)^2 - 4ab}{2(a + b)}\,; \\
\frac{(a - b)^2}{2\left(\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\right)} &\geq \frac{(a - b)^2}{2(a + b)}\,; \\
a + b &\geq \sqrt{\frac{a^2 + b^2}{2}} + \sqrt{ab}\,;
\end{aligned}$$

$$\frac{a^2+b^2}{2}+ab+2\sqrt{ab}\sqrt{\frac{a^2+b^2}{2}} \leq a^2+b^2+2ab\,;$$
$$\frac{a^2+b^2}{2}+ab-2\sqrt{ab}\sqrt{\frac{a^2+b^2}{2}} \geq 0\,;$$
$$\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)^2 \geq 0\,;$$

and the last inequality is true.

(b) Taking $a=1$, $b=2$, and $c=3$ the inequality becomes

$$\sqrt{\frac{14}{3}}+\frac{18}{11} \geq 2+\sqrt[3]{6}\,,$$

which is false, as one may verify by hand calculation that $\sqrt{\frac{14}{3}}+\frac{18}{11}<\frac{19}{5}$ and $\frac{19}{5}<2+\sqrt[3]{6}$.

[*Ed.:* Michel Bataille, Rouen, France comments that by coincidence, this problem is similar to problem 3266 proposed in the October 2007 issue.]

3. Triangle ABC is acute angled, $\angle BAC=60^\circ$, $AB=c$, and $AC=b$ with $b>c$. The orthocentre and the circumcentre of ABC are M and O, respectively. The line OM intersects AB and CA at X and Y, respectively.

(a) Prove that the perimeter of triangle AXY is $b+c$.

(b) Prove that $OM=b-c$.

Solved by Michel Bataille, Rouen, France; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comănești, Romania. We give the solution of Smeenk.

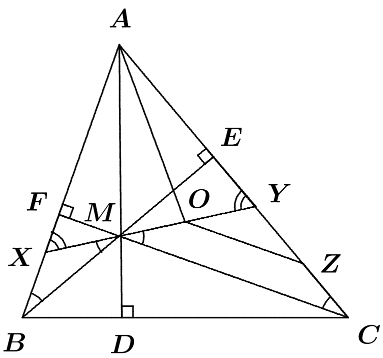
(a) Let α and β be the angles at A and B, respectively. Since

$$AM = 2R\cos\alpha = R = AO\,,$$

we have that $\angle AMO=\angle AOM$. Also, since

$$\angle XAM = \angle YAO = 90^\circ-\beta\,,$$

it follows that $\triangle XAM \sim \triangle YAO$. Therefore, $\angle AXY=\angle AYX=60^\circ$ and $\triangle AXY$ is equilateral.



Now $\angle ABM=30^\circ$, hence $\angle XMB=30^\circ$ and $XB=XM$. In the same way we can prove that $YC=YM$. It now follows that the perimeter of $\triangle AXY$ is $b+c$.

(b) Let Z be the point on AC such that $AZ = AB = c$. Then we have $ZC = b - c$, and from part (a)

$$YZ = BX = MX = OY .$$

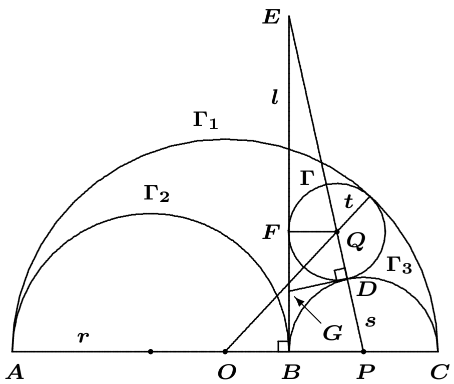
Therefore, $\triangle YOZ$ is isosceles, and $\angle YOZ = \angle YZO = 30^\circ$. Since we have $\angle MCZ = \angle CMO = 30^\circ$, we conclude that $MCZO$ is an isosceles trapezoid. This establishes that $OM = ZC = b - c$, as desired.

Now we turn to solutions to problems of the Indian Team Selection Test to the IMO 2002, given at [2008 : 149-151].

1. Let A, B, and C be three points on a line with B between A and C. Let Γ_1, Γ_2, and Γ_3 be semicircles, all on the same side of AC, and with AC, AB, and BC as diameters, respectively. Let l be the line perpendicular to AC through B. Let Γ be the circle which is tangent to the line l, tangent to Γ_1 internally, and tangent to Γ_3 externally. Let D be the point of contact of Γ and Γ_3. The diameter of Γ through D meets l in E. Show that $AB = DE$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let O, P, and Q, be the centres of Γ_1, Γ_3, and Γ, respectively. Let r, s, and t be the radii of Γ_2, Γ_3, and Γ, respectively. Let Γ meet l at the point F. The tangent to Γ at D meets l at the point G. Let u denote the common length of the tangent segments GB, GD, and GF. Let $\alpha = \angle BPD$.



By the Law of Cosines in $\triangle OPQ$ we have

$$\begin{aligned}(r+s-t)^2 &= OQ^2 = OP^2 + PQ^2 - 2OP \cdot PQ \cos \angle OPQ , \\ &= r^2 + (s+t)^2 - 2r(s+t)\cos\alpha ;\end{aligned}$$

hence

$$\cos\alpha = \frac{2st + rt - rs}{rs + rt} . \tag{1}$$

Applying the Law of Cosines to the triangles BPD, BGD, DGF, and DQF we have

$$\begin{aligned} 2s^2(1-\cos\alpha) &= BD^2 = 2u^2(1+\cos\alpha)\,, \\ 2u^2(1-\cos\alpha) &= FD^2 = 2t^2(1+\cos\alpha)\,; \end{aligned}$$

thus, $(1+\cos\alpha)t = (1-\cos\alpha)s$. We substitute for $\cos\alpha$ the expression in equation (1) and simplify successively to obtain

$$\begin{aligned} \left(\frac{rt+st}{rs+rt}\right)t &= \left(\frac{rs-st}{rs+st}\right)s\,, \\ (r+s)t^2+s^2t-rs^2 &= 0\,, \\ \big((r+s)t-rs\big)(t+s) &= 0\,. \end{aligned}$$

Since $t>0$, we have $t = \dfrac{rs}{r+s}$.

Let $d = ED$. We observe that the triangles EFQ and EBP are homothetic. Hence,

$$d-t = EQ = \frac{FQ\cdot EP}{BP} = \frac{t(d+s)}{s}\,,$$

so that $s(d-t) = t(d+s)$.

Therefore,

$$DE = d = \frac{2st}{s-t} = \frac{2s\cdot\dfrac{rs}{r+s}}{s-\dfrac{rs}{r+s}} = \frac{2rs^2}{s^2} = 2r = AB\,,$$

and the proof is complete.

4. Let ABC be an acute triangle with orthocentre H and circumcentre O. Show that there are points D, E, and F on BC, CA, and AB, respectively, such that AD, BE, and CF are concurrent and

$$DO+DH = EO+EH = FO+FH\,.$$

Solved by Oliver Geupel, Brühl, NRW, Germany. Remark by Michel Bataille, Rouen, France.

This problem has already been solved in this *Corner*. See Vol. 31, No. 5 (September 2005) p. 295.

5. Let a, b, and c be positive real numbers such that $a^2+b^2+c^2 = 3abc$. Prove that

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2} \geq \frac{9}{a+b+c}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give Apostolopoulos' solution.

Applying the Cauchy–Schwarz Inequality we have

$$\begin{aligned}&(a+b+c)\cdot\left(\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\right)\\ \geq\ &\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)^2 = \left(\frac{a^2+b^2+c^2}{abc}\right)^2 = \left(\frac{3abc}{abc}\right)^2 = 9\,,\end{aligned}$$

because $a^2+b^2+c^2=3abc$.

Therefore, $\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq\frac{9}{a+b+c}$.

(*Ed.:* Zvonaru remarks that this problem is #2859 at [2003 : 318] with a solution at [2004 : 314].)

11. Let ABC be a triangle and let P be an exterior point in the plane of the triangle. Let AP, BP, and CP meet the (possibly extended) sides BC, CA, and AB in D, E, and F, respectively. If the areas of the triangles PBD, PCE, and PAF are all equal, prove that their common area is equal to the area of the triangle ABC.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $[XYZ]$ denote the area of $\triangle XYZ$. We assume without loss of generality that $[ABC]=1$. Let (a,b,c) be the barycentric coordinates of P with respect to $\triangle ABC$. The oriented (signed) area of $\triangle PBD$ is then

$$[PBD] = \begin{pmatrix} a & b & c\\ 0 & 1 & 0\\ 0 & \frac{b}{b+c} & \frac{c}{b+c}\end{pmatrix} = \frac{ac}{b+c}\,.$$

Similarly, we have $[PCE]=\frac{ab}{a+c}$ and $[PAF]=\frac{ab}{a+c}$. The assumption $|[PBD]|=|[PCE]|=|[PAF]|$ implies $|a(a+b)|=|b(b+c)|=|c(c+a)|$, where at least two of the real numbers $a(a+b)$, $b(b+c)$, $c(c+a)$ are equal. Without loss of generality assume that $a(a+b)=c(c+a)$. We can scale the barycentric coordinates so that $a=1$. Therefore, $1(1+b)=c(c+1)$, that is

$$b = c^2+c-1\,. \tag{1}$$

The equation $|b(b+c)|=|a(a+b)|=|1+b|$ leads to two cases.

Case 1. $b(b+c)=1+b$. We subsitute for b as in (1) and obtain

$$\begin{aligned}\left(c^2+c-1\right)\left(c^2+2c-1\right) &= c^2+c\,;\\ c^4+2c^3-c^2-4c+1 &= (c-1)\left(c^3+4c^2+3c-1\right) = 0\,.\end{aligned}$$

If $c = 1$ then also $b = a = 1$, which contradicts the hypothesis that P is an exterior point of $\triangle ABC$. Consequently, $c^3 + 4c^2 + 3c - 1 = 0$. It follows from (1) that

$$\begin{aligned} [PBD] &= \frac{c}{(c^2+2c)(c^2+2c-1)} \\ &= \frac{1}{c^3+4c^2+3c-2} \;=\; -1 \;=\; -[ABC]\,, \end{aligned}$$

which is the desired conclusion.

Case 2. $b(b+c) = -1 - b$. We substitute according to (1) to obtain

$$\begin{aligned} (c^2+c-1)\,(c^2+2c-1) &= -c^2 - c\,; \qquad (2) \\ c^4 + c^3 + c^2 - 2c + 1 &= 0\,. \end{aligned}$$

We will prove that $f(x) = x^4 + 3x^3 + x^2 - 2x + 1$ is positive for $x \in \mathbb{R}$, thus proving that (3) is impossible. Since $f(x) = x^4 + 3x^3 + (x-1)^2$, we see that $f(x) > 0$ for $x \in (-\infty, -3] \cup [0, \infty)$.

It suffices to show that $f(x) > 0$ for $x \in [-3, 0]$. Let $t = -x$. The AM–GM Inequality yields

$$\begin{aligned} f(x) &= -4t\left(\frac{t}{2}\right)\left(\frac{t}{2}\right)(3-t) \;+\; (t+1)^2 \\ &\geq -4t\left(\frac{1}{3}\left(\frac{t}{2}+\frac{t}{2}+3-t\right)\right)^3 \;+\; (t+1)^2 \;=\; (t-1)^2 \;\geq\; 0\,, \end{aligned}$$

where at least one of the last two inequalities is strict. This completes the proof.

15. Let x_1, x_2, …, x_n be real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2} < \sqrt{n}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give the solution of Díaz-Barrero.

For vectors $\vec{u} = (a_1, a_2, \ldots, a_n)$ and $\vec{v} = (1, 1, \ldots, 1)$ an application of the Cauchy–Schwartz Inequality yields

$$a_1 + a_2 + \cdots + a_n \;\leq\; \sqrt{n}\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\,.$$

Setting $a_i = \dfrac{x_i}{1+x_1^2+\cdots+x_i^2}$ for $1 \leq i \leq n$, it suffices to prove that

$$\left(\frac{x_1}{1+x_1^2}\right)^2 + \left(\frac{x_2}{1+x_1^2+x_2^2}\right)^2 + \cdots + \left(\frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2}\right)^2 < 1\,.$$

We have

$$\frac{x_1^2}{\left(1+x_1^2\right)^2} \;\le\; \frac{x_1^2}{\left(1+x_1^2\right)} \;=\; 1-\frac{1}{1+x_1^2}\,,$$

and for $2 \le i \le n$ we can assert that

$$\begin{aligned}\frac{x_i^2}{\left(1+x_1^2+\cdots+x_i^2\right)^2} \;&\le\; \frac{x_i^2}{\left(1+x_1^2+\cdots+x_{i-1}^2\right)\left(1+x_1^2+\cdots+x_i^2\right)} \\ &=\; \frac{1}{1+x_1^2+\cdots+x_{i-1}^2} - \frac{1}{1+x_1^2+\cdots+x_i^2}\,.\end{aligned}$$

Adding the preceding expressions, we obtain

$$\sum_{i=1}^{n}\frac{x_i^2}{\left(1+x_1^2+\cdots+x_i^2\right)^2} \;\le\; 1-\frac{1}{1+x_1^2+\cdots+x_n^2} \;<\; 1\,,$$

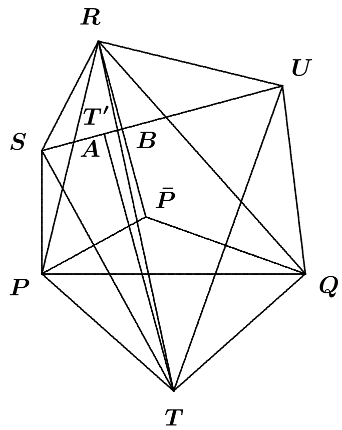
and we are done.

19. Let PQR be an acute triangle. Let SRP, TPQ, and UQR be isosceles triangles exterior to PQR, with $SP = SR$, $TP = TQ$, and $UQ = UR$, such that $\angle PSR = 2\angle QPR$, $\angle QTP = 2\angle RQP$, and $\angle RUQ = 2\angle PRQ$. Let S', T', and U' be the points of intersection of SQ and TU, TR and US, and UP and ST, respectively. Determine the value of

$$\frac{SQ}{SS'} + \frac{TR}{TT'} + \frac{UP}{UU'}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $\angle RPQ = \alpha$, $\angle PQR = \beta$, and $\angle QRP = \gamma$. Let $\bar{P}$, $\bar{Q}$, and $\bar{R}$ be the reflections of P, Q, and R, with respect to the axes ST, TU, and US, respectively. Since $PT = \bar{P}T$, $\bar{P}$ is on the circle Γ_T with centre T and radius TP. Since $PS = \bar{P}S$, $\bar{P}$ is on the circle Γ_S with centre S and radius SP. Therefore, $\angle P\bar{P}Q = 180^\circ = \beta$ and $\angle R\bar{P}P = 180^\circ - \alpha$. We now have that $\angle Q\bar{P}R = 360^\circ - (180^\circ - \alpha) - (180^\circ - \beta) = 180^\circ - \gamma$. Hence, $\bar{P}$ is also on the circle Γ_U with centre U and radius UR.



We conclude that the three circles Γ_S, Γ_T, and Γ_U intersect at $\bar{P}$. By symmetry, $\bar{P}$ coincides with $\bar{Q}$ and with $\bar{R}$, so we have $\triangle PST \cong \triangle \bar{P}ST$, $\triangle QTU \cong \triangle \bar{P}TU$, and $\triangle RUS \cong \triangle \bar{P}US$.

Let A and B be the feet of the perpendiculars from T and R to SU, respectively. Then we have

$$\frac{RT'}{TT'} = \frac{RB}{TA} = \frac{[RUS]}{[STU]} = \frac{[\bar{P}US]}{[STU]},$$

thus

$$\frac{TR}{TT'} = 1+\frac{[\bar{P}US]}{[STU]}.$$

We finally obtain

$$\begin{aligned}&\frac{SQ}{SS'}+\frac{TR}{TT'}+\frac{UP}{UU'}\\ &= \left(1+\frac{[\bar{P}TU]}{[STU]}\right) + \left(1+\frac{[\bar{P}US]}{[STU]}\right) + \left(1+\frac{[\bar{P}ST]}{[STU]}\right) = 4.\end{aligned}$$

20. Let a, b, and c be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

Solved by Arkady Alt, San Jose, CA, USA; and George Apostolopoulos, Messolonghi, Greece. We give Alt's solution.

Due to the cyclic symmetry, we can suppose that $c = \min\{a, b, c\}$.

Let x, y, and z be nonzero real numbers. Since

$$x^2z+y^2x+z^2y = 3xyz+z(x-y)^2+y(x-z)(y-z),$$

we obtain

$$\begin{aligned}\frac{x}{y}+\frac{y}{z}+\frac{z}{x} &= \frac{x^2z+y^2x+z^2y}{xyz}\\ &= 3+\frac{(x-y)^2}{xy}+\frac{(x-z)(y-z)}{xz}.\end{aligned}$$

Setting $x=a$, $y=b$, $z=c$ and then $x=c+a$, $y=c+b$, $z=a+b$, we obtain (respectively) the two equations

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{a} &= 3+\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{ac},\\ \frac{c+a}{c+b}+\frac{a+b}{a+c}+\frac{b+c}{b+a} &= 3+\frac{(a-b)^2}{(c+a)(c+b)}+\frac{(a-c)(b-c)}{(c+a)(a+b)}.\end{aligned}$$

Comparing the last two equations gives the result, since $(c+a)(c+b) > ab$, $(a+b)(c+a) > ac$, $(a-b)^2 \geq 0$, and $(b-c)(a-c) \geq 0$.

21. Given a prime p, show that there is a positive integer n such that the decimal representation of p^n has a block of 2002 consecutive zeros.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove the following generalization: Given a prime p and a positive integer N, there are infinitely many natural numbers n such that the decimal representation of p^n has a block of N consecutive zeroes. To prove this we distinguish three cases.

Case 1. $p \notin \{2, 5\}$. Let $k > N$, $k \in \mathbb{Z}$, and let $n = \phi\left(10^k\right) = \frac{2}{5} \cdot 10^k$, where ϕ is Euler's totient function. We have $p^n > 1$ and, by Euler's theorem, $p^n \equiv 1 \pmod{10^k}$; hence n has the desired property.

Case 2. $p = 2$. Let $k > 2N$, $k \in \mathbb{Z}$, and let $n = \phi\left(5^k\right) + k = 4 \cdot 5^{k-1} + k$. We see that $2^n > 2^k$ and, by Euler's theorem, that $2^{\phi(5^k)} \equiv 1 \pmod{5^k}$. Therefore, $2^n = 2^{\phi(5^k)} \cdot 2^k \equiv 2^k \pmod{10^k}$. Since $k > 2N > \log_5 10 \cdot N$, we have $\dfrac{10^k}{2^k} = 5^k > 10^N$. Consequently, the number 2^n contains a block of N consecutive zeros to the left of the rightmost 2^k digits.

Case 3. $p = 5$. Let $k > 4N$, $k \in \mathbb{Z}$, and let $n = \phi\left(2^k\right) + k = 2^{k-1} + k$. We note that $5^n > 5^k$, and again by Euler's theorem, that $5^{\phi(2^k)} \equiv 1 \pmod{2^k}$. Therefore, $5^n = 5^{\phi(2^k)} \cdot 5^k \equiv 5^k \pmod{10^k}$. Since $k > 4N > \log_2 10 \cdot N$, we have $\dfrac{10^k}{5^k} = 2^k > 10^N$. Thus, the number 5^n contains a block of N consecutive zeroes to the left of the rightmost 5^k digits.

That completes the material for this number of the *Corner*. Please send me Olympiad contests as well as your nice solutions and generalizations.

BOOK REVIEWS

Amar Sodhi

Benjamin Franklin's Numbers: An Unsung Mathematical Odyssey
By Paul C. Pasles, Princeton University Press, 2008
ISBN 13:978-0-691-12956-3; 254+xii pages, US$26.95
Reviewed by **Jeff Hooper**, *Acadia University, Wolfville, NS*

Biographies of Benjamin Franklin are not new. He holds a unique place in American history and culture: printer and author, scientist, inventor, philosopher, diplomat, and legend. Who hasn't heard the stories of Franklin's electricity experiments using a wet kite in a thunder storm? His tenure as Ambassador to France occurred at a crucial time in U.S. history, and even into his 80s he served as the President of Pennsylvania. Yet one area in which many historians seem to have paid Benjamin Franklin short shrift is the field of mathematics. The impression given by biographers is that mathematics was Franklin's one glaring weakness. He was indeed once referred to as a "polymath who excelled at everything *except* mathematics."

The central theme of Paul Pasles' book is that historians have done Benjamin Franklin a tremendous injustice. He actually possessed a strong mathematical mind: he was skilled in logical argument and adept at the sort of systematic and creative thinking about numbers, arrangements, and relationships that characterize mathematical thought. He developed an algebra for everyday living, a sort of decision-making technique reminiscent of modern utility theory. His almanac regularly proposed mathematical challenges for his readers. But the most fascinating side to Franklin was his work on recreational mathematics.

Pasles does a thorough job of resurrecting Franklin's mathematical reputation, and this is no simple feat. Pasles returns to primary sources, drawing on Franklin's letters and journals, as well as reconstructions of his library. In doing so Pasles establishes that Franklin not only taught himself basic mathematical skills, but developed a keen sense for recreational mathematics which lasted throughout his long life.

Central to the author's argument are the examples. Franklin has long been known for creating magic squares, and Pasles examines their numerous unusual symmetries, as well as Franklin's ingenious methods for constructing them. This includes in particular some wonderful 8×8 examples (including one newly re-discovered) and even an enormous 16×16 square. Pasles also includes a similar discussion of the incredible symmetries of Franklin's 'Magic Circle of Circles.' Some of these can be viewed at Pasles' website (`www.pasles.org`) where the author has collected links to additional material and examples. All of these topics are discussed thoroughly, and carefully placed in historical context.

This gem of a book is an excellent addition for anyone interested in recreational mathematics or more generally, the history of science.

Geometric Constructions of Mixtilinear Incircles

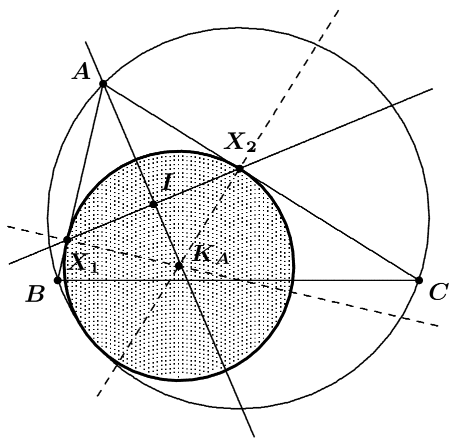
Cosmin Pohoaţă

L. Bankoff [1] introduced the term *mixtilinear incircle* of a triangle to name the three circles each tangent to two sides and to the circumcircle internally. In the same paper, Bankoff establishes the *fundamental formula* (as P. Yiu [2] refers to it) expressing the radius of a mixtilinear incircle in terms of the inradius of the triangle. More precisely, consider a triangle ABC and its mixtilinear incircle in the angle A with centre K_A and radius ρ_A. Then,

$$r = \rho_A \cos^2 \frac{\alpha}{2},$$

where r is the inradius of the triangle and $\alpha = \angle BAC$. This formula leads to a first construction of the mixtilinear incircle (see [2]).

Construction 1 Denote by I the incentre of triangle ABC, and let the perpendicular through I to the internal bisector of angle A intersect the sides AB and AC at X_1 and X_2, respectively. The perpendiculars at these points to the respective sides of the triangle intersect again on the angle bisector, at the mixtilinear incentre K_A. The circle with centre K_A and passing through X_1 and X_2 is the mixtilinear incircle in angle A.

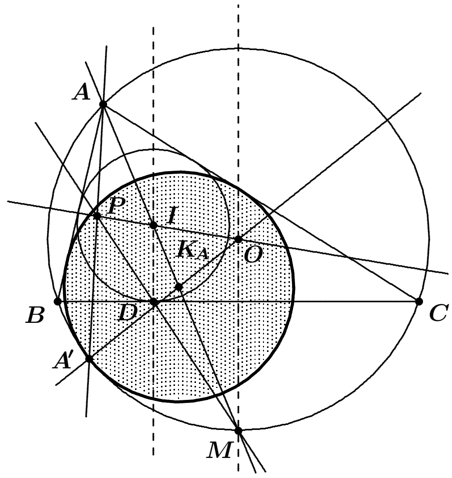


In the same paper, Yiu [2] gives an alternative construction based on the following result, whose proof we omit.

Theorem 1 (Yiu). The three lines each joining a vertex of a triangle to the point of contact of the circumcircle with the respective mixtilinear incircle are concurrent at the external center of similitude of the circumcircle and incircle.

Consider the circumcenter O, the midpoint M of the arc BC not containing the vertex A, and the tangency point D of the incircle with the side BC. Since the lines OM and ID are parallel and $OM : ID = R : r$, the intersection Q of the lines MD and OI is the external centre of similitude of the circumcircle and the incircle. This leads us to the following construction.

Construction 2 ([2]) Given a triangle ABC, let Q be the external centre of similitude of the circumcircle with centre O and incircle with centre I. Extend AQ to intersect the circumcircle at A'. The intersection of AI and $A'O$ is the centre K_A of the mixtilinear incircle in angle A.



We now give a new construction, similar at first sight to Construction 1, although it does not use Bankoff's formula or the collinearity of the incentre with the tangency points of the sides AB and AC with the mixtilinear incircle in angle A. First we will prove the following result.

Theorem 2 Let N and P be the midpoints of arcs CA and AB, not containing the vertices B and C, respectively. The reflection of A with respect to the midpoint of the segment NP lies on the line determined by the tangency points of the sides AC and AB with the mixtilinear incircle in angle A.

Proof: Let A', X_1 and X_2 be as above, and let A_1 be the reflection of the point A in the midpoint of the segment NP (see the diagram on the next page). Since A', X_1, and P are collinear and also A', X_2, and N are collinear, we have

$$\begin{aligned}\angle AA'P &= \tfrac{1}{2}\angle AA'B = \tfrac{1}{2}\angle ACB\,,\\ \angle AA'N &= \tfrac{1}{2}\angle AA'C = \tfrac{1}{2}\angle ABC\,.\end{aligned}$$

Since the quadrilateral $APA'N$ is cyclic,

$$\begin{aligned}\angle APA_1 &= \angle ANA_1 = \angle APN + \angle ANP\\ &= \angle AA'N + \angle AA'P = 90^\circ - \tfrac{1}{2}\angle BAC\,.\end{aligned}$$

On the other hand, since the triangle AX_1X_2 is isosceles,

$$\angle AX_1X_2 = \angle AX_2X_1 = 90^\circ - \frac{1}{2}\angle BAC\,.$$

Denote by A_1', A_1'' the intersections of PA_1, NA_1 with the line X_1X_2, respectively. Therefore, the quadrilaterals APX_1A_1' and ANX_2A_1'' are cyclic. Assume without loss of generality that A_1 lies on the opposite side of line CA than the vertex B does. In this case,

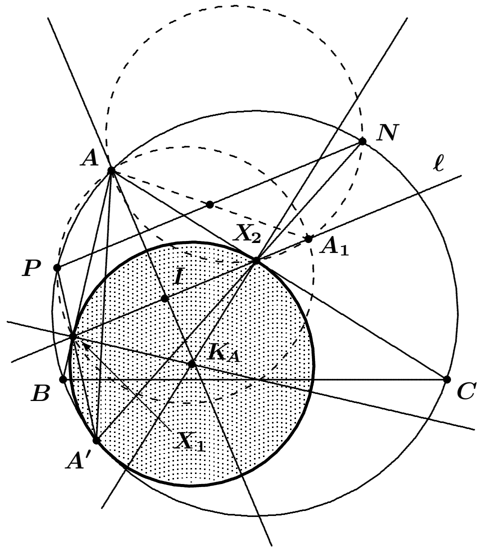
$$\angle APX_1 = 180^\circ - \angle AA_1'X_1\,, \quad \text{and} \quad \angle ANX_2 = \angle AA_1''X_2\,.$$

However, since the quadrilateral $APA'N$ is cyclic,

$$(180° - \angle AA_1'X_1) + \angle AA_1''X_2 = \angle APA' + \angle ANA' = 180° .$$

Hence, $\angle AA_1'X_1 = \angle AA_1''X_2$ and $A_1' = A_1''$, so A_1 lies on X_1X_2. ∎

This result leads us to our third construction:



Construction 3 Given the triangle ABC, construct the midpoints N and P of the circumcircle arcs CA and AB not containing the vertices B and C, respectively. Let A_1 be the reflection of the point A in the midpoint of the segment NP. Draw the line ℓ through A_1 perpendicular to the internal angle bisector of angle A; this line intersects AB and AC at the points X_1 and X_2, respectively. The perpendiculars at X_1 and X_2 to the sides AB and AC (respectively) intersect at K_A, the centre of the mixtilinear incircle in angle A.

We give two other related constructions involving the isogonality of the external centre of similitude of the circumcircle and incircle with the Nagel Point of the triangle. The first one is implicit in the literature.

Construction 4 In a triangle ABC construct the midpoints N and P of the circumcircle arcs CA and AB not containing the vertices C and B, respectively. Draw the tangents at N and P to the circumcircle. The second intersection of the line determined by their intersection point and the vertex A coincides with the tangency point A' of the circumcircle with the mixtilinear incircle in angle A. Hence, the lines $A'O$ and AI meet at K_A, the centre of the mixtilinear incircle in angle A.

Proof: Let M' be the antipodal point of M with respect to the circumcircle and let N_0 and P_0 be the intersection points of the tangent at M to the circumcircle with the tangents in N and P, respectively. Suppose the tangents at N and P to the circumcircle meet at the point M_0. Since the internal angle bisector AI is perpendicular to NP and to its corresponding external angle bisector AM', the lines NP and AM' are parallel. Therefore, the cyclic quadrilateral $NPAM'$ is an isosceles trapezoid, and so the lines M_0A, M_0M' are isogonal with respect to angle NM_0P. Since M_0M' is the Nagel Cevian corresponding to vertex M_0 in triangle $M_0N_0P_0$ and because triangles $M_0N_0P_0$ and ABC are homothetic, the lines M_0A and the Nagel Cevian corresponding to vertex A in triangle ABC are isogonal. ∎

We will use the following known theorem to give an argument for the isogonality of the external centre of similitude of the circumcircle and incircle with the Nagel Point of the triangle.

Theorem 3 Given a triangle ABC, let P be a point with homogeneous barycentric coordinates $(x : y : z)$.

(a) The reflection of the line AP in the internal angle bisector of vertex A intersects the line BC at the point $P_A = (0 : \frac{b^2}{y} : \frac{c^2}{z})$.

(b) If P_B and P_C are the intersection points of the reflections of the lines BP and CP in the internal angle bisectors corresponding to the vertices B and C with the lines CA and AB, respectively, then P_A, P_B, and P_C are the traces of $P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right) = \left(a^2yz : b^2zx : c^2xy\right)$, the isogonal conjugate of P. ■

The Nagel point X_8, and the external centre of similitude of the circumcircle and incircle X_{56} have (see [3]) homogeneous barycentric coordinates

$$\begin{aligned} X_8 &= (b+c-a : c+a-b : a+b-c)\,, \\ X_{56} &= \left(\frac{a^2}{b+c-a} : \frac{b^2}{c+a-b} : \frac{c^2}{a+b-c}\right) . \end{aligned}$$

Hence, by Theorem 3, they are isogonal with respect to triangle ABC.

Inspired by the preceding result, we give a last simple construction based on the following theorem.

Theorem 4. Let D be the point of tangency on side BC of a triangle with its incircle. Let I_A and M be the intersection point of the internal angle bisector AI with the side BC and with the circumcircle, respectively. Then the point of tangency A' of the mixtilinear incircle in angle A lies on the circumcircle of triangle DI_AM.

Proof: Since the Nagel Cevian corresponding to vertex A and the line AA' are isogonal with respect to angle BAC, the lines $A'A_0$ and BC are parallel, where A'' is the second intersection of the Nagel Cevian through A with the circumcircle. Let A_0 be the intersection point of the parallel through A to BC with the circumcircle. Hence, the cyclic quadrilateral $AA_0A''A'$ is an isosceles trapezoid, and by symmetry the line A_0A' passes through D. On other hand, also due to the symmetry, triangles A_0CB and ABC are congruent and therefore $A_0M = AM$. Hence,

$$\angle DA'M = \angle A_0A'M = \angle A_0BC + \angle MBC = \angle C + \frac{1}{2}\angle A\,,$$

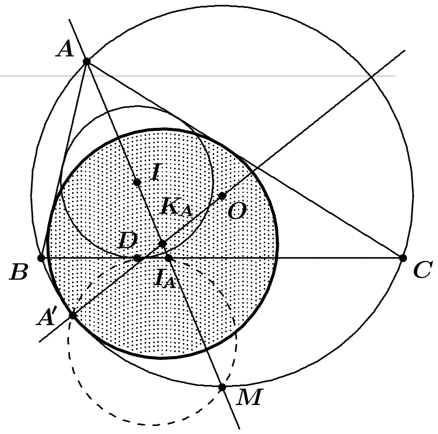
and since

$$\angle CDM = \angle CAM + \angle ACB = \angle C + \frac{1}{2}\angle A\,,$$

we conclude that the quadrilateral $DA'MI_A$ is cyclic. ■

This being said, we can now formulate our last construction.

Construction 5 Given a triangle ABC, let D be the point of tangency of the incircle with the side BC. Let I_A and M be the intersection points of the internal angle bisector corresponding to vertex A with BC and with the circumcircle, respectively. The second intersection of the circumcircles of triangles DI_AM and ABC coincides with A', the tangency point of the circumcircle of ABC with the mixtilinear incircle in angle A.



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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er septembre 2009**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB et Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3393. Correction. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit le triangle ABC, où $a = BC$, $b = AC$, $c = AB$ et où s est le demi périmètre. Montrer que

$$\frac{y+z}{x}\cdot\frac{A}{a(s-a)}+\frac{z+x}{y}\cdot\frac{B}{b(s-b)}+\frac{x+y}{z}\cdot\frac{C}{c(s-c)}\ \geq\ \frac{9\pi}{s^2}\,,$$

où les angles A, B et C sont mesurés en radians et x, y et z sont des nombres réels positifs quelconques.

3414. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Un triangle ABC varie de sorte que son cercle circonscrit $\gamma_1(O,R)$ et son cercle inscrit $\gamma_2(I,r)$ restent fixes, O et I, R et r étant leur centre et leur rayon respectifs. Trouver le lieu de l'orthocentre H du triangle ABC.

3415. *Proposé par Cezar Lupu, étudiant, Université de Bucarest, Bucarest, Roumanie.*

Soit a, b et c des nombres réels tels que $abc = 1$. Montrer que

$$\sqrt[3]{a}\ +\ \sqrt[3]{b}\ +\ \sqrt[3]{c}\ \leq\ \sqrt[3]{3(3+a+b+c+ab+bc+ca)}\,.$$

3416. *Proposé par Michel Bataille, Rouen, France.*

Soit (a_n) la suite définie par $a_0 = 6$ et la récursion

$$a_{n+1}\ =\ \frac{1}{13}\left(8a_n\sqrt{3a_n^2+13}-6a_n^2-13\right)$$

pour $n \geq 0$. Montrer que tout a_n est un entier positif, et que $a_n^2 - a_{n+1}$ est divisible par 13 pour tout $n \geq 0$.

3417. *Proposé par Michel Bataille, Rouen, France.*

Posons $S_p(n) = 1^p + 2^p + \cdots + n^p$. On aurait alors

$$\sum_{n=1}^{\infty} \frac{\left(S_{-1}(n)\right)^2}{S_1(n)} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_3(n)}\right) + 2\left(\sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_1(n)}\right) .$$

3418. *Proposé par Pantelimon George Popescu, Bucarest, Roumanie et José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $\mathcal{I}(\phi)$ l'ensemble de toutes les anti-dérivées de la fonction continue ϕ.

(a) Déterminer la fonction continue $f : I_p \subset \mathbb{R} \to \mathbb{R}\backslash\{0\}$ telle que $f(0) = 1$ et $f^{-p} \in \mathcal{I}(f)$ où p est un nombre naturel impair et l'intervalle I_p a zéro à l'intérieur et est maximal pour cettes propriétés de f.

(b) Montrer que $p = q$ si et seulement si $I_p = I_q$.

3419. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Soit a, b et c trois nombres réels positifs.

(a) Montrer que $\displaystyle\sum_{\text{cyclique}} \sqrt{\frac{a^2 + 4bc}{b^2 + c^2}} \geq 2 + \sqrt{2}$.

(b) Montrer que $\displaystyle\sum_{\text{cyclique}} \sqrt{\frac{a^2 + bc}{b^2 + c^2}} \geq 2 + \frac{1}{\sqrt[3]{2}}$.

3420. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que

$$\prod_{k=1}^{n} \left(\frac{(k+1)^2}{k(k+2)}\right)^{k+1} < n + 1 < \prod_{k=1}^{n} \left(\frac{k^2 + k + 1}{k(k+1)}\right)^{k+1} .$$

3421. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit a, b et c trois nombres réels positifs tels que $abc \leq 1$. Montrer que

$$\frac{a}{b^2 + b} + \frac{b}{c^2 + c} + \frac{c}{a^2 + a} \geq \frac{3}{2} .$$

3422. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit a, b et c trois nombres réels positifs tels que $a+b+c \leq 1$. Montrer que

$$\frac{a}{a^3+a^2+1} + \frac{b}{b^3+b^2+1} + \frac{c}{c^3+c^2+1} \leq \frac{27}{31}\,.$$

3423. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit $n \geq 2$ un entier et x_1, x_2, $\ldots$, x_n n nombres réels positifs tels que $x_1 + x_2 + \cdots + x_n = 2n$. Montrer que

$$\sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{x_j}{\sqrt{x_i^3+1}} \right) \geq \frac{2n(n-1)}{3}\,.$$

3424. *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Pour un entier positif m, soit σ la permutation de $\{0, 1, \ldots, 2m-1\}$ définie par $\sigma(2i) = i$ pour $i = 0, 1, 2, \ldots, m$ et $\sigma(2i-1) = m+i$ pour $i = 1, 2, \ldots, m$. Montrer qu'il existe un entier positif k tel que $\sigma^k = \sigma$ et $k \leq 2m+1$.

3425. *Proposé par Slavko Simic, Institut de Mathématiques SANU, Belgrade, Serbie.*

Pour x réel différent de -1, on définit la fonction f par $f(x) = \dfrac{e^x}{x+1}$. Montrer que si $f(x) = f(y)$ pour certains $x \neq y$, alors

$$\left(\sqrt{x+1} - \sqrt{y+1}\right)^2 \geq \ln f(y)\,.$$

. . . .

3393. Correction. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let ABC be a triangle with $a = BC$, $b = AC$, $c = AB$, and semiperimeter s. Prove that

$$\frac{y+z}{x} \cdot \frac{A}{a(s-a)} + \frac{z+x}{y} \cdot \frac{B}{b(s-b)} + \frac{x+y}{z} \cdot \frac{C}{c(s-c)} \geq \frac{9\pi}{s^2}\,,$$

where the angles A, B, and C are measured in radians and x, y, and z are any positive real numbers.

3414. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

As triangle ABC varies, its circumcircle $\gamma_1(O,R)$ and its incircle $\gamma_2(I,r)$ are fixed, where O and I are the respective centres and R and r are the respective radii. Find the locus of the orthocentre H of triangle ABC.

3415. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.*

Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \le \sqrt[3]{3(3+a+b+c+ab+bc+ca)}\,.$$

3416. *Proposed by Michel Bataille, Rouen, France.*

Let the sequence (a_n) be defined by $a_0 = 6$ and the recursion

$$a_{n+1} = \frac{1}{13}\left(8a_n\sqrt{3a_n^2+13} - 6a_n^2 - 13\right)$$

for $n \ge 0$. Prove that each a_n is a positive integer, and that $a_n^2 - a_{n+1}$ is divisible by 13 for each $n \ge 0$.

3417. *Proposed by Michel Bataille, Rouen, France.*

Let $S_p(n) = 1^p + 2^p + \cdots + n^p$. Prove that

$$\sum_{n=1}^{\infty} \frac{\left(S_{-1}(n)\right)^2}{S_1(n)} = \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{S_{-1}(n)}{S_3(n)}\right) + 2\left(\sum_{n=1}^{\infty}\frac{S_{-1}(n)}{S_1(n)}\right).$$

3418. *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $\mathcal{I}(\phi)$ be the set of all antiderivatives of a continuous function ϕ.

(a) Determine the continuous function $f\colon I_p \subset \mathbb{R} \to \mathbb{R}\backslash\{0\}$ such that $f(0) = 1$ and $f^{-p} \in \mathcal{I}(f)$, where p is an odd natural number and the interval I_p contains zero and is maximal for the given properties of f.

(b) Prove that $p = q$ if and only if $I_p = I_q$.

3419. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, and c be positive real numbers.

(a) Prove that $\displaystyle\sum_{\text{cyclic}} \sqrt{\frac{a^2+4bc}{b^2+c^2}} \ge 2+\sqrt{2}$.

(b) Prove that $\displaystyle\sum_{\text{cyclic}} \sqrt{\frac{a^2+bc}{b^2+c^2}} \ge 2+\frac{1}{\sqrt[3]{2}}$.

3420. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\prod_{k=1}^{n}\left(\frac{(k+1)^2}{k(k+2)}\right)^{k+1} < n+1 < \prod_{k=1}^{n}\left(\frac{k^2+k+1}{k(k+1)}\right)^{k+1}.$$

3421. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $abc \le 1$. Prove that

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a} \ge \frac{3}{2}.$$

3422. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $a+b+c \le 1$. Prove that

$$\frac{a}{a^3+a^2+1} + \frac{b}{b^3+b^2+1} + \frac{c}{c^3+c^2+1} \le \frac{27}{31}.$$

3423. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let $n \ge 2$ be an integer and $x_1, x_2, \ldots, x_n$ positive real numbers such that $x_1 + x_2 + \cdots + x_n = 2n$. Prove that

$$\sum_{j=1}^{n}\left(\sum_{\substack{i=1\\ i\ne j}}^{n}\frac{x_j}{\sqrt{x_i^3+1}}\right) \ge \frac{2n(n-1)}{3}.$$

3424. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

For a positive integer m, let σ be the permutation of $\{0, 1, \ldots, 2m-1\}$ defined by $\sigma(2i) = i$ for each $i = 0, 1, 2, \ldots, m$ and $\sigma(2i-1) = m+i$ for each $i = 1, 2, \ldots, m$. Prove that there exists a positive integer k such that $\sigma^k = \sigma$ and $k \le 2m+1$.

3425. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

For real $x \ne -1$, let $f(x) = \dfrac{e^x}{x+1}$. Prove that if $f(x) = f(y)$ for some $x \ne y$, then

$$\left(\sqrt{x+1} - \sqrt{y+1}\right)^2 \ge \ln f(y).$$

SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

3313. [2008 : 104, 106] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_k > 1$ for $1 \leq k \leq n$. If we set $x_{n+1} = x_1$, prove that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\log_{x_k} x_{k+1} + \log_{x_{k+1}} x_k\right) \;\leq\; \left(\prod_{k=1}^{n}\left(1 + \log^n_{x_k} x_{k+1}\right)\right)^{\frac{1}{n}}.$$

Similar solutions by George Apostolopoulos, Messolonghi, Greece; Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA; and the proposers.

We prove the following more general result of which the given inequality is a special case: If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $\prod\limits_{k=1}^{n} a_k = 1$, then

$$\frac{1}{n}\left(\sum_{k=1}^{n} a_k + \sum_{k=1}^{n}\frac{1}{a_k}\right) \;\leq\; \left(\prod_{k=1}^{n}\left(1 + a_k^n\right)\right)^{1/n}. \tag{1}$$

Applying the AM–GM Inequality, we have for each fixed j, $1 \leq j \leq n$,

$$\left(\sum_{\substack{k=1\\k\neq j}}^{n}\frac{a_k^n}{1+a_k^n}\right) + \frac{1}{1+a_j^n} \;\geq\; \frac{n}{a_j\left(\prod\limits_{k=1}^{n}\left(1+a_k^n\right)\right)^{1/n}} \tag{2}$$

and

$$\left(\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{1+a_k^n}\right) + \frac{a_j^n}{1+a_j^n} \;\geq\; \frac{na_j}{\left(\prod\limits_{k=1}^{n}\left(1+a_k\right)\right)^{1/n}}. \tag{3}$$

Adding the inequalities (2) and (3) we obtain

$$\left(a_j + \frac{1}{a_j}\right)\frac{n}{\left(\prod\limits_{k=1}^{n}\left(1+a_k\right)\right)^{1/n}} \;\leq\; n\,,$$

or

$$a_j + \frac{1}{a_j} \;\le\; \left(\prod_{k=1}^{n}\left(1+a_k^n\right)\right)^{1/n} . \qquad (4)$$

Inequality (1) follows upon summing inequality (4) on $j = 1, 2, \ldots, n$.

If $a_k = \log_{x_k} x_{k+1}$, then $a_k > 0$ for all k and $\prod_{k=1}^{n} a_k = \prod_{k=1}^{n} \frac{\ln x_{k+1}}{\ln x_k} = 1$, so the proposed inequality follows from inequality (1).

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Both Oliver Geupel, Brühl, NRW, Germany and Ricardo pointed out that the proposed inequality is a special case of (1) *which was attributed to Gabriel Dospinescu and can be found in the book* Old and New Inequalities *by T. Andreescu, V. Cîrtoaje, G. Dospinescu, and M. Lascu (GIL Publishing House, Zalau, Romania, 2004).*

3314. [2008 : 102, 104] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\sum_{\text{cyclic}} \frac{a}{b} \;\ge\; \frac{3}{4} + \sum_{\text{cyclic}} \frac{(a+c)^2 + (a+b)c}{(b+c)(2a+b+c)} .$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

The solution makes use of the following two lemmas.

Lemma 1 For all positive real numbers x, y, and z,

$$\sum_{\text{cyclic}} \left(\frac{x}{y} - \frac{z+x}{z+y}\right) \;\ge\; 0 .$$

Proof: By a shift of the cycle (x, y, z) , we can force the number z to be the minimum value of x, y, and z. Then we have

$$\sum_{\text{cyclic}} \left(\frac{x}{y} - \frac{z+x}{z+y}\right) \;=\; \frac{(x-y)^2}{xy} + \frac{(x-z)(y-z)}{xz} - \frac{(x-y)^2}{(x+z)(y+z)} - \frac{(x-z)(y-z)}{(x+y)(z+x)} \;\ge\; 0 ,$$

as required. ∎

Lemma 2 For all positive real numbers a, b, and c,

$$\sum_{\text{cyclic}} \frac{(a+b)a}{(b+c)(2a+b+c)} \;\ge\; \frac{3}{4} . \qquad (1)$$

Proof: By Lemma 1 and Nesbitt's Inequality, for all positive real numbers x, y, and z we have

$$\sum_{\text{cyclic}} \frac{x(x-y+z)}{y(x+z)} = \sum_{\text{cyclic}} \left(\frac{x}{y} - \frac{z+x}{z+y}\right) + \sum_{\text{cyclic}} \frac{y}{x+z} \geq \frac{3}{2}.$$

We let $x = a+b$, $y = b+c$, and $z = c+a$ to obtain the inequality (1). ■

From the two lemmas, we conclude that

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a}{b} &\geq \sum_{\text{cyclic}} \frac{c+a}{c+b} \\ &= \sum_{\text{cyclic}} \frac{(a+b)a}{(b+c)(2a+b+c)} + \sum_{\text{cyclic}} \frac{(a+b)^2+(a+b)c}{(b+c)(2a+b+c)} \\ &\geq \frac{3}{4} + \sum_{\text{cyclic}} \frac{(a+b)^2+(a+b)c}{(b+c)(2a+b+c)}.\end{aligned}$$

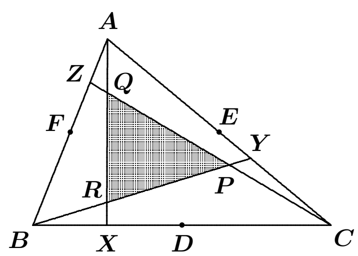
This is the desired inequality.

Also solved by the proposer.

Geupel indicated that Lemma 1 also appeared as an exercise in the Indian team selection test for the IMO 2002, see **Crux with Mayhem** *34 (2008) p. 151, Problem 20. Lemma 2 also appeared as Problem 28 on p. 203 in a compilation by Eckard Specht,* `http://www.imomath.com/othercomp/Journ/ineq.pdf` *with two references: T. Andreescu, V. Cîrtoaje, G. Dospinescu, and M. Lascu,* Old and New Inequalities, *and* Gazeta Matematica *[D. Olteanu]. He could not verify any of these sources. Indeed, Lemma 2 appears as problem 28, p. 11 of* Old and New Inequalities, *attributed there to D. Olteanu and as appearing in* Gazeta Matematica. *However, we could not verify the latter sources.*

3315. [2008 : 102, 105] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Let D, E, and F be the mid-points of sides BC, CA, and AB, respectively, in $\triangle ABC$. Let X, Y, and Z be points on the segments BD, CE, and AF, respectively. The lines AX, BY, and CZ bound a central triangle (shaded in the diagram). Let X', Y', and Z' be the reflections of X, Y, and Z in the points D, E, and F, respectively. The points X', Y', and Z' determine in a similar manner another central triangle $P'Q'R'$.



Prove that

$$\frac{2+\sqrt{3}}{4} \leq \frac{[PQR]}{[P'Q'R']} \leq 8 - 4\sqrt{3},$$

where $[STU]$ represents the area of $\triangle STU$.

Solution by Michel Bataille, Rouen, France.

Let $\mathcal{K} = [0,1]\times[0,1]\times[0,1]$ and let $x = \dfrac{BX}{XC}$, $y = \dfrac{CY}{YA}$, and $z = \dfrac{AZ}{ZB}$. Note that $(x,y,z) \in \mathcal{K}$ and that $x = \dfrac{CX'}{X'B}$, $y = \dfrac{AY'}{Y'C}$, and $z = \dfrac{BZ'}{Z'A}$.

Using Routh's Theorem (see problem 2752 [2002 : 329; 2003 : 331]), we see that $\rho(x,y,z) = \frac{[PQR]}{[P'Q'R']}$ is given by

$$\rho(x,y,z) \;=\; \frac{(1+x+zx)(1+y+xy)(1+z+yz)}{(1+z+zx)(1+x+xy)(1+y+yz)}\,.$$

The continuous function ρ attains its maximum M and its minimum m on the compact set $\mathcal{K}$; we will show that $m = \dfrac{2+\sqrt{3}}{4}$ and $M = 8-4\sqrt{3}$.

First, we restrict ρ to an edge of the cube $\mathcal{K}$, characterized by one variable equal to 0 and another equal to 1, say $x=1$, $y=0$. A quick study of $\phi(z) = \rho(1,0,z) = \dfrac{2+3z+z^2}{2(1+2z)}$ on $[0,1]$ shows that

$$1 \;\geq\; \phi(z) \;\geq\; \phi\left(\frac{\sqrt{3}-1}{2}\right) \;=\; \frac{2+\sqrt{3}}{4}\,.$$

Similarly, on the edge $x=0$, $y=1$, we have

$$1 \;\leq\; \rho(0,1,z) \;=\; \frac{1}{\phi(z)} \;\leq\; 8-4\sqrt{3} \;=\; \rho(0,1,\frac{\sqrt{3}-1}{2})\,.$$

It follows that $m \leq \dfrac{2+\sqrt{3}}{4}$ and $M \geq 8-4\sqrt{3}$ and that it suffices to prove that M, m cannot be attained on either

(a) an edge of the cube $\mathcal{K}$ where two variables are equal,

(b) the interior of the cube $\mathcal{K}$, or

(c) the interior of a face of the cube $\mathcal{K}$.

A key remark is that $m<1$ and $M>1$, so that m, M cannot be attained at a triple containing two equal numbers, because ρ takes the value 1 at such a triple (it is easily checked, for example, that $\rho(x,x,z)=1$). This remark settles case (a).

(b) Assume that ρ has an extremum at an interior point (x,y,z) of $\mathcal{K}$. Then by the previous case x, y, z are distinct and we have that $\dfrac{\partial\rho}{\partial x}$, $\dfrac{\partial\rho}{\partial y}$, and $\dfrac{\partial\rho}{\partial z}$ all vanish at (x,y,z). Using logarithmic derivatives, we may rewrite the latter condition as

$$\frac{p(z,z)}{p(x,z)p(z,x)} \;=\; \frac{p(y,y)}{p(x,y)p(y,x)}\,; \qquad \frac{p(z,z)}{p(y,z)p(z,y)} \;=\; \frac{p(x,x)}{p(x,y)p(y,x)}\,;$$

$$\frac{p(x,x)}{p(x,z)p(z,x)} \;=\; \frac{p(y,y)}{p(y,z)p(z,y)}\,;$$

where $p(u,v) = 1+u+uv$. Since $p(x,x)p(y,y) = p(x,y)p(y,x)+(x-y)^2$, the preceding equations are equivalent to

$$\left(1+x+x^2\right)(y-z)^2 \;=\; \left(1+y+y^2\right)(z-x)^2 \;=\; \left(1+z+z^2\right)(x-y)^2 \,.$$

Solving the first equation for z leads to

$$(1+x+y)z + 1 - xy \;=\; \sqrt{\left(1+x+x^2\right)\left(1+y+y^2\right)}\,,$$

and similarly $(1+y+z)x + 1 - yz = \sqrt{\left(1+y+y^2\right)\left(1+z+z^2\right)}$. By subtracting these two expressions and dividing by $(z-x)$, we arrive at

$$\frac{\sqrt{1+y+y^2}}{\sqrt{1+x+x^2}+\sqrt{1+z+z^2}}(1+z+x) \;=\; -(1+2y)\,,$$

a contradiction since $1+z+x > 0$ and $1+2y > 0$ in the interior of $\mathcal{K}$.

(c) We can limit the study to the faces $x = 0$ and $x = 1$. Consider first $\rho(0,y,z) = \dfrac{(1+y)(1+z+yz)}{(1+z)(1+y+yz)}$. The necessary condition for an extremum now reads as

$$(1+y)\left(1+z+z^2\right) \;=\; (1+z)\left(1+y+y^2\right) \;=\; (1+z+yz)(1+y+yz)$$

and the first equation yields $(z-y)(z+y+yz) = 0$, which is impossible when $y \neq z$. Similarly, the study of possible extrema for $\rho(1,y,z)$ leads to the condition $\dfrac{(2+y)(1+2y)}{1+y+y^2} = \dfrac{(2+z)(1+2z)}{1+z+z^2}$, again impossible for $y \neq z$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; and the proposer.

3316. [2008 : 103, 105] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\sum_{\text{cyclic}} \frac{a}{b} + \Big(\sum_{\text{cyclic}} a^2\Big)^{\frac{1}{2}}\Big(\sum_{\text{cyclic}} \frac{1}{a^2}\Big)^{\frac{1}{2}} \;\geq\; \frac{2}{3}\Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} \frac{1}{a}\Big)\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Applying the Cauchy-Schwarz Inequality and the AM–GM Inequality, we obtain:

$$\begin{aligned}
&\sum_{\text{cyclic}} \frac{a}{b} + \left(\sum_{\text{cyclic}} a^2\right)^{\frac{1}{2}} \left(\sum_{\text{cyclic}} \frac{1}{a^2}\right)^{\frac{1}{2}} \\
&= \left(\sum_{\text{cyclic}} \frac{a}{b}\right) + \left((a^2+b^2+c^2)\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)\right)^{\frac{1}{2}} \\
&\geq \left(\sum_{\text{cyclic}} \frac{a}{b}\right) + \left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right) \\
&= \frac{2}{3}\left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right) + \frac{1}{3}\left(\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-6\right) \\
&\geq \frac{2}{3}\left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right) + \frac{1}{3}(2+2+2-6) \\
&= \frac{2}{3}\left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right).
\end{aligned}$$

Equality holds if and only if $a = b = c$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3317. [2008 : 103, 105] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\begin{aligned}
&\left(\sum_{\text{cyclic}} \frac{a^3}{b^2-bc+c^2}\right)\left(\sum_{\text{cyclic}} \frac{b^2c^2}{a^3(b^2-bc+c^2)}\right) \\
&\quad\geq \left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right) \geq 16\left(\sum_{\text{cyclic}} \frac{ab}{a+b+2c}\right)\left(\sum_{\text{cyclic}} \frac{c}{2ab+bc+ca}\right).
\end{aligned}$$

Similar solutions by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam and by the proposer.

It suffices for us to prove the two inequalities on the following page:

$$\sum_{\text{cyclic}} \frac{a^3}{b^2 - bc + c^2} \;\geq\; \sum_{\text{cyclic}} a \;\geq\; 4 \sum_{\text{cyclic}} \frac{ab}{a+b+2c}\,, \tag{1}$$

$$\sum_{\text{cyclic}} \frac{b^2c^2}{a^3\left(b^2 - bc + c^2\right)} \;\geq\; \sum_{\text{cyclic}} \frac{1}{a} \;\geq\; 4 \sum_{\text{cyclic}} \frac{c}{2ab+bc+ca}\,. \tag{2}$$

Hence,

$$\begin{aligned} 4 \sum_{\text{cyclic}} \frac{ab}{a+b+2c} &= 4 \sum_{\text{cyclic}} \frac{ab}{(c+a)+(c+b)} \\ &\leq \sum_{\text{cyclic}} \left(\frac{ab}{c+a} + \frac{ab}{c+b} \right) \\ &= \frac{c(a+b)}{a+b} + \frac{a(b+c)}{b+c} + \frac{b(c+a)}{c+a} \;=\; \sum_{\text{cyclic}} a\,. \end{aligned}$$

This establishes the last inequality of (1). By the Cauchy–Schwartz Inequality, we have

$$\begin{aligned} \sum_{\text{cyclic}} \frac{a^3}{\left(b^2 - bc + c^2\right)} &= \sum_{\text{cyclic}} \frac{a^4}{a\left(b^2 - bc + c^2\right)} \\ &\geq \left(\sum_{\text{cyclic}} a^2 \right)^2 \left(\sum_{\text{cyclic}} a\left(b^2 - bc + c^2\right) \right)^{-1}\,. \end{aligned}$$

However, the following inequalities are equivalent

$$\begin{aligned} \left(\sum_{\text{cyclic}} a^2 \right)^2 \left(\sum_{\text{cyclic}} a\left(b^2 - bc + c^2\right) \right)^{-1} &\geq \sum_{\text{cyclic}} a\,; \\ \left(\sum_{\text{cyclic}} a \right) \left(\sum_{\text{cyclic}} a\left(b^2 - bc + c^2\right) \right) &\leq \left(\sum_{\text{cyclic}} a^2 \right)^2\,; \\ \sum_{\text{cyclic}} a^4 \;+\; abc \sum_{\text{cyclic}} a &\geq \sum_{\text{cyclic}} ab\left(a^2 + b^2\right)\,; \\ \sum_{\text{cyclic}} a^2(a-b)(a-c) &\geq 0\,. \end{aligned}$$

The last inequality is Schur's inequality, and this completes the proof of (1).

To prove (2) we again use the inequality $\dfrac{4}{x+y} \leq \left(\dfrac{1}{x} + \dfrac{1}{y} \right)$ for positive

x and y, as follows

$$\begin{aligned} 4\sum_{\text{cyclic}} \frac{c}{2ab+bc+ca} &= 4\sum_{\text{cyclic}} \frac{c}{(ab+bc)+(ab+ac)} \\ &\le \sum_{\text{cyclic}} \left(\frac{c}{b(a+c)}+\frac{c}{a(b+c)}\right) \\ &= \frac{(a+c)}{b(a+c)} + \frac{(b+c)}{a(b+c)} + \frac{(a+b)}{c(a+b)} = \sum_{\text{cyclic}} \frac{1}{a}\,. \end{aligned}$$

This proves the second inequality in (2).

By setting $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$, the first inequality in (2) becomes

$$\sum_{\text{cyclic}} \frac{x^3}{y^2-yz+z^2} \ \ge\ \sum_{\text{cyclic}} x\,,$$

which follows from (1). This completes the proof of (2).

Equality holds if and only if $a = b = c$.

Also solved by ARKADY ALT, San Jose, CA, USA; and TITU ZVONARU, Comăneşti, Romania. There was one incorrect solution submitted.

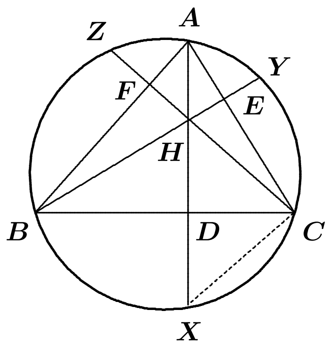
3318. [2008 : 103, 105] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

The altitudes AD, BE, and CF of $\triangle ABC$ are produced to meet the circumcircle at X, Y, and Z, respectively. Prove that

$$\frac{AX}{AD}+\frac{BY}{BE}+\frac{CZ}{CF} \ =\ 4\,.$$

Similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Ricardo Barroso Campos, University of Seville, Seville, Spain; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Apostolis K. Demis, Varvakeio High School, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; and George Tsapakidis, Agrinio, Greece.

Let H be the orthocentre of $\triangle ABC$. The proof is trivial when the triangle ABC is right-angled. We now assume that $\triangle ABC$ is acute. The right triangles CDH and CDX are congruent, because



$$\begin{aligned} \angle DHC &= 90^\circ - \angle DCH \\ &= 90^\circ - \angle BCF \\ &= 90^\circ - (90^\circ - \angle B) \\ &= \angle B \ =\ \angle AXC \\ &= \angle DXC\,. \end{aligned}$$

Hence, $XD = HD$. Similarly, $YE = HE$ and $ZF = HF$. Let $[ABC]$ denote the area of triangle ABC. We have

$$\begin{aligned}\frac{AX}{AD}+\frac{BY}{BE}+\frac{CZ}{CF} &= \frac{AD+XD}{AD}+\frac{BE+YE}{BE}+\frac{CF+ZF}{CF}\\ &= \frac{AD+HD}{AD}+\frac{BE+HE}{BE}+\frac{CF+HF}{CF}\\ &= 3+\frac{HD}{AD}+\frac{HE}{BE}+\frac{HF}{CF}\\ &= 3+\frac{[HBC]}{[ABC]}+\frac{[HAC]}{[ABC]}+\frac{[HAB]}{[ABC]}\\ &= 3+\frac{[ABC]}{[ABC]} = 4\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA (2 solutions); KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Bataille, Geupel, and Peiró gave solutions using signed distances, for which the featured solution is valid for all triangles, but the statement (as given) is not true for obtuse triangles. For example, if $AB = AC$, then AX is fixed (it is the diameter of the circumcircle), so the ratio $\frac{AX}{AD}$ can be larger than 4; in fact, it can be made arbitrarily large. The correct statement for an obtuse triangle, say, with $\angle A > 90^\circ$, B between E and Y, and C between F and Z would be

$$\frac{AX}{AD}-\frac{BY}{BE}-\frac{CZ}{CF} = 4\,.$$

3319. [2008 : 103, 106] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let m be a natural number, $m \geq 2$, and let r be any real number such that $r \geq 1/m$. If a and b are positive real numbers satisfying $ab = r^2$, prove that

$$\frac{1}{(1+a)^m}+\frac{1}{(1+b)^m} \geq \frac{2}{(1+r)^m}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

First we show that for all positive real numbers x,

$$\frac{(x+1)^2}{x^{2/(m+1)}} \geq \frac{m+1}{m-1}\cdot\frac{1-r^2}{r^{2/(m+1)}}\,. \qquad (1)$$

To prove (1), consider the function $h(x) = \frac{(x+1)^2}{x^{2/(m+1)}}$ for positive x. From the derivative

$$h'(x) \;=\; \frac{2(x+1)}{x^{(m+3)/(m+1)}}\left(x - \frac{x+1}{m+1}\right),$$

we see that $h'(x) \le 0$ for $x \in \left(0, \frac{1}{m}\right]$ while $h'(x) \ge 0$ for $x \in \left[\frac{1}{m}, \infty\right)$. Therefore, h takes its minimum value at $x = \frac{1}{m}$ and we have

$$h\left(\frac{1}{m}\right) \;=\; \frac{m+1}{m-1}\left(1 - \frac{1}{m^2}\right)m^{2/(m+1)} \;\ge\; \frac{m+1}{m-1}\cdot\frac{1-r^2}{r^{2/(m+1)}},$$

which completes the proof of (1).

Next we claim that

$$k(x) \;=\; r^2x^{m-1}(1+x)^{m+1} - \left(x+r^2\right)^{m+1}$$

is not positive if $0 < x \le r$ and is not negative if $x \ge r$. For this, consider the function $g(x) = r^{2/(m+1)}x^{(m-1)/(m+1)} - \frac{x+r^2}{x+1}$, for which we have

$$g'(x) \;=\; \frac{m-1}{m+1}\cdot\frac{r^{2/(m+1)}}{x^{2/(m+1)}} - \frac{1-r^2}{(x+1)^2}.$$

Observe that $g(r) = 0$ and, by (1), $g'(x) \ge 0$ for all $x > 0$, which implies our claim regarding $k(x)$.

Finally, we prove the required inequality by writing $x = a$, $b = \frac{r^2}{x}$, and by considering for $x > 0$ the function $f(x) = \frac{1}{(1+x)^m} + \frac{1}{(1+r^2/x)^m}$. We have

$$f'(x) \;=\; \frac{m}{\left((1+x)(x+r^2)\right)^{m+1}}\,k(x).$$

From what we know about $k(x)$ we have that $f'(x) \le 0$ for $0 < x \le r$, and $f'(x) \ge 0$ for $x \ge r$. Therefore, f takes its minimum value at $x = r$, which is $f(r) = \frac{2}{(1+r)^m}$, as desired.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete and three incorrect solutions submitted.

3320. [2008 : 103, 106] *Proposed by Michel Bataille, Rouen, France.*

Let $\triangle ABC$ be right-angled at A and let O be the midpoint of BC. Let M be a point in the plane of $\triangle ABC$, and let M', M'', N, N', and N'' denote the orthocentres of $\triangle MAB$, $\triangle MAC$, $\triangle AM'M''$, $\triangle NAB$, and $\triangle NAC$, respectively. If O is the midpoint of $M'M''$, show that O is also the midpoint of $N'N''$.

Solution by the proposer.

Let us assume that M lies on neither AB nor AC so that M' and M'' will be well defined. We first find the locus of points M such that O is the midpoint of $M'M''$. To this end we choose a system of axes with origin at O, such that the vertices have coordinates $A(-b,-c)$, $B(b,-c)$, and $C(-b,c)$. Let M have coordinates $M(x,y)$. Since $MM' \perp AB$ and $MM'' \perp AC$, we have coordinates $M'(x,\beta)$ and $M''(\alpha,y)$ for some pair of real numbers α and β. For BM' and CM'' perpendicular to AM we obtain

$$(x+b)(x-b)+(y+c)(\beta+c) \;=\; 0 \;=\; (x+b)(\alpha+b)+(y+c)(y-c)\,. \quad (1)$$

If O is the midpoint of $M'M''$, then $\alpha=-x$, $\beta=-y$, and (1) becomes

$$x^2-y^2 \;=\; b^2-c^2\,. \quad (2)$$

Conversely, if (2) holds then by (1) we have $(\alpha+b)(x+b)=c^2-y^2=b^2-x^2$, which (since we assume $x\neq -b$) implies $\alpha=-x$. Similarly, $\beta=-y$ and O is the midpoint of MM'. Note that (2) is the equation of a rectangular hyperbola $\mathcal{H}$ which contains A, B, and C. (It degenerates into the pair of lines OA and BC if $\triangle ABC$ is isosceles, in which case $b=\pm c$.)

We now appeal to the standard theorem that if a triangle is inscribed in a rectangular hyperbola, its orthocentre is also on the hyperbola: If O is the midpoint of $M'M''$, then M is on $\mathcal{H}$ (as we have just seen), and since A and B are also on $\mathcal{H}$, we conclude that M' is on $\mathcal{H}$ as well. Similarly M'' is on $\mathcal{H}$, and so $AM'M''$ forms a triangle that is inscribed in $\mathcal{H}$. As a consequence, N is likewise on $\mathcal{H}$, whence O is the midpoint of $N'N''$. This concludes the argument for all positions of M where the M's and N's are all well defined; the remaining positions can be handled easily by noting that as M moves continuously along $\mathcal{H}$, so do N, N', and N''.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3321. [2008 : 103, 108] *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of $\triangle ABC$ have centre I and meet the sides AC and AB at E and F, respectively. For a point M on the line segment EF, show that $\triangle MAB$ and $\triangle MCA$ have the same area if and only if $MI \perp BC$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let P and Q be the projections of M on the sides AC and AB, respectively. Note that because M is on the segment EF, it lies inside $\triangle ABC$; moreover, because $AF=AE$ the right triangles MQF and MPE are similar, so that

$$\frac{MQ}{MP} \;=\; \frac{MF}{ME}\,. \quad (1)$$

Triangles MAB and MCA have equal areas if and only if $c \cdot MQ = b \cdot MP$, or $\frac{b}{c} = \frac{MQ}{MP}$. Consequently, from (1) we see that it is sufficient to prove that MI is perpendicular to BC if and only if

$$\frac{b}{c} = \frac{MF}{ME}. \qquad (2)$$

Let D be the projection of I on BC. Suppose first that M lies on ID. Since $BDIF$ is cyclic, $\angle MIF = \angle DBF = B$, and $\angle MIE = \angle ECD = C$. Note that $IE = IF$ (which equal the inradius) whence, by the Law of Sines applied to triangles MFI and MEI,

$$\frac{MF}{\sin B} = \frac{FI}{\sin \angle IMF} = \frac{EI}{\sin(\pi - \angle IMF)} = \frac{ME}{\sin C}.$$

We can conclude that $\frac{MF}{ME} = \frac{\sin B}{\sin C} = \frac{b}{c}$ holds by applying the Law of Sines to $\triangle ABC$, thus satisfying the requirement of equation (2). Conversely, suppose that equation (2) holds, and define M' to be the point where the line ID intersects EF. Then from the previous argument applied to M', it must be the unique point on EF for which $\frac{M'F}{M'E} = \frac{b}{c}$. We conclude that $M = M'$, and the proof is complete.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3322. [2008 : 104, 106] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be nonnegative real numbers such that $a \le b \le c$, and let n be a positive integer. Prove that

$$\big(a + (n+1)b\big)\big(b + (n+2)c\big)(c + na) \ \ge\ (n+1)(n+2)(n+3)abc.$$

I. *Solution by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.*

We will prove a more general result: If $0 < a_1 \le a_2 \le \cdots \le a_n$ and $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, then

$$\begin{aligned}(a_1 + \lambda_2 a_2)(a_2 + \lambda_3 a_3) \cdots (a_n + \lambda_1 a_1) \\ \ge\ (1+\lambda_1)(1+\lambda_2) \cdots (1+\lambda_n) a_1 a_2 \cdots a_n. \qquad (1)\end{aligned}$$

To prove this, we will use Karamata's inequality applied to the concave function $f(x) = \log x$.

Let r be such that

$$a_r + \lambda_{r+1}a_{r+1} \;\geq\; a_n + \lambda_1 a_1 \;\geq\; a_{r-1} + \lambda_r a_r\,.$$

Then, it is not difficult to see that

$$\begin{aligned}&(a_n + \lambda_n a_n, a_{n-1} + \lambda_{n-1}a_{n-1}, \ldots, a_1 + \lambda_1 a_1)\\ &\quad\succ\;\; (a_{n-1} + \lambda_n a_n, \ldots, a_r + \lambda_{r+1}a_{r+1}; a_n + \lambda_1 a_1, \ldots, a_1 + \lambda_2 a_2)\,.\end{aligned}$$

By Karamata's inequality,

$$\begin{aligned}&\log(a_1 + \lambda_2 a_2) + \log(a_2 + \lambda_3 a_3) + \cdots + \log(a_n + \lambda_1 a_1)\\ &\quad\geq\;\; \log(a_1 + \lambda_1 a_1) + \log(a_2 + \lambda_2 a_2) + \cdots + \log(a_n + \lambda_n a_n)\end{aligned}$$

and exponentiating yields inequality (1).

II. *Solution by Titu Zvonaru, Comăneşti, Romania.*

By the AM–GM Inequality we have

$$\frac{a + (n+1)b}{n+2} \;\geq\; a^{\frac{1}{n+2}}b^{\frac{n+1}{n+2}}\,; \qquad \frac{b + (n+2)c}{n+3} \;\geq\; b^{\frac{1}{n+3}}c^{\frac{n+2}{n+3}}\,;$$

$$\frac{c + na}{n+1} \;\geq\; c^{\frac{1}{n+1}}a^{\frac{n}{n+1}}\,.$$

These three inequalities imply that

$$\big(a + (n+1)b\big)\big(b + (n+2)c\big)(c + na) \;\geq\; (n+1)(n+2)(n+3)a^p b^q c^r\,,$$

where $p = \dfrac{n^2+3n+1}{(n+1)(n+2)}$, $q = \dfrac{n^2+5n+5}{(n+2)(n+3)}$, and $r = \dfrac{n^2+4n+5}{(n+1)(n+3)}$. Now, since $0 \leq a \leq b \leq c$ and $r = 1 + \dfrac{1}{(n+1)(n+2)} + \dfrac{1}{(n+2)(n+3)}$, we have

$$c^r \;=\; c^1\, c^{\frac{1}{(n+1)(n+2)}}\, c^{\frac{1}{(n+2)(n+3)}} \;\geq\; a^{\frac{1}{(n+1)(n+2)}}\, b^{\frac{1}{(n+2)(n+3)}}\, c\,,$$

and hence,

$$(n+1)(n+2)(n+3)a^p b^q c^r \;\geq\; (n+1)(n+2)(n+3)abc\,.$$

This proves the proposed inequality. Equality holds if and only if $a = b = c$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer. There was one incorrect solution submitted.

Several solvers noted that the inequality holds if the positive integer n is replaced by a nonnegative real number.

3323. [2008 : 104, 106] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be nonnegative real numbers with $a^2 + b^2 + c^2 = 1$. Prove that

$$\sum_{\text{cyclic}} \left(1 - 2a^2\right)(b - c)^2 \ \geq\ 0\,.$$

Solution by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain, expanded by the editor.

Since $1 = a^2 + b^2 + c^2$, we have

$$\begin{aligned}
\sum_{\text{cyclic}} \left(1 - 2a^2\right)(b - c)^2 &= \sum_{\text{cyclic}} \left(-a^2 + b^2 + c^2\right)(b - c)^2 \\
&= \sum_{\text{cyclic}} a^2 \left[(a - b)^2 - (b - c)^2 + (c - a)^2\right] \\
&= \sum_{\text{cyclic}} a^2 \left(2a^2 - 2ab + 2bc - 2ca\right) \\
&= 2\sum_{\text{cyclic}} a^2(a - b)(a - c)\,,
\end{aligned}$$

which is nonnegative by Schur's inequality. Equality holds if and only if one of the numbers a, b, or c is 0 and the other two are $\frac{\sqrt{2}}{2}$, or if all three of them are $\frac{\sqrt{3}}{3}$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; GEORGE TSAPAKIDIS, Agrinio, Greece; GEORGE VELISARIS, medical student, Athens, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3324. [2008 : 104, 106] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be nonnegative real numbers with $a^2 + b^2 + c^2 = 1$. Prove that

$$3 - 5(ab + bc + ca) + 6abc(a + b + c) \ \geq\ 0\,.$$

Essentially similar solutions by George Apostolopoulos, Messolonghi, Greece; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

We use a standard homogenization idea: Since $1 = a^2 + b^2 + c^2$, we can write the inequality to be proved as

$$3\left(a^2+b^2+c^2\right)^2 - 5(ab+bc+ca)\left(a^2+b^2+c^2\right) + 6abc(a+b+c) \;\geq\; 0\,.$$

The above inequality is equivalent to

$$3\sum_{\text{cyclic}} a^4 \;-\; 5\sum_{\text{cyclic}}\left(a^3b+ab^3\right) \;+\; 6\sum_{\text{cyclic}} a^2b^2 \;+\; \sum_{\text{cyclic}} a^2bc \;\geq\; 0\,,$$

which we group as

$$\left(\sum_{\text{cyclic}} a^4 \;+\; \sum_{\text{cyclic}} a^2bc \;-\; \sum_{\text{cyclic}} ab\left(a^2+b^2\right)\right) \;+\; \sum_{\text{cyclic}}\left[a^4+b^4-4ab\left(a^2+b^2\right)+6a^2b^2\right] \;\geq\; 0\,,$$

which reduces to

$$\sum_{\text{cyclic}} a^2(a-b)(a-c) \;+\; \sum_{\text{cyclic}}(a-b)^4 \;\geq\; 0\,.$$

The last inequality is true, because the second sum is obviously nonnegative and the first sum is nonnegative by Schur's inequality. Equality holds if and only if $a = b = c = \sqrt{3}/3$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; GEORGE VELISARIS, medical student, Athens, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer. There was one incomplete solution submitted.

3325. [2008 : 104, 106] *Proposed by Manuel Benito Muñoz, IES P.M. Sagasta, Logroño, Spain.*

Let $\sigma(n)$ denote the sum of the divisors of the natural number n.

(a) Find a natural number n such that

$$\sigma(n) + 500 \;=\; \sigma(n+2)\,.$$

(b)★ How many solutions are there to part (a)?

Solution to (a) *by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.*

The prime power decompositions of 2005 and 2007 are $2005 = 5 \times 401$ and $2007 = 3^2 \times 223$, respectively.

Hence, $\sigma(2005) = 1 + 5 + 401 + 2005 = 2412$ and

$$\sigma(2007) = 1 + 3 + 9 + 223 + 669 + 2007 = 2912 = \sigma(2005) + 500\,.$$

Therefore, $n = 2005$ is a solution.

Also solved (part (a) *only) by MOHAMMED AASSILA, Strasbourg, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; OLIVER GEUPEL, Brühl, NRW, Germany; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer. No solution to part* (b) *was received, hence part* (b) *remains open.*

All solvers gave the solution $n = 2005$*. By using Mathematica, Wagon found that* $n = 2005$ *is the only solution for* $n \le 42213628$*. However,* $n = 42213629$ *is a second solution. To see this, observe that the prime power decompositions of* n *and* $n + 2$ *are* $42213629 = 109 \times 387281$*, and* $42213631 = 229 \times 337 \times 547$*, respectively. Hence,* $\sigma(42213629) = 1 + 109 + 387281 + 42213629 = 42601020$ *and*

$$\begin{aligned}\sigma(42213631) &= 1 + 229 + 337 + 547 + 77173 + 125263 + 184339 + 42213631 \\ &= 42601520 \;=\; \sigma(42213629) + 500\,.\end{aligned}$$

By further searching, he also found a third solution: $n = 60992425$ *for which* $\sigma(n + 2) = 81448780 = \sigma(n) + 500$*. He remarked that these three solutions are the only ones for* n *up to* 10^8*, and stated that "*$\cdots$ *perhaps there are infinitely many [solutions]".*

The density of a set $S \subseteq \mathbb{Z}^+$ *is* $\lim\limits_{n\to\infty} \frac{|S\cap[1,n]|}{n}$*, provided the limit exists. If* $n = 2^3 m$*,* m *odd, then* $\sigma(n + 2) \equiv \sigma(n) \equiv 0 \pmod 3$*, hence* $\sigma(n + 2) \ne \sigma(n) + 500$*. Thus,* $S_3 = \{n : n = 2^3 m,\ m$ *an odd positive integer*$\}$ *has density* $\frac{1}{16}$ *yet contains no solution of the equation. For various integers* m *there are sets* $S_m \subseteq \mathbb{Z}^+$ *of positive density such that* $\sigma(n + 2) \equiv \sigma(n) \equiv 0 \pmod m$ *for* $n \in S_m$ *but* 500 *is not divisible by* m*. This suggests sieving as an attempt to answer the question "how many solutions?" in terms of density, and for reducing the search space for solutions.*